Some New Fixed Point Results for Multi-Valued Weakly Picard Operators

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Abstract. In this paper, we present some extensions of Banach contraction principle for multi-valued maps. Corresponding convergence theorems for the Picard iteration associated to a class of multi-valued operators are obtained in the setting of modular metric spaces. The presented results improve many recent fixed points results in the setting of modular metric spaces and also generalize some classical known results. Moreover, some examples are given.

1. Introduction

In 1969, Nadler [21] introduced the notion of multi-valued Lipschitz mappings as a generalization of the Banach contraction principle in the setting of complete metric spaces. Since then, several authors investigated fixed point results in this direction, see [5–7, 9, 10, 13, 14, 20, 28]. In 2010, Chistyakov [11, 12] introduced the concept of modular metric spaces. There are different approaches for this concept. The class of modular metric spaces is viewed as the nonlinear version of the classical modular spaces introduced in [18, 19, 24] (see also [8, 16]). Recently, Abdou and Khamsi [1] investigated the fixed point property in the setting of modular spaces and introduced the analog of the Banach contraction principle theorem in the setting of modular metric spaces. In 2014, Abdou and Khamsi [2] established some fixed point theorems for multi-valued Lipschitzian mappings defined on some subsets of modular metric spaces. In 2012, Samet \textit{et al.} [27] introduced the notion of $\alpha - \psi$–contractive mappings and $\alpha$–admissible mappings in metric spaces and obtained many fixed point results. Recently, Ali \textit{et al.} [3] generalized and extended the notion of $\alpha - \psi$–contractive mappings by introducing the notion of $(\alpha, \psi, \xi)$–contractive multi-valued mappings and gave fixed point theorems for such type mappings in metric spaces.

Motivated by [2], the purpose of this paper is to extend the results of Abdou and Khamsi [1] by using the concept of $\alpha$–admissible contractive mappings. We will establish some fixed point theorems involving such contractions in the setting of modular metric spaces.

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2. Preliminaries

Throughout the paper, \( \mathbb{N} \) and \( \mathbb{R} \) denote the set of positive integers and the set of all real numbers, respectively. In what follows, we recall some definitions and results we will need in the sequel.

Let \( X \) be a nonempty set. For a function \( \omega : (0, \infty) \times X \times X \to (0, \infty) \), we denote by

\[
\omega_\lambda(x, y) := \omega(\lambda, x, y) \quad \text{for all } \lambda > 0 \text{ and } x, y \in X.
\]

**Definition 2.1.** [11] A function \( \omega : (0, \infty) \times X \times X \to [0, \infty] \) is said to be a modular metric on \( X \) (or simply a modular if no ambiguity arises) if for all \( x, y, z \in X \), the following three axioms hold:

(i) \( x = y \) if and only if \( \omega_\lambda(x, y) = 0 \) for all \( \lambda > 0 \);

(ii) \( \omega_\lambda(x, y) = \omega_\lambda(y, x) \) for all \( \lambda > 0 \);

(iii) \( \omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y) \) for all \( \lambda, \mu > 0 \).

The pair \((X, \omega)\) is then called a modular metric space.

**Definition 2.2.** [11] Let \((X, \omega)\) be a modular metric space.

(1) \( \omega \) is said regular if the axiom (i) in Definition 2.1 is replaced by the following axiom:

\[
\lim_{\lambda \to \infty} \omega_\lambda(x, y) = 0 \quad \text{if and only if } x = y.
\]

(2) \( \omega \) is said convex if for \( \lambda, \mu > 0 \) and \( x, y, z \in X \), the following inequality is satisfied:

\[
\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu} \omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu} \omega_\mu(z, y).
\]

**Definition 2.3.** [11] Let \((X, \omega)\) be a modular metric space and \( x_0 \in X \) an arbitrarily element. Let

\[
X_\omega = X_\omega(x_0) = \{ x \in X : \lim_{\lambda \to \infty} \omega_\lambda(x, x_0) = 0 \}
\]

and

\[
X^*_\omega = X^*_\omega(x_0) = \{ x \in X : \exists \lambda = \lambda(x) \text{ such that } \omega_\lambda(x, x_0) < \infty \}.
\]

The sets \( X_\omega \) and \( X^*_\omega \) are called modular spaces (around \( x_0 \)).

Note that if \( \omega \) is a convex modular metric on \( X \), then \( X^*_\omega = X_\omega \).

**Proposition 2.4.** [11] If \((X, \omega)\) is a modular metric space, then the modular set \( X_\omega \) is a metric space with metric given by

\[
d_\omega(x, y) = \inf\{ \lambda > 0 : \omega_\lambda(x, y) \leq \lambda \}, \quad x, y \in X_\omega.
\]

**Proposition 2.5.** [11] Given a convex modular space \((X, \omega)\). Define

\[
d^*_\omega(x, y) = \inf\{ \lambda > 0 : \omega_\lambda(x, y) \leq 1 \}, \quad x, y \in X^*_\omega.
\]

Then \((X^*_\omega, d^*_\omega)\) is a metric space.

**Definition 2.6.** [11, 12] (Topological concepts)

Let \((X, \omega)\) be a modular metric space.

(1) The sequence \( \{x_n\} \) in \( X_\omega \) is said \( \omega \)-convergent to an element \( x \in X_\omega \) if and only if

\[
\lim_{n \to \infty} \omega_1(x_n, x) = 0.
\]

We say then that \( x \) is an \( \omega \)-limit of \( \{x_n\} \).
(2) The sequence \( \{x_n\} \) in \( X_\omega \) is said \( \omega \)-Cauchy if
\[
\lim_{n,m \to \infty} \omega_1(x_n, x_m) = 0.
\]

(3) We say that a subset \( M \) of \( X_\omega \) is \( \omega \)-closed if for any sequence \( \{x_n\} \) in \( M \) such that \( \lim_{n \to \infty} \omega_1(x_n, x) = 0 \), then, \( x \in M \).

(4) We say that a subset \( M \) of \( X_\omega \) is \( \omega \)-complete if any \( \omega \)-Cauchy sequence in \( M \) is \( \omega \)-convergent in \( M \).

(5) We say that a subset \( M \) of \( X_\omega \) is \( \omega \)-bounded if
\[
\delta_\omega(M) = \sup \{\omega_1(x, y) : x, y \in M\} < \infty.
\]

(6) We say that a subset \( M \) of \( X_\omega \) is \( \omega \)-compact if any sequence in \( M \) has a subsequence \( \omega \)-convergent in \( M \).

(7) We say that \( \omega \) satisfies the Fatou property if for every sequence \( \{x_n\} \) in \( X_\omega \) and all \( x, y \in X_\omega \), we have
\[
\lim_{n \to \infty} \omega_1(x_n, x) = 0 \Rightarrow \omega_1(x, y) \leq \liminf_{n \to \infty} \omega_1(x_n, y).
\]

**Definition 2.7.** [2] We say that \( \omega \) satisfies the \( \Delta_2 \)-condition, if \( \lim_{n \to \infty} \omega_1(x_n, x) = 0 \), for some \( \lambda > 0 \) implies \( \lim_{n \to \infty} \omega_1(x_n, x) = 0 \), for all \( \lambda > 0 \).

