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On C-Parallel Legendre Curves in Non-Sasakian Contact Metric Manifolds

Cihan Özgür[a]

[a] *Balıkesir University, Department of Mathematics, 10145, Balıkesir, TURKEY*

Abstract. In $(2n+1)$-dimensional non-Sasakian contact metric manifolds, we consider Legendre curves whose mean curvature vector fields are C-parallel or C-proper in the tangent or normal bundles. We obtain the curvature characterizations of these curves. Moreover, we give some examples of these kinds of curves which satisfy the conditions of our results.

1. Introduction

In [6] and [7], Chen studied submanifolds whose mean curvature vector fields H satisfy the condition $\Delta H = \lambda H$, where λ is a non-zero differentiable function on the submanifold and Δ denotes the Laplacian. Later, in [1], Arroyo, Barros and Garay defined the notion of a submanifold with a proper mean curvature vector field H in the normal bundle as a submanifold whose mean curvature vector field H satisfies the condition $\Delta^{\perp} H = \lambda H$, where $\Delta^{\perp}$ denotes the Laplacian in the normal bundle. Furthermore, when the mean curvature vector field H of the submanifold satisfies the condition $\Delta H = \lambda H$, they called the submanifold as a *submanifold with a proper mean curvature vector field*. In a Riemannian space form, curves with a proper mean curvature vector field in the tangent and normal bundles were studied in [1]. In [2], Kılıç and Arslan studied Euclidean submanifolds satisfying $\Delta^{\perp} H = \lambda H$. In [12], Kocayiğit and Hacısalihoğlu studied curves satisying $\Delta H = \lambda H$ in a 3-dimensional Riemannian manifold. For Legendre curves in Sasakian manifolds, same problems were studied by Inoguchi in [10]. In [3], Baikoussis and Blair considered submanifolds in Sasakian space forms $M(c) = (M, \varphi, \xi, \eta, g)$. They defined the mean curvature vector field H as *C-parallel* if $\nabla H = \lambda \xi$, where λ is a non-zero differentiable function on M and ∇ the induced Levi-Civita connection. Later, in [13], Lee, Suh and Lee studied curves with C-parallel and C-proper mean curvature vector fields in the tangent and normal bundles. A curve γ has *C-parallel mean curvature vector field H* if $\nabla_T H = \lambda \xi$, *C-proper mean curvature vector field H* if $\Delta H = \lambda \xi$, *C-parallel mean curvature vector field H in the normal bundle* if $\nabla^{\perp}_T H = \lambda \xi$, *$C$-proper mean curvature vector field H in the normal bundle* if $\Delta^{\perp} H = \lambda \xi$, where λ is a non-zero differentiable function along the curve γ, T the unit tangent vector field of γ, ∇ the Levi-Civita connection, $\nabla^{\perp}$ the normal connection [13].

Let $M = (M, \varphi, \xi, \eta, g)$ be a contact metric manifold and $\gamma : I \to M$ a Frenet curve in M parametrized by the arc-length parameter s. The contact angle $\alpha(s)$ is a function defined by $cos[\alpha(s)] = g(T(s), \xi)$. If $\alpha(s)$ is a constant, then the curve is called a *slant curve* [8]. If $\alpha(s) = \frac{\pi}{2}$, then γ is called a *Legendre curve* [5].

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Email address: cozgur@balikesir.edu.tr (Cihan Özgür)

In [13], Lee, Suh and Lee studied slant curves with $\mathcal{C}$-parallel and $\mathcal{C}$-proper mean curvature vector fields in Sasakian 3-manifolds. In [9], Güvenç and the present author studied $\mathcal{C}$-parallel and $\mathcal{C}$-proper slant curves in $(2n+1)$-dimensional trans-Sasakian manifolds. Since the paper [9] includes the Legendre curves in Sasakian manifolds, in the present paper, we consider $\mathcal{C}$-parallel and $\mathcal{C}$-proper Legendre curves in $(2n+1)$-dimensional non-Sasakian contact metric manifolds.

The paper is organized as follows: In Section 2 and Section 3, in non-Sasakian contact metric manifolds, we consider Legendre curves with $\mathcal{C}$-parallel and $\mathcal{C}$-proper mean curvature vector fields, respectively. In the final section, we give some examples of Legendre curves which support our theorems.

