Filomat 33:14 (2019), 4353–4360
https://doi.org/10.2298/FIL1914353T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Some Upper Bounds for the Berezin Number of Hilbert Space Operators

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**Abstract.** In this paper, we obtain some Berezin number inequalities based on the definition of Berezin symbol. Among other inequalities, we show that if $A, B$ be positive definite operators in $B(H)$, and $A\sharp B$ is the geometric mean of them, then

$$\mathbf{ber}^2(A\sharp B) \leq \mathbf{ber}\left(\frac{A^2+B^2}{2}\right) - \frac{1}{2}\inf_{\lambda\in\Omega} \zeta(\hat{k}_\lambda),$$

where $\zeta(\hat{k}_\lambda) = \langle (A-B)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2$, and $\hat{k}_\lambda$ is the normalized reproducing kernel of the space $H$ for $\lambda$ belong to some set $\Omega$.

## 1. Introduction and preliminaries

Let $B(H)$ stand for $C^*$-algebra of all bounded linear operators on a complex Hilbert space $H$ with inner product $\langle\cdot,\cdot\rangle$. An operator $A \in B(H)$ is called positive semi-definite and write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Also, it is called positive definite if $A > 0$. The numerical range and numerical radius of $A \in B(H)$ are defined by

$$W(A) := \{\langle Ax, x\rangle : x \in H, \|x\| = 1\} \text{ and } w(A) := \sup\{|\lambda| : \lambda \in W(A)\},$$

respectively. It is well-known that $w(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\|.\|$. In fact, $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$, for any $A \in B(H)$. A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Omega$ such that the evaluation functional $\varphi_\lambda(f) = f(\lambda), \lambda \in \Omega$, are continuous on $H$. Then by the Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in H$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for all $f \in H$. The family $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $H$. For $A$ a bounded linear operator on $H$, the Berezin symbol of $A$ is the function $\tilde{A}$ on $\Omega$ defined by

$$\tilde{A}(\lambda) = \langle A\hat{k}_\lambda(z), \hat{k}_\lambda(z)\rangle,$$

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2010 *Mathematics Subject Classification*. Primary 47A30; Secondary 15A60, 47B20.
*Keywords*. Berezin number, Numerical radius, Geometric mean.
Received: 23 February 2019; Accepted: 15 September 2019
Communicated by Fuad Kittaneh
The third author is supported by the Talented Young Scientist Program of Ministry of Science and Technology of China (Iran-19-001).
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where $\hat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of the space $H$ [8, 9, 13].
Berezin set and Berezin number of operator $A$ are defined respectively by

$$\mathbf{Ber}(A) := \{\tilde{A}(\lambda) : \lambda \in \Omega\} \text{ and } \mathbf{ber}(A) := \sup\{|\tilde{A}(\lambda)| : \lambda \in \Omega\}.$$

It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on $\Omega$ whose value lies in the numerical range of the operator $A$ and hence for any $A \in B(H)$,

$$\mathbf{Ber}(A) \subseteq W(A) \text{ and } \mathbf{ber}(A) \leq w(A).$$

We remark that this numerical characteristic of operator deserve large investigations. We refer the reader to [2, 3, 5, 6, 8–13, 18, 19] as a sample of recent work in this literature.
The Berezin number of an operator $A$ satisfies the following properties:

(i) $\mathbf{ber}(\alpha A) = |\alpha|\mathbf{ber}A,$ for all $\alpha \in \mathbb{C}$,
(ii) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$.

For two positive definite operators $A, B \in B(H)$, define $A\sharp_t B$ to be

$$A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$$

with $t \in \mathbb{R}$, which is a positive definite operator in $B(H)$. When $0 \leq t \leq 1$, the operator $A\sharp_t B$ is called the $t-$ *weighted geometric mean* of $A$ and $B$. In particular, for $t = \frac{1}{2}$, the operator $A\sharp B := A\sharp_{\frac{1}{2}} B$ is called the *geometric mean* of $A$ and $B$. If $AB = BA$, then $A\sharp_t B = A^{1-t}B^t$.
In this paper we obtain some upper bounds for the Berezin number of the geometric mean of $A$ and $B$, and in the sequel, we establish some inequalities involving generalization of Berezin number inequalities.