Following [2, 11, 12], the \( \omega \)-convergence and \( d_\omega \)-convergence are equivalent if and only if \( \omega \) satisfies the \( \Delta_2 \)-condition. Furthermore, if \( \omega \) is a convex modular, then \( d_\omega \) and \( d_\omega^* \) are equivalent.

**Definition 2.8.** [2] Let \( (X, \omega) \) be a modular metric space. We say that \( \omega \) satisfies the \( \Delta_2 \)-type condition, if for every \( \mu > 0 \), there exits \( C_\mu > 0 \) such that
\[
\omega_{\lambda/\mu}(x, y) \leq C_\mu \omega_\lambda(x, y), \quad \text{for all } \lambda > 0, x, y \in X, \text{ with } x \neq y.
\]

Obviously, if \( \omega \) satisfies the \( \Delta_2 \)-type condition, then it satisfies the \( \Delta_2 \)-condition.

**Definition 2.9.** [2] Let \( (X, \omega) \) be a modular metric space. Define the function \( \Omega \) as
\[
\Omega(t) = \sup \left\{ \frac{\omega_{\lambda/t}(x, y)}{\omega_\lambda(x, y)} : \lambda > 0, x, y \in X_\omega, x \neq y \right\} \quad \text{for every } t > 0.
\]

**Lemma 2.10.** [2] Let \( (X, \omega) \) be a modular metric space. Assume that \( \omega \) is a convex regular modular satisfying the \( \Delta_2 \)-type condition. Then
\begin{enumerate}
  
  
  
  
  (1) \( \Omega(t) < \infty \), for each \( t > 0 \);
  
  (2) \( \Omega \) is an increasing function with \( \Omega(1) = 1 \);
  
  (3) \( \Omega(st) \leq \Omega(s) \Omega(t) \) for each \( s, t > 0 \);
  
  (4) \( \Omega^{-1}(s) \Omega^{-1}(t) \leq \Omega^{-1}(st) \), where \( \Omega^{-1} \) is the function inverse of \( \Omega \);
  
  (5) for each \( x, y \in X_\omega \), with \( x \neq y \), we have
  \[
  d_\omega^*(x, y) \leq \frac{1}{\Omega^{-1}(1/\omega_1(x, y))}.
  \]
\end{enumerate}

We have the useful lemmas.

**Lemma 2.11.** [2] Let \( (X, \omega) \) be a modular metric space. Assume that \( \omega \) is a convex regular modular satisfying the \( \Delta_2 \)-type condition. Let \( \{x_n\} \) be a sequence in \( X_\omega \) such that
\[
\omega_1(x_{n+1}, x_n) \leq Ca^n \quad \forall n = 0, 1, 2, \ldots
\]

where \( C \) is an arbitrary constant and \( \alpha \in [0, 1) \). Then \( \{x_n\} \) is Cauchy for both \( \omega \) and \( d_\omega^* \).
Let \((X, \omega)\) be a modular metric space and \(M\) be a nonempty subset of \(X\).

We denote by

(a) \(K(M)\) the family of all nonempty and \(\omega\)-compact subsets of \(M\);

(b) \(CL(M)\) the family of all nonempty and \(\omega\)-closed subsets of \(M\);

(c) \(CB(M)\) the family of all nonempty, \(\omega\)-closed and \(\omega\)-bounded subsets of \(M\).

For \(A, B \in CB(M)\) and \(x \in X\), set

\[ H_\omega(A, B) = \max \{\sup\{\omega_1(x, B) : x \in A\}, \sup\{\omega_1(y, A) : y \in B\}\} \]

where \(\omega_1(x, A) = \inf\{\omega_1(x, y) : y \in A\}\). \(H_\omega\) is the Hausdorff modular metric on \(CB(M)\), induced by the modular metric \(\omega\).

**Proposition 2.12.** Let \((X, \omega)\) be a modular metric space. Consider the metric \(D_\omega\) as

\[ D_\omega((x, y), (u, v)) = d_\omega(x, u) + d_\omega(y, v) \quad \text{for all} \quad (x, y), (u, v) \in X_\omega \times X_\omega. \]

Then \((X_\omega \times X_\omega, D_\omega)\) is a metric space.

**Definition 2.13.** Let \((X, \omega)\) be a modular metric space, \(M\) be a nonempty subset of \(X\), and \(T : M \rightarrow CL(M)\) be a multi-valued mapping. The graph of \(T\) denoted by \(G(T)\) is the subset \(\{(x, y) : x \in M, y \in Tx\}\) of \(M \times M\). Then \(T\) is said to be closed if the graph of \(G(T)\) is a closed subset of \((M \times M, D_\omega)\).

**Definition 2.14.** A function \(f : X_\omega \rightarrow [0, \infty)\) is called lower semi-continuous (l.s.c) if, for any \(x \in X_\omega\), and \(\{x_n\} \subseteq X_\omega\) with \(\lim_{n \rightarrow \infty} x_n = x\) in \(X_\omega\), we have

\[ f(x) \leq \lim_{n \rightarrow \infty} f(x_n). \]

For a multi-valued map \(T : M \rightarrow CB(M)\), let \(f_T : M \rightarrow [0, \infty)\) be a function defined by

\[ f_T(x) = \omega_1(x, Tx). \]

**Definition 2.15.** [26] Let \((X, d)\) be a metric space and \(T : X \rightarrow CL(X)\) be a multivalued operator. We say that \(T\) is a multivalued weakly Picard (briefly, MWP) operator if for all \(x \in X\) and \(y \in Tx\), there exists a sequence \(\{x_n\}\) such that

(i) \(x_0 = x\) and \(x_1 = y\);

(ii) \(x_{n+1} \in Tx_n\) for all \(n = 0, 1, 2, \ldots\);

(iii) \(\{x_n\}\) is convergent and its limit is a fixed point of \(T\).

A sequence \(\{x_n\}\) satisfying conditions (i) and (ii) in Definition 2.15 is said a sequence of successive approximations of \(T\), starting from \(x_0\).

As in [4, 17], we give the following definition.

**Definition 2.16.** Let \((X, \omega)\) be a modular metric space and \(M\) be a nonempty subset of \(X\). Let \(T : M \rightarrow CB(M)\) be a multi-valued mapping. Such \(T\) is called \(\alpha\)-admissible if, for each \(x \in M\) and \(y \in Tx\) with \(\alpha(x, y) \geq 1\), we have \(\alpha(y, z) \geq 1\) for all \(z \in Ty\).

In this paper, we investigate several types of multivalued weakly Picard operators, and so we ensure the existence of fixed points. Some consequences and examples have been provided.
3. Fixed point of multi-valued weak contraction mappings

We start with the following useful technical lemmas (corresponding to the ones given in [21] on modular metric spaces).