2. Legendre Curves with $\mathcal{C}$-parallel Mean Curvature Vector Fields

Let $M = (M, \varphi, \xi, \eta, g)$ be a contact metric manifold. The contact metric structure of M is said to be *normal* if

$$[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi,$$

where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ and X, Y are vector fields on M. A normal contact metric manifold is called a *Sasakian manifold* [5].

Given a contact Riemannian manifold M, the operator h is defined by $h = \frac{1}{2}(L_\xi \varphi)$, where L denotes the Lie differentiation. The operator h is self adjoint and satisfies

$$h\xi = 0 \quad \text{and} \quad h\varphi = -\varphi h,$$

$$\nabla_X \xi = -\varphi X - \varphi h X. \tag{1}$$

In a Sasakian manifold, it is clear that

$$\nabla_X \xi = -\varphi X.$$

For more details about contact metric manifolds and their submanifolds, we refer to [5] and [16].

Let (M, g) be an n-dimensional Riemannian manifold. A unit-speed curve $\gamma : I \to M$ is said to be *a Frenet curve of osculating order r*, if there exists positive functions $k_1, ..., k_{r-1}$ on I satisfying

$$\begin{aligned} T &= v_1 = \gamma', \\ \nabla_T T &= k_1 v_2, \\ \nabla_T v_2 &= -k_1 T + k_2 v_3, \\ &\;\;\, ... \\ \nabla_T v_r &= -k_{r-1} v_{r-1}, \end{aligned}$$

where $1 \leq r \leq n$ and $T, v_2, ..., v_r$ are a g-orthonormal vector fields along the curve. The positive functions $k_1, ..., k_{r-1}$ are called *curvature functions* and $\{T, v_2, ..., v_r\}$ is called the *Frenet frame field*. A *geodesic* is a Frenet curve of osculating order $r = 1$. A *circle* is a Frenet curve of osculating order $r = 2$ with a constant curvature function k_1. A *helix of order r* is a Frenet curve of osculating order r with constant curvature functions $k_1, ..., k_{r-1}$. A helix of order 3 is simply called a *helix*.

Now let (M, g) be a Riemannian manifold and $\gamma : I \to M$ a unit speed Frenet curve of osculating order r. By a simple calculations, it can be easily seen that

$$\nabla_T \nabla_T T = -k_1^2 T + k_1' v_2 + k_1 k_2 v_3,$$

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T &= -3k_1 k_1' T + \left(k_1'' - k_1^3 - k_1 k_2^2\right) v_2 \\ &\quad + \left(2k_1' k_2 + k_1 k_2'\right) v_3 + k_1 k_2 k_3 v_4, \end{aligned}$$

$$\nabla_T^\perp \nabla_T^\perp T = k_1' v_2 + k_1 k_2 v_3,$$

$$\nabla^{\perp}_T\nabla^{\perp}_T\nabla^{\perp}_T T = \left(k_1'' - k_1k_2^2\right)v_2 + \left(2k_1'k_2 + k_1k_2'\right)v_3 + k_1k_2k_3v_4,$$

(see [9]). Then we have

$$\nabla_T H = -k_1^2 T + k_1'v_2 + k_1k_2v_3, \tag{2}$$

$$\begin{aligned}\Delta H &= -\nabla_T\nabla_T\nabla_T T \\ &= 3k_1k_1'T + \left(k_1^3 + k_1k_2^2 - k_1''\right)v_2 \\ &\quad -(2k_1'k_2 + k_1k_2')v_3 - k_1k_2k_3v_4, \end{aligned} \tag{3}$$

$$\nabla^{\perp}_T H = k_1'v_2 + k_1k_2v_3, \tag{4}$$

$$\begin{aligned}\Delta^{\perp} H &= -\nabla^{\perp}_T\nabla^{\perp}_T\nabla^{\perp}_T T \\ &= \left(k_1k_2^2 - k_1''\right)v_2 - \left(2k_1'k_2 + k_1k_2'\right)v_3 \\ &\quad -k_1k_2k_3v_4, \end{aligned} \tag{5}$$

(see [1]).

Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Frenet curve in a contact metric manifold M. From [9], we give the following relations:

i) γ is a curve with C-parallel mean curvature vector field H if and only if

$$-k_1^2 T + k_1'v_2 + k_1k_2v_3 = \lambda\xi; \text{ or} \tag{6}$$

ii) γ is a curve with C-proper mean curvature vector field H if and only if

$$3k_1k_1'T + \left(k_1^3 + k_1k_2^2 - k_1''\right)v_2 - (2k_1'k_2 + k_1k_2')v_3 - k_1k_2k_3v_4 = \lambda\xi; \text{ or} \tag{7}$$

iii) γ is a curve with C-parallel mean curvature vector field H in the normal bundle if and only if

$$k_1'v_2 + k_1k_2v_3 = \lambda\xi; \text{ or} \tag{8}$$

iv) γ is a curve with C-proper mean curvature vector field H in the normal bundle if and only if

$$\left(k_1k_2^2 - k_1''\right)v_2 - \left(2k_1'k_2 + k_1k_2'\right)v_3 - k_1k_2k_3v_4 = \lambda\xi, \tag{9}$$

where λ is a non-zero differentiable function along the curve γ.

Now, let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order r in an n-dimensional contact metric manifold. By the use of the definition of a Legendre curve and (1), we have

$$\eta(T) = 0, \tag{10}$$

$$\nabla_T\xi = -\varphi T - \varphi hT. \tag{11}$$

Differentiating (10) and using (11), we obtain

$$k_1\eta(v_2) = g(T, \varphi hT). \tag{12}$$

If the osculating order $r = 2$, then we have the following results:

Theorem 2.1. *There does not exist a non-geodesic Legendre curve $\gamma : I \subseteq \mathbb{R} \to M$ of osculating order 2, which has C-parallel mean curvature vector field in a contact metric manifold M.*

Proof. Assume that γ have C-parallel mean curvature vector field. From (6), we have

$$-k_1^2 T + k_1' v_2 = \lambda \xi. \tag{13}$$

Then taking the inner product of (13) with T, we find $k_1 = 0$, this means that γ is a geodesic. This completes the proof. □

In the normal bundle, we can state the following theorem:

Theorem 2.2. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 2 in a non-Sasakian contact metric manifold. Then γ has C-parallel mean curvature vector field in the normal bundle if and only if*

$$k_1 = \pm g(\varphi hT, T),\ \xi = \pm v_2,\ \lambda = k_1'. \tag{14}$$

Proof. Let γ have C-parallel mean curvature vector field in the normal bundle. From (8), we have

$$k_1' v_2 = \lambda \xi. \tag{15}$$

So we have

$$\lambda = \pm k_1',$$

$$\xi = \pm v_2. \tag{16}$$

Differentiating (16), we find

$$-\varphi T - \varphi hT = \mp k_1 T, \tag{17}$$

which gives us

$$k_1 = \pm g(\varphi hT, T).$$

The converse statement is trivial. Then we complete the proof. □

If the osculating order $r \geq 3$, then similar to the proof of Theorem 2.1, we have the following theorem:

Theorem 2.3. *There does not exist a non-geodesic Legendre curve $\gamma : I \subseteq \mathbb{R} \to M$ of osculating order $r \geq 3$, which has C-parallel mean curvature vector field in a contact metric manifold M.*

In the normal bundle, we have the following theorem:

Theorem 2.4. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 3 in a non-Sasakian contact metric manifold. Then γ has C-parallel mean curvature vector field in the normal bundle if and only if*

$$k_1 \neq constant,$$

$$k_2 = \mp \frac{k_1' \sqrt{k_1^2 - g(T, \varphi hT)^2}}{k_1 g(T, \varphi hT)},$$

$$\xi = \frac{g(T, \varphi hT)}{k_1} v_2 + \frac{k_2}{k_1'} g(T, \varphi hT) v_3$$