## 2. Main results

To prove our Berezin number inequalities, we need the following well-known results.
For $a, b > 0$ and $0 \leq \nu \leq 1$, the Young's inequality says that

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b. \tag{1}$$

Recently Kittaneh and Manasrah in [15] refined inequality (1) as following

$$a^\nu b^{1-\nu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu)b, \tag{2}$$

where $r_0 = \min\{\nu, 1-\nu\}$.
Furthermore, in [1] they generalized inequality (2) in the following form.

$$(a^\nu b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\nu a + (1-\nu)b)^m, \tag{3}$$

for $m = 1, 2, 3, \cdots$.
From the spectral theorem for positive operators and Jensen's inequality we have:

**Lemma 2.1.** *[14] Let $A$ be a positive operator in $B(H)$ and let $x \in H$ be any unit vector. Then*

*(a)* $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ *for* $r \geq 1$,
*(b)* $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for* $0 < r \leq 1$.

Dragomir in [4] obtained an useful extension for four operators of the Schwarz inequality as following.

**Theorem 2.2.** *Let $A, B, C, D \in B(H)$. Then for $x, y \in H$ we have the inequality*

$$|\langle DCBAx, y\rangle|^2 \leq \langle A^*|B|^2Ax, x\rangle\langle D|C^*|^2D^*y, y\rangle. \tag{4}$$

From now on, our means of $r_0$ and $R_0$ are $\min\{\nu, 1-\nu\}$ and $\max\{\nu, 1-\nu\}$, respectively.
Now we are in a position to present our first result.

**Theorem 2.3.** *Let $A, B, X \in B(H)$ such that $A, B > 0$ and $\nu \in [0,1]$. Then for all $r \geq 2m$ $(m = 1, 2, 3, \cdots)$, and $\alpha \geq 0$*

$$\mathbf{ber}^r((A\sharp_\alpha B)X) \leq \mathbf{ber}\left(\nu(X^*AX)^{\frac{r}{2m\nu}} + (1-\nu)(A\sharp_{2\alpha}B)^{\frac{r}{2m(1-\nu)}}\right)^m - r_0^m \inf_{\lambda\in\Omega} \zeta(\hat{k}_\lambda), \tag{5}$$

*where* $\zeta(\hat{k}_\lambda) = \left(\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2\nu}} - \langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2(1-\nu)}}\right)^2.$

*Proof.* Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $H(\Omega)$, then

$$\begin{aligned}
|\langle (A\sharp_\alpha B)X\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r &= |\langle A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}X\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r \\
&\quad \text{By Theorem 2.2} \\
&\leq \langle X^*AX\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{2}} \langle A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\alpha}A^{\frac{1}{2}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{2}} \\
&= \left(\langle X^*AX\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{2m}} \langle (A\sharp_{2\alpha}B)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{2m}}\right)^m \\
&\leq \left(\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle \langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^m. \quad \text{By Lemma 2.1(a)}
\end{aligned}$$

Now, by refinement of Young's inequality (3) we have

$$\begin{aligned}
&\left(\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle \langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^m \\
&\leq \left(\nu\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{\nu}} + (1-\nu)\langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{1-\nu}}\right)^m \\
&\quad - r_0^m\left(\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2\nu}} - \langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2(1-\nu)}}\right)^2 \\
&\leq \left(\nu\langle (X^*AX)^{\frac{r}{2m\nu}}\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\nu)\langle (A\sharp_{2\alpha}B)^{\frac{r}{2m(1-\nu)}}\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^m \\
&\quad - r_0^m\left(\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2\nu}} - \langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2(1-\nu)}}\right)^2 \quad \text{By Lemma 2.1 (b)} \\
&= \left\langle\left(\nu(X^*AX)^{\frac{r}{2m\nu}} + (1-\nu)(A\sharp_{2\alpha}B)^{\frac{r}{2m(1-\nu)}}\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^m \\
&\quad - r_0^m\left(\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2\nu}} - \langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2(1-\nu)}}\right)^2 \\
&\leq \mathbf{ber}\left(\nu(X^*AX)^{\frac{r}{2m\nu}} + (1-\nu)(A\sharp_{2\alpha}B)^{\frac{r}{2m(1-\nu)}}\right)^m \\
&\quad - r_0^m\left(\langle (X^*AX)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2\nu}} - \langle (A\sharp_{2\alpha}B)^{\frac{r}{2m}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{m}{2(1-\nu)}}\right)^2
\end{aligned}$$