**Lemma 3.1.** Let \((X, \omega)\) be a modular metric space, \(M\) be a nonempty subset of \(X\), and \(B \subseteq CB(M)\). If \(a \in M\) and \(\omega_1(a, B) < c\) with \(c > 0\), then there exists \(b \in B\) such that \(\omega_1(a, b) < c\).

**Lemma 3.2.** [2] Let \((X, \omega)\) be a modular metric space and \(M\) be a nonempty subset of \(X\). Let \(A, B \subseteq CB(M)\), then for every \(\varepsilon > 0\) and \(x \in A\), there exists \(y \in B\) such that
\[
\omega_1(x, y) \leq H_\omega(A, B) + \varepsilon.
\]
Moreover, if \(B\) is \(\omega\)-compact and \(\omega\) satisfies the Fatou property, then for every \(x \in A\) there exists \(y \in B\) such that
\[
\omega_1(x, y) \leq H_\omega(A, B).
\]

**Lemma 3.3.** [2] Let \((X, \omega)\) be a modular metric space. Assume that \(\omega\) satisfies the \(\Delta_2\)-condition. Let \(M\) be a nonempty subset of \(X\), and let \(A_n\) be a sequence of sets in \(CB(M)\). Suppose that \(\lim_{n \to \infty} H_\omega(A_n, A) = 0\), for \(A \subseteq CB(M)\). Then if \(x_0 \in A_0\) and \(\lim_{n \to \infty} x_n = x\), it follows that \(x \in A\).

### 3.1. Result-I

In this subsection, we first introduce the notion of \(\omega\)-quasi-contractions in modular metric spaces.

**Definition 3.4.** Let \((X, \omega)\) be a modular metric space and \(M\) be a nonempty subset of \(X\). A multi-valued \(T : M \to CL(M)\) is said an \(\omega\)-quasi-contraction if there exists a constant \(k \in [0, 1)\) such that for any \(x, y \in M\) with \(y \in Tx\), there exists \(z \in Ty\) such that
\[
\omega_1(x, z) \leq k \max(\omega_1(x, y), \omega_1(x, Tx), \omega_1(y, Ty)).
\]  

Now, we state and prove our first result.

**Theorem 3.5.** Let \((X, \omega)\) be a modular metric space. Assume that \(\omega\) is a convex regular modular satisfying the \(\Delta_2\)-condition. Let \(M\) be a nonempty \(\omega\)-complete subset of \(X\). Let \(T : M \to CL(M)\) be a closed \(\omega\)-quasi-contraction. Then \(T\) is a MWP operator.

**Proof.** Let \(x_0 \in M\) and \(x_1 \in Tx_0\). Clearly, if \(x_0 = x_1\), then \(x_1\) is a fixed point of \(T\) and so this completes the proof. Now, we assume that \(x_0 \neq x_1\). Since \(T\) is \(\omega\)-quasi-contraction, there exists \(x_2 \in Tx_1\) such that
\[
\omega_1(x_1, x_2) \leq k \max(\omega_1(x_0, x_1), \omega_1(x_0, Tx_0), \omega_1(x_1, Tx_1)).
\]
If \(x_2 = x_1\), then \(x_2\) is a fixed point of \(T\) and so the proof is finished.

From now on, we assume that \(x_2 \neq x_1\). It follows that
\[
\omega_1(x_1, x_2) \leq k \max(\omega_1(x_0, x_1), \omega_1(x_0, x_1), \omega_1(x_1, x_2)) = k \max(\omega_1(x_0, x_1), \omega_1(x_1, x_2)).
\]
If \(\max(\omega_1(x_0, x_1), \omega_1(x_1, x_2)) = \omega_1(x_1, x_2)\), then we obtain
\[
\omega_1(x_1, x_2) \leq k \omega_1(x_1, x_2) < \omega_1(x_1, x_2)
\]
which is a contradiction. Then we get
\[
\omega_1(x_1, x_2) \leq k \omega_1(x_0, x_1).
\]
Continuing in this fashion, we construct a sequence \(\{x_n\}\) in \(M\) such that \(x_{n+1} \in Tx_n, x_{n+1} \neq x_n\) and
\[
\omega_1(x_n, x_{n+1}) \leq k \omega_1(x_{n-1}, x_n) \quad \forall n = 1, 2, 3, \ldots
\]
By induction, we get
\[ \omega_1(x_n, x_{n+1}) \leq k^n \omega_1(x_0, x_1) \quad \forall n = 0, 1, 2, \ldots \]

(3)

By Lemma 2.11, we conclude that \( \{x_n\} \) is \( \omega \)-Cauchy. By completeness of \((M, \omega)\), there exists \( u \in M \) such that
\[ \lim_{n \to \infty} x_n = u. \]

We shall prove that \( i \) is a fixed point of \( T \). Since \( T \) is a closed multi-valued mapping and \( x_{n+1} \in Tx_n \), then \( (x_n, x_{n+1}) \in G(T) \). Moreover, we have
\[ D_\omega((x_n, x_{n+1}), (u, u)) = d_\omega(x_n, u) + d_\omega(x_{n+1}, u). \]

Since \( \omega \) satisfies the \( \Delta_2 \)-condition, we get
\[ \lim_{n \to \infty} D_\omega((x_n, x_{n+1}), (u, u)) = 0. \]

Finally, since \( G(T) \) is closed, it follows that \( (u, u) \in G(T) \). Hence \( u \in Tu \), that is, \( u \) is a fixed point of \( T \).

3.2. Result II

In this subsection, we give another characterization of MWP operators.

**Definition 3.6.** Let \((X, \omega)\) be a modular metric space and \( M \) be a nonempty subset of \( X_\omega \). A multi-valued \( T : M \to CB(M) \) is said an \( \alpha - \omega \)-weak contraction if there exist a function \( \alpha : M \times M \to [0, \infty) \) and two constants \( k \in [0, 1) \) and \( L \geq 0 \) such that for any \( x, y \in M \), with \( \alpha(x, y) \geq 1 \), we have
\[ H_\omega(Tx, Ty) \leq k \omega_1(x, y) + L \omega_1(y, Tx). \]

(4)

**Definition 3.7.** Let \((X, \omega)\) be a modular metric space and \( M \) be a nonempty subset of \( X_\omega \). The pair \((M, \alpha)\) is said \( \omega \)-regular if the following condition holds: for any sequence \( \{x_n\} \) in \( M \) with \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( x_n \to x \in M \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, x) \geq 1 \) for all \( k \in \mathbb{N} \).

**Remark 3.8.** Definition 3.7 is valid for Result II and Result III.

We provide the following result.