and

$$\lambda = \frac{k_1' k_1}{g(T, \varphi hT)}$$

or

$$k_1 = constant,$$

$$k_2 = \sqrt{1 + 2g(T, hT) + g(hT, hT)},$$

$$\lambda = k_1 k_2 \quad and \quad \xi = v_3.$$

Proof. If $k_1 \neq constant$, then from (8), we have

$$k_1' v_2 + k_1 k_2 v_3 = \lambda \xi. \tag{18}$$

Then taking the inner product of (18) with v_2 and using (12), we find

$$k_1' = \lambda \eta(v_2) = \lambda \frac{g(T, \varphi hT)}{k_1}. \tag{19}$$

This gives us

$$\lambda = \frac{k_1' k_1}{g(T, \varphi hT)}.$$

Taking the inner product of (18) with v_3, we have

$$\eta(v_3) = \frac{k_2 g(T, \varphi hT)}{k_1'}. \tag{20}$$

Since $\xi \in span\{v_2, v_3\}$, using (12) and (20), we get

$$\xi = \frac{g(T, \varphi hT)}{k_1} v_2 + \frac{k_2}{k_1'} g(T, \varphi hT) v_3.$$

Since ξ is a unit vector field, we obtain

$$k_2 = \mp \frac{k_1' \sqrt{k_1^2 - g(T, \varphi hT)^2}}{k_1 g(T, \varphi hT)}.$$

If $k_1 = constant$, then from (8), we have

$$k_1 k_2 v_3 = \lambda \xi,$$

which gives us $\lambda = k_1 k_2$ and $\xi = v_3$. So by a differentiation of $\xi = v_3$, using (1), we have $-k_2 v_2 = -\varphi T - \varphi hT$. Hence, we obtain

$$k_2 = \sqrt{1 + 2g(T, hT) + g(hT, hT)}.$$

The converse statement is trivial. This completes the proof of the theorem. □

3. Legendre Curves with *C*-proper Mean Curvature Vector Fields

If the osculating order $r = 2$, then we have the following theorems:

Theorem 3.1. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 2 in a non-Sasakian contact metric manifold. Then γ has C-proper mean curvature vector field if and only if*

$$k_1 = \pm g(T, \varphi hT) = constant,$$

$$\xi = \pm v_2$$

and

$$\lambda = g(T, \varphi hT)^3.$$

Proof. Let γ have C-proper mean curvature vector field. From (7), we have

$$3k_1k_1'T + \left(k_1^3 - k_1''\right) \upsilon_2 = \lambda\xi. \tag{21}$$

Then taking the inner product of (21) with T, we have $k_1k_1' = 0$. Since γ is not a geodesic, we obtain $k_1' = 0$, which means that k_1 is a constant. Taking the inner product of (21) with υ_2, we have

$$k_1^3 - k_1'' = \lambda\eta(\upsilon_2).$$

Since k_1 is a constant, using (12), we get

$$\lambda = \frac{k_1^4}{g(T, \varphi hT)}. \tag{22}$$

Furthermore, taking the inner product of (21) with ξ and using (12), we have

$$\lambda = k_1^2 g(T, \varphi hT). \tag{23}$$

Then comparing (22) and (23), we obtain

$$k_1 = \mp g(T, \varphi hT).$$

Since $\xi \in span\{\upsilon_2\}$, we have

$$\xi = \pm\upsilon_2. \tag{24}$$

The converse statement is trivial. Hence, the proof is finished. □

In the normal bundle, we can state the following theorem:

Theorem 3.2. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 2 in a non-Sasakian contact metric manifold. Then γ is a curve with C-proper mean curvature vector field in the normal bundle if and only if*
i) $k_1(s) = as + b$, where a and b are arbitrary real constants and $\lambda = 0$ or
ii) $k_1 = \mp g(T, \varphi hT)$, $\xi = \pm\upsilon_2$ and $\lambda = k_1''$.