Now, by taking supremum over $\lambda \in \Omega$, we get the desired inequality. □

choosing $m = 1$ in the proof of Theorem 2.3 we have:

**Corollary 2.4.** *Let $A, B, X \in B(H)$ such that $A, B > 0$ and $\nu \in [0,1]$. Then for all $r \geq 2R_0$*

$$\mathbf{ber}^r((A\sharp_\alpha B)X) \leq \mathbf{ber}\left(\nu(X^*AX)^{\frac{r}{2\nu}} + (1-\nu)(A\sharp_{2\alpha}B)^{\frac{r}{2(1-\nu)}}\right) - r_0 \inf_{\lambda\in\Omega} \zeta(\hat{k}_\lambda), \tag{6}$$

*where* $\zeta(\hat{k}_\lambda) = (\langle (X^*AX)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{4\nu}} - \langle (A\sharp_{2\alpha}B)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{4(1-\nu)}})^2.$

By letting $\alpha = \frac{1}{2}$ and $m = 1$ in the proof of Theorem 2.3, since $A\sharp B = B\sharp A$ we obtain the following corollary which was proved earlier in [17] for the numerical radius in $(p, q)$−version.

**Corollary 2.5.** *Let $A, B, X \in B(H)$ such that $A, B > 0$ and $\nu \in [0,1]$. Then for all $r \geq 2R_0$*

$$\mathbf{ber}^r((A\sharp B)X) \leq \mathbf{ber}\left(\nu A^{\frac{r}{2\nu}} + (1-\nu)(X^*BX)^{\frac{r}{2(1-\nu)}}\right) - r_0 \inf_{\lambda\in\Omega} \zeta(\hat{k}_\lambda), \tag{7}$$

*where* $\zeta(\hat{k}_\lambda) = (\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{4\nu}} - \langle X^*BX\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{4(1-\nu)}})^2.$

**Remark 2.6.** *Note that, if we set $X = I$, $r = 2$ and $\nu = \frac{1}{2}$ in (7), then we have*

$$\mathbf{ber}^2(A\sharp B) \leq \mathbf{ber}\left(\frac{A^2+B^2}{2}\right) - \frac{1}{2}\inf_{\lambda\in\Omega}\zeta(\hat{k}_\lambda), \tag{8}$$

*where $\zeta(\hat{k}_\lambda) = \langle (A-B)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2$. Actually, (8) is an operator Berezin number version for arithmetic-geometric mean.*

The next result reads as follows.

**Theorem 2.7.** *Let $A, B$ be positive definite operators in $B(H)$ and $\nu \in [0,1]$. Then for $\alpha \in [0,1]$ and all $r \geq R_0/\alpha$*

$$\mathbf{ber}^r(A\sharp_\alpha B) \leq \mathbf{ber}\left(\nu A^{\frac{(1-\alpha)r}{\nu}} + (1-\nu)B^{\frac{\alpha r}{1-\nu}}\right) - r_0 \inf_{\lambda\in\Omega}\zeta(\hat{k}_\lambda), \tag{9}$$

*where $\zeta(\hat{k}_\lambda) = (\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{(1-\alpha)r}{2\nu}} - \langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{\alpha r}{2(1-\nu)}})^2$.*

*Proof.* If $\hat{k}_\lambda$ is the normalized reproducing kernel of $H(\Omega)$, then