**Theorem 3.9.** Let \((X, \omega)\) be a modular metric space. Assume that \( \omega \) is a convex regular modular satisfying the \( \Delta_2 \)-condition. Let \( M \) be a nonempty \( \omega \)-complete subset of \( X_\omega \). Let \( T : M \to CB(M) \) be an \( \alpha - \omega \)-weak contraction. Suppose also that
(1) \( T \) is \( \alpha \)-admissible;
(2) there exit \( x_0 \in M \) and \( x_1 \in Tx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \);
(3) \((M, \alpha)\) is \( \omega \)-regular or \( f_\omega \) is lower semi-continuous.

Then \( T \) is a MWP operator.

**Proof.** Let \( r \) be a real number such that \( 0 < k < r < 1 \). By condition (2), there exist \( x_0 \in M \) and \( x_1 \in Tx_0 \) such that
\[ \alpha(x_0, x_1) \geq 1. \]

If \( x_0 = x_1 \), then \( x_1 \) is a fixed point of \( T \) and the proof is finished.

Now, we assume that \( x_0 \neq x_1 \). Since \( x_1 \in Tx_0 \), by (9), we have
\[ \omega_1(x_1, Tx_1) \leq H_\omega(Tx_0, Tx_1) \leq k \omega_1(x_0, x_1) + L \omega_1(x_1, Tx_0) \leq k \omega_1(x_0, x_1) + L \omega_1(x_1, x_1) = k \omega_1(x_0, x_1) < r \omega_1(x_0, x_1). \]

By Lemma 3.1, there exists \( x_2 \in Tx_1 \) such that
\[ \omega_1(x_1, x_2) < r \omega_1(x_0, x_1). \]

\( T \) is \( \alpha \)-admissible and \( x_2 \in Tx_1 \), so
\[ \alpha(x_1, x_2) \geq 1. \]
If \( x_2 = x_1 \), then \( x_2 \) is a fixed point of \( T \) and again the proof is finished. Now, we assume that \( x_2 \neq x_1 \).

Since \( x_2 \in Tx_1 \) and \( \alpha(x_1, x_2) \geq 1 \), by (9), we have

\[
\omega_1(x_2, Tx_2) \leq H_\alpha(Tx_1, Tx_2) \\
\leq k\omega_1(x_1, x_2) + L\omega_1(x_2, Tx_1) \leq k\omega_1(x_1, x_2) + L\omega_1(x_2, x_2) \\
= k\omega_1(x_1, x_2) < r\omega_1(x_1, x_2).
\]

Iterating this process, we can define a sequence \( \{x_n\} \) such that

\[
\alpha(x_n, x_{n+1}) \geq 1, \quad x_{n+1} \neq x_n, \quad x_{n+1} \in Tx_n
\]

and

\[
\omega_1(x_n, x_{n+1}) \leq r^n\omega_1(x_0, x_1) \quad \forall n = 1, 2, 3, \ldots
\]

Hence we obtain

\[
\omega_1(x_n, x_{n+1}) \leq r^n\omega_1(x_0, x_1) \quad \forall n = 0, 1, 2, \ldots
\]

By Lemma 2.11, we deduce that \( \{x_n\} \) is \( \omega \)-Cauchy. Since \( M \) is \( \omega \)-complete, there exists \( v \in M \) such that

\[
\lim_{n \to \infty} x_n = v.
\]

We shall prove that \( v \) is a fixed point of \( T \). Since \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( x_n \to v \), in view of the fact that \((M, \alpha)\) is \( \omega \)-regular, there exists a subsequence \( \{x_{n(m)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(m)}, v) \geq 1 \) for all \( m \in \mathbb{N} \).

We have for any \( m \geq 0 \)

\[
H_\alpha(Tx_{n(m)},Tv) \leq k\omega_1(x_{n(m)}, v) + L\omega_1(x_{n(m)}, Tx_{n(m)}) \\
\leq k\omega_1(x_{n(m)}, v) + L\omega_1(v, x_{n(m)+1})
\]

Passing to limit as \( m \to \infty \), we get \( \lim_{m \to \infty} H_\alpha(Tx_{n(m)},Tv) = 0 \). Since \( x_{n(m)+1} \in Tx_{n(m)} \), by Lemma 3.3, we conclude that \( v \in Tv \), that is, \( v \) is a fixed point of \( T \).

Now, passing to the case where \( f_T \) is lower semi-continuous, we have

\[
\omega_1(v, Tv) = f_T(v) \leq \lim_{n \to \infty} \omega_1(x_n, Tx_n) \leq \lim_{n \to \infty} \omega_1(x_n, x_{n+1}) = 0.
\]

Thus, \( \omega_1(v, Tv) = 0 \), and so \( v \in Tv \).

\( \Box \)

We give the following illustrated examples.

**Example 3.10.** Let \( X = \mathbb{R}^+ \), \( M = [0, 1] \) and \( \omega_1(x, y) = \frac{|x - y|}{\lambda}, \quad \forall x, y \in X, \quad \forall \lambda > 0 \). Mention that \( \omega \) is a convex regular modular and satisfies the \( \Delta_2 \)-condition. Also, \( M \) is an \( \omega \)-complete subset of \( X_\omega \). Define a mapping \( T : M \to CB(M) \) by

\[
Tx = \begin{cases} \left[ \frac{1}{3}, \frac{5}{3} \right], & 0 \leq x \leq \frac{1}{2} \\ \left[ 1, \frac{5}{3} \right], & \frac{1}{2} < x \leq 1. \end{cases}
\]

Let \( \alpha : M \times M \to [0, \infty) \) be defined by \( \alpha(x, y) = 1 \) for all \( x, y \in M \).

Condition (2) of Theorem 3.9 is satisfied and \((M, \alpha)\) is \( \omega \)-regular.

We show that (9) of Theorem 3.9 is satisfied for all \( x, y \in M \) with \( k \in \left[ \frac{1}{3}, 1 \right] \) and for \( L \geq 3 \).

We consider the following cases:

**Case 1:** \( (x, y) \in [0, \frac{1}{3}] \times [\frac{1}{3}, 1] \). In this case, condition (9) reduces

\[
H_\omega(Tx, Ty) = \max \{ \max \{ \omega_1(x, y), \frac{y + 2}{3} \}, \sup \{ \omega_1(x, \frac{5}{6} \alpha \omega_1(x, \frac{5}{3}) : 0 \leq a \leq \frac{y + 2}{3} \} \}
\]

\[
= \max \{ \frac{|x|}{3} - \frac{5}{6} |x - \frac{2}{3}|, \frac{y}{3} - \frac{2}{3} \} = \frac{|x - y|}{3}.
\]

\[
\leq k|x - y| + L|y - x| = \frac{|x - y|}{3}.
\]
We have $|\frac{5}{3} - \frac{y}{3} - \frac{2}{3}| \leq 1$ and $|y - \frac{3}{2}| \geq \frac{1}{2}$ for all $x, y \in [0, 1] \times [\frac{1}{2}, 1]$. In order the previous inequality holds, it suffices to take $L \geq 3$ and $k \in [0, 1)$ to be arbitrary. Case 2: $(x, y) \in (\frac{1}{2}, 1] \times [0, 1]$. In this case, condition (9) is reduced to

$$H_w(Tx, Ty) = \frac{x}{3} + \frac{2}{3} \cdot \frac{y}{3} - \frac{y}{3} \leq k|x - y| + L\frac{5}{6} - y.$$ 