Proof. Let γ have C-proper mean curvature vector field in the normal bundle. From (9), we have

$$-k_1''\upsilon_2 = \lambda\xi. \tag{25}$$

Taking the inner product of (25) with υ_2 and using (12), we have

$$\lambda = -\frac{k_1''k_1}{g(T, \varphi hT)}. \tag{26}$$

Taking the inner product of (25) with ξ and using (12), we find

$$\lambda = -\frac{k_1''g(T, \varphi hT)}{k_1}. \tag{27}$$

Then comparing (26) and (27), we obtain either $k_1'' = 0$, in this case $k_1(s) = as + b$, where a and b are arbitrary real constants and $\lambda = 0$ or $k_1 = \mp g(T, \varphi hT)$. If $k_1 = \mp g(T, \varphi hT)$, it is easy to see that $\xi = \pm\upsilon_2$ and $\lambda = k_1''$.
The converse statement is trivial. This completes the proof of the theorem. □

If the osculating order $r = 3$, then we have the following theorems:

Theorem 3.3. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 3 in a non-Sasakian contact metric manifold. Then γ is a curve with C-proper mean curvature vector field if and only if*

$$k_1 = \text{constant},$$

$$\lambda = \frac{k_1^2\left(k_1^2 + k_2^2\right)}{g(T, \varphi hT)},$$

$$\xi = \frac{g(T, \varphi hT)}{k_1} v_2 - \frac{k_1 k_2'}{\lambda} v_3$$

and

$$\eta(v_2)^2 + \eta(v_3)^2 = 1.$$

Proof. Let γ have C-proper mean curvature vector field. Then, from (7), we have

$$3k_1k_1'T + \left(k_1^3 + k_1k_2^2 - k_1''\right)v_2 - (2k_1'k_2 + k_1k_2')v_3 = \lambda\xi. \tag{28}$$

Taking the inner product of (28) with T, we have $k_1k_1' = 0$. Since γ is not a geodesic, we find $k_1' = 0$, which gives us k_1 is a constant. Now taking the inner product of (28) with v_2 and using (12), we have

$$\lambda = \frac{k_1^2\left(k_1^2 + k_2^2\right)}{g(T, \varphi hT)}.$$

Taking the inner product of (28) with v_3, we have

$$\eta(v_3) = -\frac{k_1k_2'}{\lambda}. \tag{29}$$

Since $\xi \in span\{v_2, v_3\}$, using (12) and (29), we obtain

$$\xi = \frac{g(T, \varphi hT)}{k_1} v_2 - \frac{k_1 k_2'}{\lambda} v_3.$$

Since ξ is a unit vector field, we have $\eta(v_2)^2 + \eta(v_3)^2 = 1$. The converse statement is trivial. So we get the result as required. □

In the normal bundle, we can give the following result:

Theorem 3.4. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order 3 in a non-Sasakian contact metric manifold. Then γ is a curve with C-proper mean curvature vector field in the normal bundle if and only if*

$$\lambda = \frac{k_1^2k_2^2 - k_1k_1''}{g(T, \varphi hT)},$$

$$\xi = \frac{g(T, \varphi hT)}{k_1} v_2 - \frac{\left(2k_1'k_2 + k_1k_2'\right)}{\lambda} v_3$$

and

$$\eta(v_2)^2 + \eta(v_3)^2 = 1.$$

Proof. Let γ have C-proper mean curvature vector field in the normal bundle. From (9), γ is a Legendre curve with

$$\left(k_1k_2^2 - k_1''\right)v_2 - \left(2k_1'k_2 + k_1k_2'\right)v_3 = \lambda\xi. \tag{30}$$

Taking the inner product of (30) with v_2 and using (12), we have

$$\lambda = \frac{k_1^2k_2^2 - k_1k_1''}{g(T,\varphi hT)}.$$

Taking the inner product of (30) with v_3, we get

$$\eta(v_3) = -\frac{2k_1'k_2 + k_1k_2'}{\lambda}. \tag{31}$$

Since $\xi \in span\{v_2, v_3\}$, using (12) and (31), we obtain

$$\xi = \frac{g(T,\varphi hT)}{k_1}v_2 - \frac{2k_1'k_2 + k_1k_2'}{\lambda}v_3.$$

Since ξ is a unit vector field, we have $\eta(v_2)^2 + \eta(v_3)^2 = 1$. The converse statement is trivial. Hence, we complete the proof. □

If the osculating order $r \geq 4$, then we can state the following theorem:

Theorem 3.5. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order $r \geq 4$ in a non-Sasakian contact metric manifold. Then γ is a curve with C-proper mean curvature vector field if and only if it satisfies*

$$k_1 = \ constant,$$

$$\lambda = \frac{k_1^2\left(k_1^2 + k_2^2\right)}{g(T,\varphi hT)},$$

$$\xi = \frac{g(T,\varphi hT)}{k_1}v_2 - \frac{k_1k_2'}{\lambda}v_3 - \frac{k_1k_2k_3}{\lambda}v_4$$

and

$$\eta(v_2)^2 + \eta(v_3)^2 + +\eta(v_4)^2 = 1.$$

Proof. Since γ has C-proper mean curvature vector field, by the use of (7), we have

$$3k_1k_1'T + \left(k_1^3 + k_1k_2^2 - k_1''\right)v_2 - (2k_1'k_2 + k_1k_2')v_3 - k_1k_2k_3v_4 = \lambda\xi. \tag{32}$$

Taking the inner product of (32) with T, we have $k_1k_1' = 0$. Since γ is not a geodesic, we find $k_1' = 0$, which gives us k_1 is a constant. Now taking the inner product of (32) with v_2 and using (12), we find

$$\lambda = \frac{k_1^2\left(k_1^2 + k_2^2\right)}{g(T,\varphi hT)}.$$

Taking the inner product of (32) with v_3 and v_4, we get

$$\eta(v_3) = -\frac{k_1k_2'}{\lambda} \tag{33}$$

and

$$\eta(v_4) = -\frac{k_1k_2k_3}{\lambda}, \tag{34}$$

respectively. Since $\xi \in span\{v_2, v_3, v_4\}$, using (33) and (34), we obtain

$$\xi = \frac{g(T,\varphi hT)}{k_1}v_2 - \frac{k_1k_2'}{\lambda}v_3 - \frac{k_1k_2k_3}{\lambda}v_4.$$

Since ξ is a unit vector field, we have $\eta(v_2)^2 + \eta(v_3)^2 + +\eta(v_4)^2 = 1$. The converse statement is trivial. Thus we get the result as required. □

In the normal bundle, we can give the following theorem:

Theorem 3.6. *Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic Legendre curve of osculating order $r \geq 4$ in a non-Sasakian contact metric manifold. Then γ is a curve with $\mathcal{C}$-proper mean curvature vector field in the normal bundle if and only if*

$$\lambda = \frac{k_1^2k_2^2 - k_1k_1''}{g(T,\varphi hT)},$$

$$\xi = \frac{g(T,\varphi hT)}{k_1}v_2 - \frac{2k_1'k_2 + k_1k_2'}{\lambda}v_3 - \frac{k_1k_2k_3}{\lambda}v_4$$

and

$$\eta(v_2)^2 + \eta(v_3)^2 + +\eta(v_4)^2 = 1.$$

Proof. The proof is similar to the proof of Theorem 3.5. □

4. Examples

Let us take $M = \mathbb{R}^3$ and denote the standard coordinate functions with (x, y, z). We define the following vector fields on $\mathbb{R}^3$:

$$e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y},\ e_3 = 2y\frac{\partial}{\partial x} + \left(\frac{1}{4}e^{2x} - y^2\right)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

It is seen that e_1, e_2, e_3 are linearly independent at all points of M. We define a Riemannian metric on M such that e_1, e_2, e_3 are orthonormal. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = \frac{e^{2x}}{2}e_2,\ [e_2, e_3] = -2ye_2 + 2e_1.$$

Let η be defined by $\eta(W) = g(W, e_1)$ for all $W \in \chi(M)$. Let φ be the $(1,1)$-type tensor field, defined by $\varphi e_1 = 0$, $\varphi e_2 = e_3$, $\varphi e_3 = -e_2$. Then $(M, \varphi, e_1, \eta, g)$ is a contact metric manifold. Let us set $\xi = e_1$, $X = e_2$ and $\varphi X = e_3$. Let ∇ be the Levi-Civita connection corresponding to g which is calculated as