$$\begin{aligned}
\langle (A\sharp_\alpha B)\hat{k}_\lambda, \hat{k}_\lambda\rangle^r &= \langle A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \\
&= \langle (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}\hat{k}_\lambda, A^{\frac{1}{2}}\hat{k}_\lambda\rangle^r \\
&\leq \|A^{\frac{1}{2}}\hat{k}_\lambda\|^{(2-2\alpha)r}\langle (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}\hat{k}_\lambda, A^{\frac{1}{2}}\hat{k}_\lambda\rangle^{\alpha r} \quad \text{By Lemma 2.1(b)} \\
&= \langle A^{\frac{1}{2}}\hat{k}_\lambda, A^{\frac{1}{2}}\hat{k}_\lambda\rangle^{(1-\alpha)r}\langle (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}\hat{k}_\lambda, A^{\frac{1}{2}}\hat{k}_\lambda\rangle^{\alpha r} \\
&= \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{(1-\alpha)r}\cdot\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\alpha r} \\
&\leq \nu\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{(1-\alpha)r}{\nu}} + (1-\nu)\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{\alpha r}{1-\nu}} \\
&\quad - r_0\left(\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{(1-\alpha)r}{2\nu}} - \langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{\alpha r}{2(1-\nu)}}\right)^2 \quad \text{By Inequality(2)} \\
&\leq \nu\langle A^{\frac{(1-\alpha)r}{\nu}}\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\nu)\langle B^{\frac{\alpha r}{1-\nu}}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\quad - r_0\left(\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{(1-\alpha)r}{2\nu}} - \langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{\alpha r}{2(1-\nu)}}\right)^2 \quad \text{By Lemma 2.1(a)} \\
&= \left\langle\left(\nu A^{\frac{(1-\alpha)r}{\nu}} + (1-\nu)B^{\frac{\alpha r}{1-\nu}}\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\quad - r_0\left(\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{(1-\alpha)r}{2\nu}} - \langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{\alpha r}{2(1-\nu)}}\right)^2 \\
&\leq \mathbf{ber}\left(\nu A^{\frac{(1-\alpha)r}{\nu}} + (1-\nu)B^{\frac{\alpha r}{1-\nu}}\right) \\
&\quad - r_0\left(\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{(1-\alpha)r}{2\nu}} - \langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{\alpha r}{2(1-\nu)}}\right)^2
\end{aligned}$$

Now, by taking supremum over $\lambda \in \Omega$, we get the inequality. □

**Remark 2.8.** *If we put $\alpha = \frac{1}{2}$, $r = 2$ and $\nu = \frac{1}{2}$ in (9), we get the inequality in (8).*

Finally, we end this section by the following results.

**Theorem 2.9.** *Let $A, B \in B(H)$ be positive definite operators and $\alpha \in [0,1]$, then*

$$\mathbf{ber}(A\sharp_\alpha B) \leq \mathbf{ber}^{1-\alpha}(A)\mathbf{ber}^\alpha(B).$$

*In particular,*

$$\mathbf{ber}(A\sharp B) \leq \sqrt{\mathbf{ber}(A)\mathbf{ber}(B)}.$$

*Proof.* let $\hat{k}_\lambda$ be the normalized reproducing kernel of $H(\Omega)$, then

$$\begin{aligned}\langle (A\sharp_\alpha B)\hat{k}_\lambda, \hat{k}_\lambda\rangle &= \langle A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &= \langle (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}\hat{k}_\lambda, A^{\frac{1}{2}}\hat{k}_\lambda\rangle \\ &\leq \langle (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}\hat{k}_\lambda, A^{\frac{1}{2}}\hat{k}_\lambda\rangle^{\alpha}\langle A^{\frac{1}{2}}\hat{k}_\lambda, A^{\frac{1}{2}}\hat{k}_\lambda\rangle^{(1-\alpha)} \\ &= \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle^{(1-\alpha)} \cdot \langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\alpha}\end{aligned}$$

Now, by taking supremum over $\lambda \in \Omega$, we get the first inequality. In particular, for $\alpha = \frac{1}{2}$ we obtain the second one. □

**Corollary 2.10.** *Let $A, B \in B(H)$ be positive definite operators which commute with each other and $\alpha \in [0, 1]$, then*

$$\mathbf{ber}(A^{1-\alpha}B^{\alpha}) \leq \mathbf{ber}^{1-\alpha}(A)\mathbf{ber}^{\alpha}(B).$$

*In particular, if $\alpha = \frac{1}{2}$, then*

$$\mathbf{ber}(\sqrt{AB}) \leq \sqrt{\mathbf{ber}(A)\mathbf{ber}(B)}.$$

## 3. Additional results

To prove our results in this section, the following basic lemmas are also required.

**Lemma 3.1.** *[14] Let $A$ be an operator in $B(H)$, and $f, g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for all $x, y$ in $H$,*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\| \tag{10}$$

**Lemma 3.2.** *[16] Let $a_i$ be a positive real number $(i = 1, 2, \ldots, n)$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r \qquad \forall r \geq 1 \tag{11}$$

The following result is proved in [16], for the numerical radius. We bring the proof here with a slight difference for the convenience of readers.