We have $|\frac{5}{3} + \frac{y}{3} - \frac{2}{3}| \leq 1$ and $|\frac{5}{6} - y| \geq \frac{1}{2}$ for all $x, y \in (\frac{1}{2}, 1] \times [0, \frac{5}{6}]$. Again, in order the previous inequality holds, it suffices to take $L \geq 3$ and $k \in [0, 1)$ to be arbitrary. Case 3: $(x, y) \in (\frac{1}{2}, 1] \times [0, \frac{5}{6}]$. In this case, the condition (9) is reduced to

$$H_w(Tx, Ty) = \omega(\frac{x}{3}, \frac{y}{3}) = \frac{1}{3}|x - y| \leq k|x - y| + L|y - \frac{x}{3}|$$

and so condition (9) is satisfied with $k \in [\frac{1}{2}, 1)$ and $L \geq 0$.

Case 4: $x, y \in (\frac{1}{2}, 1]$. In this case, condition (9) becomes

$$H_w(Tx, Ty) = H_w(\frac{5}{6}, \frac{x + 2}{3}, \frac{5}{6}, \frac{y + 2}{3}) = \frac{1}{3}|x - y| \leq k|x - y| + L|y - \frac{x}{3}|$$

and so condition (9) is satisfied with $k \in [\frac{1}{2}, 1)$ and $L \geq 0$. Now, by summarizing all cases, we conclude that the condition (9) is satisfied with $k \in [\frac{1}{2}, 1)$ and $L \geq 3$.

Hence, all hypotheses of Theorem 3.9 are satisfied and $T$ has fixed points. Note that $\text{Fix}(T) = \{0\} \cup [\frac{1}{5}, 1]$ where $\text{Fix}(T)$ denotes the set of fixed points of $T$.

On the other hand, the main result of Abdou and Khamsi [2] is not applicable. In fact, taking $x = 0$ and $y = 1$, we have $H_w(Tx, Ty) = 1 > k = k_0(0, 1)$ for each $k \in [0, 1)$.

**Example 3.11.** Let $X = \mathbb{R}$, $M = [0, 2]$ and $\omega_1(x, y) = \frac{|x - y|}{\lambda}$, $\forall x, y \in X$, $\forall \lambda > 0$. Mention that $\omega$ is convex regular modular satisfying the $\Delta_2$-condition. Also, $M = [0, 2]$ is an $\omega$-complete subset of $X_\omega$. Define a mapping $T : M \rightarrow \text{CB}(M)$ by

$$Tx = \begin{cases} [0, \frac{1+x}{2}], & 0 \leq x \leq \frac{1}{2} \\ [0, \frac{3+x}{2}], & \frac{1}{2} < x \leq 1 \\ [0, 1], & 1 < x \leq 2. \end{cases}$$

Let $\alpha : M \times M \rightarrow [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & 0 \leq x, y \leq 1 \\ 0, & \text{otherwise}. \end{cases}$$

Condition (2) of Theorem 3.9 is satisfied with $x_0 = 0$ and $x_1 = \frac{1}{2}$. Obviously, $(M, \alpha)$ is $\omega$-regular.

We show that (9) of Theorem 3.9 is satisfied for all $x, y \in M$ such that $\alpha(x, y) \geq 1$ with $k \in [\frac{1}{2}, 1)$ and for all $L \geq 0$.

Let $x, y \in M$ be such that $\alpha(x, y) \geq 1$. Then $0 \leq x, y \leq 1$. We consider the following cases:

- **Case 1:** $x, y \in [0, 1]$. We have

$$H_w(Tx, Ty) = \max\{\max\{\alpha_1(0, Ty), \alpha_1(\frac{1+x}{2}, Ty)\}, \max\{\alpha_1(0, Tx), \alpha_1(\frac{1+y}{2}, Tx)\}\}$$

$$= \max\{\alpha_1(\frac{1+x}{2}, Ty), \alpha_1(\frac{1+y}{2}, Tx)\}$$

$$= \max\{\min\{\frac{1+x}{2}, \frac{1+y}{2}\}|x - y|, \min\{\frac{1+x}{2}, \frac{1+y}{2}\}|x - y|\}$$

$$= \frac{1}{2}|x - y|.$$

\textbf{Corollary 3.13.} \textit{Let $\omega$ be a convex regular modular satisfying the $\Delta_2$-condition. Let $M$ be a nonempty $\omega$-complete subset of $X_\omega$. Given $T : M \to CB(M)$. Suppose that there exist a function $\alpha : M \times M \to [0, \infty)$ and two constants $k \in [0, 1)$ and $L \geq 0$ such that for any $x, y \in M$, we have}

$$\alpha(x, y)H_\omega(Tx, Ty) \leq k\omega_1(x, y) + L\omega_1(y, Tx). \quad (6)$$

\textit{Suppose also that}

1. $T$ is $\alpha$-admissible;
2. there exit $x_0 \in M$ and $x_1 \in T x_0$ such that $\alpha(x_0, x_1) \geq 1$;
3. $(M, \alpha)$ is $\omega$-regular or $\alpha$ is lower semi-continuous.

Then $T$ is a MWP operator.

\textit{Proof.} Let $x, y \in M$ be such that $\alpha(x, y) \geq 1$. Then if (7) holds, we have

$$H_\omega(Tx, Ty) \leq \alpha(x, y)H_\omega(x, y) \leq k\omega_1(x, y) + L\omega_1(y, Tx).$$

The proof is concluded from Theorem 3.9.

\hfill $\Box$

\textbf{Corollary 3.14.} \textit{Let $(X, \omega)$ be a modular metric space. Assume that $\omega$ is a convex regular modular satisfying the $\Delta_2$-condition. Let $M$ be a nonempty $\omega$-complete subset of $X_\omega$. Given $T : M \to M$. Suppose that there exist two constants $k \in [0, 1)$ and $L \geq 0$ with $0 \leq k + L < 1$ such that for any $x, y \in M$, we have}

$$\omega_1(Tx, Ty) \leq k\omega_1(x, y) + L\omega_1(y, Tx). \quad (8)$$

Then $T$ has a unique fixed point in $X_\omega$.

\textit{Proof.} We assume that there exist $x, y \in X$ such that $x = Tx$ and $y = Ty$ with $x \neq y$. We have

$$0 < \omega_1(x, y) = \omega_1(Tx, Ty) \leq k\omega_1(x, y) + L\omega_1(y, Tx) = (k + L)\omega_1(x, y) < \omega_1(x, y),$$

which is a contradiction. Hence $x = y$. \hfill $\Box$
3.3. Result III

In this part, we give a general class of MWP operators.

