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Investigations on Weak Versions of the Alster Property in Bitopological Spaces and Selection Principles

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Abstract. We investigate the relationships among the weaker forms of the properties "Menger", "Alster" and "Lindelöf" in bitopological spaces. We give some counterexamples to show the differences between these properties. Further we introduce the weak versions of the Alster property in terms of selection principles and obtain some of their topological properties.

1. Introduction

A topological space X is said to be Lindelöf, or has the Lindelöf property, if every open cover of X has a countable subcover. It is well known that the product of Lindelöf spaces is not necessarily Lindelöf. A topological space is productively Lindelöf if its product with every Lindelöf space is Lindelöf.

K. Alster introduced a property called the Alster property in [1] (see also [5]) to characterize the productively Lindelöf spaces and proved the following theorem.

Theorem: Assuming **CH**, a space of weight of at most $\aleph_1$ is productively Lindelöf if and only if it is Alster.

Lindelöf property and productively Lindelöfness have been studied for a long time and many results were obtained by many mathematicians.

There are several weakenings of the Lindelöf property. Unlike the productivity of the Lindelöf property, there are very few works on productivity of weak Lindelöf properties. Recently in [4] Babinkostova, Pansera and Scheepers have obtained some results to identify classes of spaces which are productively weakly Lindelöf and considered the weak versions of the Alster property. Additionally Kocev in [19] investigated the almost Alster spaces and proved that the product of an almost Menger and almost Alster spaces is almost Menger.

In this article we study these properties in bitopological context. The idea of bitopological spaces was first initiated by J.C. Kelly in [14] and many papers have appeared in the literature so far.

The weak versions of the Lindelöf property in bitopological spaces were mostly studied in [15]. Selective versions of the Lindelöf property for example Menger and Rothberger properties and their weaker forms as almost Menger and weakly Menger properties in bitopological spaces were discussed in [10, 25].

The layout of the paper is as follows. In Section 2 some necessary background material is recalled. In Section 3 we will consider the main generalizations of the Lindelöf property such as weakly Lindelöf and

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almost Lindelöf and investigate how these properties are related to the corresponding covering properties in bitopological spaces. We will also consider the class of productively almost Lindelöf and productively weakly Lindelöf bitopological spaces. It is natural to ask a question whether the weak version of the Alster property plays a role to characterize the weak productively Lindelöf bitopological spaces. Section 4 presents some properties of weak versions of the Alster property in bitopological spaces. Finally in Section 5 we characterize almost and weakly Alster properties in bitopological spaces in terms of selection principles.

2. Definitions and Notations

A topological space X is almost Lindelöf (weakly Lindelöf) if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$ (respectively, $\overline{\bigcup \mathcal{V}} = X$). Clearly, every Lindelöf space is almost Lindelöf and every almost Lindelöf space is weakly Lindelöf, but the converses do not hold. For spaces with the T_3 separation axiom, almost Lindelöfness implies Lindelöfness.

Many topological properties are defined or characterized in terms of the following two classical selection principles [26].

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

If O denotes the collection of all open covers of X, then the selection principle $\mathsf{S}_{fin}(O,O)$ (resp. $\mathsf{S}_1(O,O)$) is called the *Menger property* (*Rothberger property*)[11, 12, 24, 28].

Let $\mathcal{D}$ denote the collection of families of open sets with union dense in the space. Let $\overline{O}$ denotes the collection of families $\mathcal{U}$ of open subsets of the space for which $\{\overline{U} : U \in \mathcal{U}\}$ covers the space. Then $\mathsf{S}_{fin}(O,\mathcal{D})$ denotes the *weakly Menger property*, while $\mathsf{S}_{fin}(O,\overline{O})$ denotes the *almost Menger property*.

For undefined notions regarding selection principles in topological spaces we refer the reader to the papers [12, 20, 21, 27, 30]. The readers may find the most recent results on weak forms of classical selection principles of Menger, Hurewicz and Rothberger properties in the survey paper by Kočinac in [22].

Throughout the paper (X, τ_1, τ_2) will be a bitopological space, i.e. the set X endowed with two topologies τ_1 and τ_2. τ_i-open set means the open set with respect to topology τ_i in X. By τ_i-open cover we mean the cover of X by τ_i-open sets. For a subset A of X, $\mathrm{Int}_{\tau_i}(A)$ and $\mathrm{Cl}_{\tau_i}(A)$ will denote the interior and the closure of A in (X, τ_i), $(i = 1, 2)$ respectively. Let $\mathcal{P}$ be some topological property. Then $(i, j) - \mathcal{P}$ denotes an analogue of this property for τ_i with respect to τ_j, and p-$\mathcal{P}$ denotes the conjunction $(1, 2)$-$\mathcal{P} \wedge (2, 1)$-$\mathcal{P}$ where p is the abbreviation for "pairwise". We note that (X, τ_i) has a property $\mathcal{P}$ if and only if the bitopological space (X, τ_1, τ_2) has a property i-$\mathcal{P}$ and d-$\mathcal{P} \iff 1$-$\mathcal{P} \wedge 2$-$\mathcal{P}$, where "d" is the abbreviation for "double".