**Theorem 3.3.** *Let $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, \ldots, n)$, and let $f$ and $g$ be nonnegative continuous functions on $[0, \infty)$ which satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2}\mathbf{ber}\left(\sum_{i=1}^{n}([A_i^* g^2(|X_i^*|)A_i]^r + [B_i^* f^2(|X_i|)B_i]^r)\right) \tag{12}$$

*for all $r \geq 1$.*

*Proof.* If $\hat{k}_\lambda$ is the normalized reproducing kernel of $H(\Omega)$, then

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* X_i B_i\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&= \left|\sum_{i=1}^{n} \langle A_i^* X_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^r \\
&\le \left(\sum_{i=1}^{n} |\langle A_i^* X_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle|\right)^r \\
&= \left(\sum_{i=1}^{n} |\langle X_i B_i \hat{k}_\lambda, A_i \hat{k}_\lambda\rangle|\right)^r \\
&\le \left(\sum_{i=1}^{n} \langle f^2(|X_i|) B_i \hat{k}_\lambda, B_i \hat{k}_\lambda\rangle^{\frac{1}{2}} \langle g^2(|X_i^*|) A_i \hat{k}_\lambda, A_i \hat{k}_\lambda\rangle^{\frac{1}{2}}\right)^r \quad \text{By (3.1)} \\
&\le n^{r-1} \sum_{i=1}^{n} \langle B_i^* f^2(|X_i|) B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{2}} \langle A_i^* g^2(|X_i^*|) A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{r}{2}} \quad \text{By (3.2)} \\
&\le n^{r-1} \sum_{i=1}^{n} \left\langle (B_i^* f^2(|X_i|) B_i)^r \hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} \left\langle (A_i^* g^2(|X_i^*|) A_i)^r \hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} \quad \text{By Lemma 2.1} \\
&\le \frac{n^{r-1}}{2} \sum_{i=1}^{n} \left(\left\langle [B_i^* f^2(|X_i|) B_i]^r \hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle [A_i^* g^2(|X_i^*|) A_i]^r \hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) \text{ By (1)} \\
&= \frac{n^{r-1}}{2} \left\langle \sum_{i=1}^{n} \left([B_i^* f^2(|X_i|) B_i]^r + [A_i^* g^2(|X_i^*|) A_i]^r\right) \hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\le \frac{n^{r-1}}{2} \mathbf{ber}\left(\sum_{i=1}^{n} ([A_i^* g^2(|X_i^*|) A_i]^r + [B_i^* f^2(|X_i|) B_i]^r)\right)
\end{aligned}$$

Now, by taking supremum over $\lambda \in \Omega$, we get the desired inequality. □

If we take $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0, 1)$, in inequality (12), we get the following inequality.

**Corollary 3.4.** *Let $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, \dots, n)$and $0 < \alpha < 1$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \le \frac{n^{r-1}}{2} \mathbf{ber}\left(\sum_{i=1}^{n} ([A_i^* |X_i^*|^{2(1-\alpha)} A_i]^r + [B_i^* |X_i|^{2\alpha} B_i]^r)\right) \tag{13}$$

*for $r \ge 1$.*

Inequality (13) includes some special cases as follows.

**Corollary 3.5.** *Let $A, B, X \in B(H)$. Then*

*(i)* $\mathbf{ber}^r(A) \le \frac{1}{2}\mathbf{ber}(|A|^r + |A^*|^r) \quad \forall r \ge 1$,
*(ii)* $\mathbf{ber}(A^*B) \le \frac{1}{2}\mathbf{ber}(A^*A + B^*B)$,
*(iii)* $\mathbf{ber}(A^*XB) \le \frac{1}{2}\mathbf{ber}(A^*|X^*|A + B^*|X|B)$.

Now, we want to generalize inequality (12) in the following form.