**Definition 3.15.** Let $(X, \omega)$ be a modular metric space and $M$ be a nonempty subset of $X_\omega$. A multi-valued $T : M \to CB(M)$ is said an $\alpha - \omega$–generalized weak contraction if there exist a function $\alpha : M \times M \to [0, \infty)$, a function $\theta : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to \infty} \theta(r) < 1$, for every $t \in [0, \infty)$ and a constant $L \geq 0$ such that for any $x, y \in M$ with $\alpha(x, y) \geq 1$, we have

$$H_\omega(Tx, Ty) \leq \theta(\omega_1(x, y))\omega_1(x, y) + L\omega_1(y, Tx).$$

(9)

**Theorem 3.16.** Let $(X, \omega)$ be a modular metric space. Assume that $\omega$ is a convex regular modular satisfying the $\Delta_2$–condition. Let $M$ be a nonempty $\omega$–complete subset of $X_\omega$. Let $T : M \to CB(M)$ be a $\alpha - \omega$–generalized weak contraction.

Suppose that

1. $T$ is $\alpha$–admissible;
2. there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
3. $(M, \alpha)$ is $\omega$–regular or $f_T$ is lower semi-continuous.

Then $T$ is a MWP operator.

**Proof.** The proof is inspired from Theorem 2.1 in [14].

By condition (2), there exist $x_0 \in M$ and $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) \geq 1.$$ 

If $x_0 = x_1$, then $x_1$ is a fixed point of $T$ and so the proof is finished. Now, we assume that $x_0 \neq x_1$. Select a positive integer $n_1$ such that

$$\theta^{n_1}(\omega_1(x_0, x_1)) < [1 - \theta(\omega_1(x_0, x_1))]\omega_1(x_0, x_1).$$

By Lemma 3.2, we can select $x_2 \in Tx_1$ such that

$$\omega_1(x_1, x_2) \leq H_\omega(Tx_0, Tx_1) + \theta^{n_1}(\omega_1(x_0, x_1)).$$

Since $\alpha(x_0, x_1) \geq 1$ then by (9), we have

$$\omega_1(x_1, x_2) \leq H_\omega(Tx_0, Tx_1) + \theta^{n_1}(\omega_1(x_0, x_1))$$

$$\leq \theta(\omega_1(x_0, x_1))\omega_1(x_0, x_1) + L\omega_1(x_1, Tx_0) + \theta^{n_1}(\omega_1(x_0, x_1))$$

$$= \theta(\omega_1(x_0, x_1))\omega_1(x_0, x_1) + \theta^{n_1}(\omega_1(x_0, x_1))$$

$$< \omega_1(x_0, x_1).$$

The mapping $T$ is $\alpha$–admissible and $x_2 \in Tx_1$, so

$$\alpha(x_1, x_2) \geq 1.$$ 

If $x_2 = x_1$, then $x_2$ is a fixed point of $T$ and again the proof is finished. Now, we assume that $x_2 \neq x_1$. We choose a positive integer $n_2 > n_1$ such that

$$\theta^{n_2}(\omega_1(x_1, x_2)) < [1 - \theta(\omega_1(x_1, x_2))]\omega_1(x_1, x_2).$$

By Lemma 3.2, we may select $x_3 \in Tx_2$ such that

$$\omega_1(x_2, x_3) \leq H_\omega(Tx_1, Tx_2) + \theta^{n_2}(\omega_1(x_1, x_2)).$$
Since \( \alpha(x_1, x_2) \geq 1 \), by (9), we have
\[
\omega_1(x_2, x_3) \leq H_0(Tx_1, Tx_2) + \theta^{a_1}(\omega_1(x_1, x_2)) \\
\leq \theta(\omega_1(x_1, x_2))\omega_1(x_1, x_2) + L\omega_1(x_2, Tx_1) + \theta^{a_2}(\omega_1(x_1, x_2)) \\
= \theta(\omega_1(x_1, x_2))\omega_1(x_1, x_2) + \theta^{a_2}(\omega_1(x_1, x_2)) \\
< \omega_1(x_1, x_2).
\]
Continuing in this process, we may select a positive integer \( n_k \) such that
\[
\theta^{a_k}(\omega_1(x_{k-1}, x_k)) < [1 - \theta(\omega_1(x_{k-1}, x_k))]\omega_1(x_{k-1}, x_k)
\]
where \( x_{k-1} \neq x_k \) and \( \alpha(x_{k-1}, x_k) \geq 1 \). By Lemma 3.2, we may select \( x_{k+1} \in Tx_k \) such that
\[
\omega_1(x_k, x_{k+1}) \leq H_0(Tx_{k-1}, Tx_k) + \theta^{a_k}(\omega_1(x_{k-1}, x_k)).
\]
In view of \( T \) is \( \alpha \)-admissible and \( x_{k+1} \in Tx_k \), we have
\[
\omega_1(x_k, x_{k+1}) \geq 1.
\]
Since \( \alpha(x_{k-1}, x_k) \geq 1 \), by (9), we get
\[
\omega_1(x_k, x_{k+1}) < \omega_1(x_{k-1}, x_k) \quad \text{for all } k = 1, 2, 3, \ldots
\]
It follows that \( \{a_k \equiv \omega_1(x_k, x_{k+1})\} \) is a nonincreasing sequence of nonnegative numbers. Then there exists a constant \( c \geq 0 \) such that \( \lim_{k \to \infty} a_k = c \). By assumption, \( \limsup \theta(t) < 1 \). This implies that there exists a positive integer \( N \) such that for \( k \geq N \), we have \( \theta(\omega_1(x_k, x_{k+1}) < h \) where \( \limsup \theta(t) < h < 1 \).

Now, by (9), we have for \( k \geq N \)
\[
a_k = \omega_1(x_k, x_{k+1}) \leq \theta(a_k)a_{k-1} + \theta^{a_k}(a_{k-1}) \\
\leq \theta(a_k)\theta(a_{k-1})a_{k-2} + \theta(a_k)\theta^{a_{k-1}}(a_{k-2}) + \theta^{a_k}(a_{k-1}) \\
\vdots
\leq \prod_{i=1}^{k} \theta(a_i)a_0 + \sum_{n=1}^{k-1} \prod_{i=n+1}^{k} \theta(a_i)\theta^{a_n}(a_m) + \theta^{a_k}(a_{k-1}) \\
\leq \prod_{i=1}^{k} \theta(a_i)a_0 + \sum_{n=1}^{k-1} \prod_{i=n+1}^{k} \theta(a_i)\theta^{a_n}(a_m) + \theta^{a_k}(a_{k-1}) \\
\leq h^{k-1} \prod_{i=1}^{k-1} \theta(a_i)a_0 + (k_0 - 1)h^{k-2} \sum_{m=1}^{k-1} \theta^{a_m}(a_m) + \sum_{m=1}^{k-1} h^{k-m}\theta^{a_m}(a_m) + \theta^{a_k}(a_{k-1}) \\
\leq C_1h^k + C_2h^k + C_3h^k + h^a \leq Ch^k,
\]
where \( C_1, C_2, C_3, C \) are appropriate constants.