$$\begin{array}{lll} \nabla_X\xi = \left(-\frac{e^{2x}}{4} - 1\right)\varphi X, & \nabla_{\varphi X}\xi = \left(1 - \frac{e^{2x}}{4}\right)X, & \nabla_\xi\xi = 0, \\ \nabla_\xi X = \left(-\frac{e^{2x}}{4} - 1\right)\varphi X, & \nabla_\xi\varphi X = \left(1 + \frac{e^{2x}}{4}\right)X, & \nabla_X X = 2y\varphi X, \\ \nabla_X\varphi X = -2yX + \left(\frac{e^{2x}}{4} + 1\right)\xi, & \nabla_{\varphi X}X = \left(\frac{e^{2x}}{4} - 1\right)\xi, & \nabla_{\varphi X}\varphi X = 0. \end{array} \tag{35}$$

By the definition of h, it is easy to see that

$$hX = \frac{e^{2x}}{4}X, \qquad h\varphi X = -\frac{e^{2x}}{4}\varphi X.$$

Hence, M is a (κ, μ, ν)-contact metric manifold with $\kappa = 1 - \frac{e^{4x}}{16}$, $\mu = 2\left(1 + \frac{e^{2x}}{4}\right)$, $\nu = 2$ [14].

Example 4.1. *Let M be the (κ, μ, ν)-contact metric manifold given above and let $\gamma : I \subseteq \mathbb{R} \to M$ be a curve parametrized by $\gamma(s) = (\ln 2, 0, \frac{\sqrt{2}}{2}s)$, where s is the arc-length parameter on an open interval I. The unit tangent vector field T along γ is*

$$T = -\frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}\varphi X.$$

Since $\eta(T) = 0$, the curve is Legendre. Using (35), we find

$$\nabla_T T = -\xi,$$

which gives us $k_1 = 1$ and $v_2 = -\xi$. Differentiating v_2 along the curve γ, we have

$$\begin{aligned} \nabla_T v_2 &= -\sqrt{2}\varphi X \\ &= -k_1 T + k_2 v_3. \end{aligned}$$

Thus, we get

$$k_2 = 1, \ v_3 = -\frac{\sqrt{2}}{2}(X + \varphi X).$$

Finally we find

$$g(T, \varphi h T) = -1.$$

From Theorem 3.4, γ has C-proper mean curvature vector field in the normal bundle with $\lambda = -1$.

Let $M = E(2)$ be the group of rigid motions of Euclidean 2-space with left invariant Riemannian metric g. Then M admits its compatible left-invariant contact Riemannian structure if and only if there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra such that [15]:

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = c_2 e_1, \quad [e_3, e_1] = 0,$$

where we choose $c_2 > 0$. The Reeb vector field ξ is obtained by left translation of e_3. The contact distribution D is spanned by e_1 and e_2. Then using Koszul's formula, we have the following relations:

$$\begin{aligned} \nabla_{e_1} e_2 &= \tfrac{1}{2}(-c_2 + 2)e_3, \quad \nabla_{e_1} e_3 = -\tfrac{1}{2}(-c_2 + 2)e_2, \\ \nabla_{e_2} e_1 &= -\tfrac{1}{2}(c_2 + 2)e_3, \quad \nabla_{e_2} e_3 = \tfrac{1}{2}(c_2 + 2)e_1, \\ \nabla_{e_3} e_1 &= \tfrac{1}{2}(c_2 - 2)e_2, \quad \nabla_{e_3} e_2 = -\tfrac{1}{2}(c_2 - 2)e_1, \end{aligned} \tag{36}$$

all others are zero (for more details see [15] and [11]). Let us denote by $X = e_1$, $\varphi X = e_2$, $\xi = e_3$. By the definition of h, it is easy to see that

$$hX = -\tfrac{1}{2}c_2 X, \qquad h\varphi X = \tfrac{1}{2}c_2 \varphi X. \tag{37}$$

Let $\gamma : I \to M = E(2)$ be a unit speed Legendre curve with Frenet frame $\{T = v_1, v_2, v_3\}$. Let us write

$$T = T_1\xi + T_2 X + T_3 \varphi X.$$

Since γ is Legendre, $T_1 = 0$. Using (36), we find

$$\begin{aligned} \nabla_T T &= -T_2 T_3 c_2 \xi + T_2' X + T_3' \varphi X \\ &= k_1 v_2. \end{aligned}$$