We end this section by recalling from [10, 25] the definitions of almost Menger and weakly Menger properties in bitopological spaces. Our topological terminology and notations are as in the book [9] and standard reference for bitopological spaces is [7].

Definition 2.1. ([25]) A bitopological space (X, τ_1, τ_2) is said to be (i, j)*-almost Menger*, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} \mathrm{Cl}_{\tau_j}(V)$.

Definition 2.2. ([25]) A bitopological space (X, τ_1, τ_2) is said to be (i, j)*-weakly Menger*, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $X = \mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n)$.

From the definitions, it is clear that every (i, j)-almost Menger bitopological space is (i, j)-weakly Menger but the converse does not hold (see, Exp.2.3 in [10]).

3. Weak Versions of the Alster Property and Related Bitopological Spaces

In 1988, K. Alster [1] characterized the productivity of Lindelöf spaces by introducing a very nice property which is known as $(*)$ property in the literature.

In [6] we have the definition of "amply Lindelöf space" which is actually the definition of Alster spaces. In [4] Babinkostova, Pansera and Scheepers used G_δ compact cover (A family $\mathcal{F}$ of G_δ subsets of a space X is called G_δ compact cover if there is for each compact subset K of X a set $F \in \mathcal{F}$ such that $K \subset F$) to define the Alster space, i.e. each G_δ compact cover of the space has a countable subset covering the space. They underlined the fact that this definition is not identical to the definition in [5] but it is equivalent and also it is equivalent to the $(*)$ property given by Alster.

The reader is referred to [1–6] for more background material on Alster spaces. We will use the following definition and terminology used by Aurichi and Dias in [3].

Definition 3.1. Let (X, τ) be a topological space. A cover $\mathcal{U}$ of X by G_δ subsets is said to be an *Alster cover* if every compact subset of X is included in some element of $\mathcal{U}$. A topological space X is an *Alster space* if every Alster cover $\mathcal{U}$ of X has a countable subcover.

Alster spaces were introduced to characterize the class of productively Lindelöf spaces. As mentioned in the introduction, Alster spaces are productively Lindelöf and under the continuum hypothesis, productively Lindelöf spaces of weight not exceeding $\aleph_1$ are Alster spaces [1].

In [4] the authors investigated the weak productively Lindelöf properties specially focused on weakly properties and obtained that weakly Alster spaces are productively weakly Lindelöf. Furthermore it is proven that *every productively Lindelöf space of weight at most $\aleph_1$ is productively weakly Lindelöf*. The reader is suggested to refer [4, 19] for detailed informations on weakly Alster and almost Alster spaces.

Now let us turn to bitopological spaces. The generalization of the Lindelöf property in bitopological spaces namely pairwise almost Lindelöf, pairwise weakly Lindelöf spaces were introduced and studied in [15–17]. We recall the followings:

Definition 3.2. ([15]) A bitopological space (X, τ_1, τ_2) is said to be *(i, j)-almost Lindelöf* if for every τ_i-open cover $\mathcal{U}$ of X, there exists a countable subfamily $\{U_n : n \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(U_n) = X$.

It is clear that if (X, τ_1) is a Lindelöf space, then (X, τ_1, τ_2) is $(1, 2)$-almost Lindelöf.

Proposition 3.3. ([17]) *An (i, j)-regular bitopological space is (i, j)-almost Lindelöf if and only if it is i-Lindelöf.*

We first start by giving an example that the product of two (i, j)-almost Lindelöf bitopological spaces need not be (i, j)-almost Lindelöf.

Example 3.4. Consider $(\mathbb{R}, \tau_1, \tau_2)$ where $\mathbb{R}$ is the real line, τ_1 is the Sorgenfrey topology and τ_2 is the usual topology. Thus $(\mathbb{R}, \tau_1, \tau_2)$ is an $(1, 2)$-almost Lindelöf bitopological space since $(\mathbb{R}, \tau_1)$ is Lindelöf. Now consider the product bitopological space $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$. It is clear that $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is $(1, 2)$-regular and $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1)$ is not Lindelöf [29]. Thus we observe that $(\mathbb{R} \times \mathbb{R}, \tau_1 \times \tau_1, \tau_2 \times \tau_2)$ is not $(1, 2)$-almost Lindelöf by Proposition 3.3 (see Exp. 3.24 in [18]).