For \( k \geq N \), we have
\[
\omega_1(x_k, x_{k+1}) \leq Ch^k.
\]
Proceeding as in proof of Theorem 3.9, we may prove that \( x_n \) is \( \omega \)-Cauchy in \( M \). By completeness of \( (M, \omega) \), there exists a \( u \in M \) such that \( \lim_{n \to \infty} x_n = u \). We shall prove that \( u \) is a fixed point of \( T \). Since \( \alpha(x_n, x_{n+1}) \geq 1 \), for all \( n \in \mathbb{N}, x_n \to u \) and \( (M, \alpha) \) is \( \omega \)-regular, there exists a subsequence \( \{x_{n[k]}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n[k]}, u) \geq 1 \) for all \( k \in \mathbb{N} \).
We have for any $k \geq 0$

$$H_\omega(Tx_{n(k)}, Tu) \leq \theta(\omega_1(x_{n(k)}, u))\omega_1(x_{n(k)}, u) + L\omega_1(x_n, Tx_{n(k)})$$

$$\leq \omega_1(x_{n(k)}, u) + L\omega_1(u, x_{n(k)+1}).$$

Passing to limit as $k \to \infty$, we get $\lim_{k \to \infty} H_\omega(Tx_{n(k)}, Tu) = 0$.

Since $x_{n(k)+1} \in Tx_{n(k)}$, by Lemma 3.3, we conclude that $u \in Tu$, that is, $u$ is a fixed point of $T$.

Now, passing to the case where $f_T$ is lower semi-continuous, we get again $u \in Tu$. □

3.4. Result IV

In [15], Edelstein proved that if $X$ is a complete $\varepsilon$--chainable metric space and $f : X \to X$ is an $(\varepsilon, \lambda)$--uniformly locally contractive mapping, then $f$ has a fixed point. Nadler [21] extended this result to multi-valued mappings and he proved that if $(X, d)$ is a complete $\varepsilon$--chainable metric space and $F : X \to 2^X$ is an $(\varepsilon, \lambda)$--uniformly locally contractive multi-valued mapping, then $F$ has a fixed point. We generalize this result to $\alpha$--$\omega$--weak contractions in the setting of modular metric spaces.

**Definition 3.17.** Let $(X, \omega)$ be a modular metric space. A nonempty subset $M$ of $X$ is called to be finitely $\varepsilon$--chainable (where $\varepsilon > 0$ is fixed) if and only if given $x, y \in M$, there is an $\varepsilon$--chain from $x$ to $y$ (that is, a finite set of points $x_0, x_1, \ldots, x_p \in M$ such that $x_0 = x, x_p = y$, and $\omega_1(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, 2, \ldots, p$).

**Definition 3.18.** Let $(X, \omega)$ be a modular metric space and $M$ be a nonempty subset of $X$. A multi-valued $T : M \to CB(M)$ is called to be finitely $\varepsilon$--chainable, if there exist two constants $k \in [0, 1)$, $L \geq 0$ such that for any $x, y \in M$, we have

$$H_\omega(Tx, Ty) \leq k\omega_1(x, y) + L\omega_1(y, Tx), \quad \text{whenever} \quad \omega_1(x, y) < \varepsilon.$$  

(10)

**Theorem 3.19.** Let $(X, \omega)$ be a modular metric space. Assume that $\omega$ is a convex regular modular satisfying the $\Delta_2$--type condition and the Fatou property. Let $M$ be a nonempty $\omega$--compact and $\omega$--compact subset of $X$, which is finitely $\varepsilon$--chainable, for some $\varepsilon > 0$. Then $T : M \to CB(M)$ be an $(\varepsilon, k, L)$--$\omega$--uniformly locally weak contraction.

Then $T$ is a MWP operator.

Proof. Let $(x, y) \in M \times M$. Take

$$\omega^\varepsilon(x, y) = \inf\{\sum_{i=1}^p \omega_1(x_{i-1}, x_i) : x_0 = x, x_1, \ldots, x_p = y \text{ is an } \varepsilon \text{--chain from } x \text{ to } y\}.$$ 

It is clear that $\omega^\varepsilon(x, y) < \infty$ for every $x, y \in M$ and $\omega^\varepsilon(x, y) = \omega_1(x, y)$ for all $x, y \in M$ such that $\omega_1(x, y) < \varepsilon$. Moreover, by definition of $\omega$, we have for all $x, y \in M$

$$\omega^\varepsilon(x, y) \leq \inf\{\omega_1(x, x_1) + \omega_1(x_1, x_2) + \ldots + \omega_1(x_{n-1}, y)\}.$$

Let $z_0 \in M$ and $z_1 \in Tz_0$. Letting $x_0, x_1, \ldots, x_p$ be a $\varepsilon$--chain from $z_0$ to $z_1$. Since $Tx_1$ is $\omega$--compact, by Lemma 3.2, there exists $y_1 \in Tx_1$ such that

$$\omega_1(z_1, y_1) \leq H_\omega(Tz_0, Tx_1).$$

Similarly, there exists $y_2 \in Tz_2$ such that

$$\omega_1(y_1, y_2) \leq H_\omega(Tx_1, Tx_2).$$

Continuing in this fashion, we can find $y_3, \ldots, y_n$ such that $y_i \in Tx_i$ and

$$\omega_1(y_i, y_{i+1}) \leq H_\omega(Tx_i, Tx_{i+1}) \leq k\omega_1(x_i, x_{i+1}) + L\omega_1(x_i, Tx_i)$$

$$\leq k\omega_1(x_i, x_{i+1}) + L\omega_1(x_{i+1}, x_{i+1}) = k\omega_1(x_i, x_{i+1}).$$

Obviously, $z_0, y_1, \ldots, y_n$ is a $\varepsilon$--chain from $z_0$ to $y_n$ with $y_n \in Tz_1$. Take $z_2 = y_n$.

Since $T$ is an $(\varepsilon, k, L)$--$\omega$--uniformly locally weak contraction, we obtain

$$\omega^\varepsilon(z_1, z_2) \leq k\omega^\varepsilon(z_0, z_1).$$
Proceeding as above, we may construct a sequence \( \{z_n\} \) in \( M \) such that for all \( n \geq 1 \), we have \( z_{n+1} \in Tz_n \) and

\[
\omega^f(z_n, z_{n+1}) \leq k\omega^f(z_{n-1}, z_n).
\]

By induction, we get

\[
\omega^f(z_n, z_{n+1}) \leq k^n \omega^f(z_0, z_1).
\]

Considering that \( \omega \) satisfies the \( \Delta_2 \)–type condition, so there exists a constant \( C > 0 \) such that for all \( n \geq 1 \)

\[
\omega_1(z_n, z_{n+1}) \leq C\omega^n(z_n, z_{n+1}) \leq Ck^n\omega^f(z_0, z_1).
\]

Using Lemma 2.11, we conclude that \( \{z_n\} \) is \( \omega \)–Cauchy. By completeness of \( (M, \omega) \), there exists a point \( z \in M \) such that \( \lim_{n \to \infty} z_n = z \).