If we choose $v_2 = \xi$, then

$$k_1 = -T_2 T_3 c_2,$$

and we can take $T_2 = -\cos\theta$ =constant, $T_3 = -\sin\theta$ =constant such that $\cos\theta\sin\theta < 0$, $\cos 2\theta = \frac{2}{c_2}$. So we have

$$T = -\cos\theta X - \sin\theta\varphi X. \tag{38}$$

Then using (1), (36), (37) and (38), we can write

$$\nabla_T v_2 = \nabla_T \xi = -\frac{1}{2}\sin\theta\,(c_2+2)\,X + \frac{1}{2}\cos\theta\,(-c_2+2)\,\varphi X. \tag{39}$$

Moreover $g(T, \varphi hT) = -\sin\theta\cos\theta c_2 = k_1$.

So, we can state the following example:

Example 4.2. *Let $M = E(2)$ be the group of rigid motions of Euclidean 2-space with left invariant Riemannian metric g and has a compatible left-invariant contact Riemannian structure given above. Let $\gamma : I \to M$ be a unit speed Legendre curve of osculating order 2 and $\{T = v_1, v_2 = \xi\}$ the Frenet frame of γ. Then γ is a Legendre circle with curvature $k_1 = -\cos\theta\sin\theta c_2$, where the tangent vector field of γ is $T = -\cos\theta X - \sin\theta\varphi X$ and θ is a constant such that $\sin\theta\cos\theta < 0$, $\cos 2\theta = \frac{2}{c_2}$.*

Moreover, we have

$$g(T, \varphi hT) = -\sin\theta\cos\theta c_2 = k_1.$$

From Theorem 3.1, γ has C-proper mean curvature vector field with $\lambda = -\sin^3\theta\cos^3\theta c_2^3$.

Now let us assume that $\gamma : I \to M = E(2)$ is a unit speed Legendre curve of osculating order 3 with Frenet frame $\{T = v_1, v_2 = \xi, v_3\}$. Similar to the above example, if we choose $v_2 = \xi$, we find $k_1 = -T_2T_3c_2$ and we can take $T = \cos\theta X + \sin\theta\varphi X$, where θ is a constant such that $\sin\theta\cos\theta < 0$, $\cos 2\theta \neq \frac{2}{c_2}$. Define a croos product $\times$ by $e_1 \times e_2 = e_3$. So we have $v_3 = T \times \xi = \sin\theta e_1 - \cos\theta e_2$. Then using (36), we obtain

$$\nabla_T v_3 = -\frac{1}{2}\left(\cos^2\theta(-c_2+2) + \sin^2\theta(c_2+2)\right)e_3,$$

which gives us $k_2 = \frac{1}{2}\left(\cos^2\theta(-c_2+2) + \sin^2\theta(c_2+2)\right)$ =constant.

Hence, we have the following example:

Example 4.3. *Let $M = E(2)$ be the group of rigid motions of Euclidean 2-space with left invariant Riemannian metric g and has a compatible left-invariant contact Riemannian structure given above. Let $\gamma : I \to M$ be a unit speed Legendre curve of osculating order 2 and $\{T = v_1, v_2 = \xi, v_3\}$ the Frenet frame of γ. Then γ is a Legendre helix with curvatures $k_1 = -\cos\theta\sin\theta c_2$ and $k_2 = \frac{1}{2}\left(\cos^2\theta(-c_2+2) + \sin^2\theta(c_2+2)\right)$, where the tangent vector field of γ is $T = \cos\theta X + \sin\theta\varphi X$ and θ is a constant such that $\sin\theta\cos\theta < 0$, $\cos 2\theta \neq \frac{2}{c_2}$.*

Moreover, we have

$$g(T, \varphi hT) = -\sin\theta\cos\theta c_2 = k_1.$$

From Theorem 3.3, γ has C-proper mean curvature vector field in the normal bundle with $\lambda = k_1\left(k_1^2 + k_2^2\right)$. Furthermore, from Theorem 3.4, γ has C-proper mean curvature vector field in the normal bundle with $\lambda = k_1k_2^2$.

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