**Theorem 3.6.** *Let $A_i, B_i, X_i \in B(H)$ $(i = 1, 2, \dots, n)$, and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then for $\nu \in [0, 1]$ and $r \geq 2R_0$*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq n^{r-1}\mathbf{ber}\left(\sum_{i=1}^{n} \nu(B_i^* f^2(|X_i|)B_i)^{\frac{r}{2\nu}} + (1-\nu)(A_i^* g^2(|X_i^*|)A_i)^{\frac{r}{2(1-\nu)}}\right). \tag{14}$$

*Proof.* let $\hat{k}_\lambda$ be the normalized reproducing kernel of $H(\Omega)$, then

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* X_i B_i\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&= \left|\sum_{i=1}^{n} \langle A_i^* X_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^r \\
&\leq \left(\sum_{i=1}^{n} |\langle A_i^* X_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle|\right)^r \\
&= \left(\sum_{i=1}^{n} |\langle X_i B_i \hat{k}_\lambda, A_i \hat{k}_\lambda\rangle|\right)^r \\
&\leq \left(\sum_{i=1}^{n} \langle f^2(|X_i|)B_i\hat{k}_\lambda, B_i\hat{k}_\lambda\rangle^{\frac{1}{2}} \langle g^2(|X_i^*|)A_i\hat{k}_\lambda, A_i\hat{k}_\lambda\rangle^{\frac{1}{2}}\right)^r \quad \text{By (3.1)} \\
&\leq n^{r-1}\sum_{i=1}^{n} \left\langle B_i^* f^2(|X_i|)B_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{r}{2}} \left\langle A_i^* g^2(|X_i^*|)A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{r}{2}} \quad \text{By (3.2)} \\
&\qquad \text{By Inequality (1) and Lemma 2.1} \\
&\leq n^{r-1}\sum_{i=1}^{n} \left(\nu\left\langle (B_i^* f^2(|X_i|)B_i)^{\frac{r}{2\nu}}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + (1-\nu)\left\langle (A_i^* g^2(|X_i^*|)A_i)^{\frac{r}{2(1-\nu)}}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) \\
&= n^{r-1}\left\langle\sum_{i=1}^{n} \left(\nu(B_i^* f^2(|X_i|)B_i)^{\frac{r}{2\nu}} + (1-\nu)(A_i^* g^2(|X_i^*|)A_i)^{\frac{r}{2(1-\nu)}}\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\leq n^{r-1}\mathbf{ber}\left(\sum_{i=1}^{n} \left(\nu(B_i^* f^2(|X_i|)B_i)^{\frac{r}{2\nu}} + (1-\nu)(A_i^* g^2(|X_i^*|)A_i)^{\frac{r}{2(1-\nu)}}\right)\right)
\end{aligned}$$

Now, the result follows by taking the supremum over $\lambda \in \Omega$. □

By letting $A_i = B_i = I$ $(i = 1, 2, \dots, n)$, and $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0, 1)$, in inequality (14), we obtain the following inequalities.

**Corollary 3.7.** *Let $X_i \in B(H)$ $(i = 1, 2, \dots, n)$ and $0 < \alpha < 1$. Then for $\nu \in [0, 1]$ and $r \geq R_0\alpha$*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} X_i\right) \leq n^{r-1}\mathbf{ber}\left(\sum_{i=1}^{n} \nu|X_i|^{\frac{r\alpha}{\nu}} + (1-\nu)|X_i^*|^{\frac{r(1-\alpha)}{1-\nu}}\right). \tag{15}$$

*In particular, if $X_1 = X_2 = \cdots = X_n = X$, then*

$$\mathbf{ber}^r(X) \leq \mathbf{ber}\left(\nu|X|^{\frac{\alpha r}{\nu}} + (1-\nu)|X^*|^{\frac{(1-\alpha)r}{1-\nu}}\right). \tag{16}$$

As special cases of (14), (15) and (16), we present the following inequalities.

(i) $\mathbf{ber}^r(A) \leq \mathbf{ber}\left(\nu|A|^{\frac{r}{2\nu}} + (1-\nu)|A^*|^{\frac{r}{2(1-\nu)}}\right)$,

(ii) $\mathbf{ber}^r(A^*B) \leq \mathbf{ber}\left(\nu|B|^{\frac{r}{\nu}} + (1-\nu)|A|^{\frac{r}{1-\nu}}\right)$,

(iii) $\mathbf{ber}^r(A^*XB) \leq \mathbf{ber}\left(\nu(B^*|X|B)^{\frac{r}{2\nu}} + (1-\nu)(A^*|X^*|A)^{\frac{r}{2(1-\nu)}}\right)$.

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