Proceeding as the above, there exists \( y_n \in Tz \) such that

\[
\omega_1(x_{n+1}, y_n) \leq H_\omega(x_n, z).
\]

The mapping \( T \) is an \((\epsilon, k, L) - \omega \)–uniformly locally weak contraction, so

\[
\omega^f(x_{n+1}, y_n) \leq k\omega^f(x_n, z).
\]

Since \( \lim_{n \to \infty} z_n = z \), there exists \( N \geq 1 \) such that for \( n \geq N \), we have \( \omega_1(x_n, z) < \epsilon \). It follows that \( \omega^f(x_n, z) = \omega_1(x_n, z) \).

We have for \( n \geq N \),

\[
\omega_{p+1}(y_n, z) \leq \omega_p(y_n, x_{n+1}) + \omega_1(x_{n+1}, z) \leq \omega^f(x_{n+1}, y_n) + \omega_1(x_{n+1}, z) \leq \omega_1(x_{n+1}, z) + k\omega_1(x_n, z).
\]

Passing to limit as \( n \to \infty \), we get \( \lim_{n \to \infty} \omega_{p+1}(y_n, z) \). Since \( \omega \) satisfies the \( \Delta_2 \)–type condition and \( Tz \) is closed, we conclude that \( z \in Tz \).

4. Fixed point theory in ordered modular metric spaces

The study of fixed points in partially ordered sets was developed in [22, 23, 25]. In this section, we give some results of fixed point for multi-valued mappings in the concept of partially ordered modular metric spaces. We say that \( x, y \in X_\omega \) are comparable if \( x \preceq y \) or \( y \preceq x \) holds. Moreover, for \( A, B \subseteq X_\omega \), we have \( A \preceq B \) whenever for each \( x \in A \), there exists \( y \in B \) such that \( x \preceq y \). Now, we introduce the following concepts.

**Definition 4.1.** Let \( (X, \omega) \) be a modular metric space and \( M \) be a nonempty subset of \( X_\omega \). A multi-valued \( T : M \to CB(M) \) is called to be weak continuous if the following condition holds: if \( \{x_n\} \) is an \( \omega \)–convergent sequence in \( M \) to \( x \in M \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \lim_{k \to \infty} H_\omega(Tx_{n(k)}, Tx) = 0 \).

**Definition 4.2.** Let \( (X, \omega) \) be a modular metric space and \( M \) be a nonempty subset of \( X_\omega \). The pair \( (M, \preceq) \) is said to be \( \omega \)–regular if the following condition holds: for any sequence \( \{x_n\} \) in \( M \) with \( Tx_n \preceq Tx_{n+1} \), for all \( n \in \mathbb{N} \) and \( x_n \to x \in M \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( Tx_{n(k)} \preceq Tx_k \) for all \( k \in \mathbb{N} \).

We have the following theorem.

**Theorem 4.3.** Let \( (X, \omega, \preceq) \) be a partially ordered modular metric space. Assume that \( \omega \) is a convex regular modular satisfying the \( \Delta_2 \)–condition. Let \( M \) be a nonempty \( \omega \)–compact subset of \( X_\omega \). Suppose that \( T : M \to CB(M) \) is a multi-valued mapping. Assume that there exist a function \( \theta : [0, \infty) \to [0, 1) \) satisfying \( \limsup_{r \to 1^-} \theta(r) < 1 \) for every \( t \in [0, \infty) \) and a constant \( L \geq 0 \) such that

\[
H_\omega(Tx, Ty) \leq \theta(\omega_1(x, y))\omega_1(x, y) + La_1(y, Tx)
\]

for all \( x, y \in M \), with \( Tx \preceq Ty \).

Suppose also that
1. There exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $Tx_0 \preceq Tx_1$;

2. For each $x \in M$ and $y \in Tx$ with $Tx \preceq Ty$, we have $Ty \leq Tz$ for all $z \in Ty$;

3. $T$ is weak continuous.

Then $T$ is a MWP operator.

Proof. Given $\alpha : M \times M \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1 & \text{if } Tx \preceq Ty \\ 0 & \text{otherwise.} \end{cases}$$

The multi-valued mapping $T$ is $\alpha$-admissible. In fact, if $x \in M$ and $y \in Ty$ with $\alpha(x, y) \geq 1$, then $Tx \preceq Ty$. By condition (2), we have $Ty \preceq Tz$ for all $z \in Ty$, then $\alpha(y, z) = 1$. Also, by (11), $T$ verifies (9). Proceeding as in proof of Theorem 3.16, we may construct an $\omega$-convergent sequence $\{x_n\}$ to $x \in M$ such that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. Finally, by condition (3) and Lemma 3.3, we conclude that $x$ is a fixed point of $T$. □

Theorem 4.3 remains true if the continuity hypothesis is replaced by the $\omega$-regularity of $(M, \preceq)$.

This statement is given as follows.

**Theorem 4.4.** Let $(X, \omega, \preceq)$ be a partially ordered modular metric space. Assume that $\omega$ is a convex regular modular satisfying the $\Delta_2$-condition. Let $M$ be a nonempty $\omega$-complete subset of $X_\omega$. Suppose that $T : M \to CB(M)$ is a multi-valued mapping. Assume that there exist a function $\theta : [0, \infty) \to [0, 1)$ satisfying $\limsup_{r \to t+} \theta(r) < 1$ for every $t \in [0, \infty)$ and a constant $L \geq 0$ such that

$$H_\omega(Tx, Ty) \leq \theta(\omega_1(x, y))\omega_1(x, y) + L\omega_1(y, Tx)$$

(12)

for all $x, y \in M$, with $Tx \preceq Ty$.

Suppose also that

1. There exist $x_0 \in M$ and $x_1 \in Tx_0$ such that $Tx_0 \preceq Tx_1$;

2. For each $x \in M$ and $y \in Tx$ with $Tx \preceq Ty$, we have $Ty \leq Tz$ for all $z \in Ty$;

3. $(M, \preceq)$ is $\omega$-regular.

Then $T$ is a MWP operator.

Proof. As in proof of the above theorem, we define the function $\alpha : M \times M \to [0, \infty)$ as follows

$$\alpha(x, y) = \begin{cases} 1 & \text{if } Tx \preceq Ty \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the multi-valued mapping $T$ is $\alpha$-admissible. Also, by (12), $T$ verifies the contraction (9). Finally, by condition (3), the sequence $(M, \alpha)$ is $\omega$-regular. Thus, all hypotheses of Theorem 3.16 are satisfied and hence $T$ has a fixed point. □

**Authors contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.
References


[27] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings, Nonlinear Anal. 75 (2012), 2154-2165.