Definition 3.5. ([15]) A bitopological space (X, τ_1, τ_2) is said to be *(i, j)-weakly Lindelöf*, if for every τ_i-open cover $\mathcal{U}$ of X there is a countable subset $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathrm{Cl}_{\tau_j}(\bigcup \mathcal{V}) = X$

It is clear that every (i, j)-almost Lindelöf bitopological space is (i, j)-weakly Lindelöf, but the converse is not true (see Exp.3.24 in [18]).

In order to investigate the productivity of almost Lindelöf bitopological spaces we need the following definition:

Definition 3.6. A bitopological space (X, τ_1, τ_2) is said to be *(i, j)-almost Alster*, if for every τ_i-Alster cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $\bigcup_{V\in\mathcal{V}} \mathrm{Cl}_{\tau_j}(V) = X$.

A bitopological space (X, τ_1, τ_2) is (i, j)-almost σ-compact if it is a union of τ_j-closures of countably many τ_i-compact subsets. Clearly if (X, τ_1) is σ-compact then the bitopological space (X, τ_1, τ_2) is $(1, 2)$-almost σ-compact.

Theorem 3.7. *Let (X, τ_i) be a metrizable topological space. Then (X, τ_1, τ_2) is (i, j)-almost σ-compact if and only if (X, τ_1, τ_2) is (i, j)-almost Alster.*

Proof. ($\Rightarrow$) Let $\mathcal{U}$ be τ_i-Alster cover of X. Since X is (i, j)-almost σ-compact there exist a sequence $(K_n : n \in \mathbb{N})$ of τ_i-compact subsets of X such that $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(K_n) = X$. Then since $\mathcal{U}$ is τ_i-Alster cover of X, for every $n \in \mathbb{N}$ there exists $U_n \in \mathcal{U}$ such that $K_n \subseteq U_n$. On the other hand, we have

$$X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(K_n) \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(U_n)$$

thus X is (i, j)-almost Alster.

($\Leftarrow$) Let $\mathcal{U}$ be the family of all τ_i-compact subset of X. Since (X, τ_i) is metrizable then it satisfies T_2 axiom. Thus the elements of $\mathcal{U}$ are τ_i-closed. On the other hand closed subsets are G_δ sets in metrizable topological spaces hence $\mathcal{U}$ is a τ_i-Alster cover of X.

Since (X, τ_1, τ_2) is an (i, j)-almost Alster bitopological space there exists a countable subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup_{V\in\mathcal{V}} Cl_{\tau_j}(V) = X$. Thus (X, τ_1, τ_2) is (i, j)-almost σ-compact. □

Proposition 3.8. *If (X, τ_1, τ_2) is an (i, j)-almost Alster bitopological space then X is (i, j)-almost Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. Without loss of generality we may assume that every $\mathcal{U}_n$ is closed under finite unions. Now take

$$\mathcal{V} = \Big\{ \bigcap_{n\in\mathbb{N}} U_n : U_n \in \mathcal{U}_n \Big\}.$$

Clearly the elements of $\mathcal{V}$ are τ_i-G_δ subsets of X. On the other hand for every τ_i-compact subset K of X there exists $V \in \mathcal{V}$ such that $K \subset V$. Thus $\mathcal{V}$ is a τ_i-Alster cover of X.

Since X is (i, j)-almost Alster, there exists a countable subfamily $\{V_k : k \in \mathbb{N}\}$ of $\mathcal{V}$ such that

$$\bigcup_{k\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(V_k) = X.$$

Now let $V_k = \bigcap_{n\in\mathbb{N}} U_{nk}$ $(\forall n \in \mathbb{N}, U_{nk} \in \mathcal{U}_n)$. So for each $n \in \mathbb{N}$, $V_n \subset U_{nn} \in \mathcal{U}_n$ so that

$$\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(U_{nn}) = X.$$

Thus X is (i, j)-almost Menger. □

From the previous proposition (i, j)-almost Alster bitopological spaces are (i, j)-almost Menger but the following example shows the reverse implication does not hold in general.

Example 3.9. Let $\mathbb{R}$ be the real line, τ_e be the Euclidean topology and τ_{cc} be the topology of countable complements on $\mathbb{R}$. We define τ be the smallest topology generated by $\tau_e \cup \tau_{cc}$. The topology τ is called *countable complement extension topology* [29].

- A set G is open in $(\mathbb{R}, \tau)$ if and only if $G = U \setminus A$ where $U \in \tau_e$ and A is a countable subset of $\mathbb{R}$.
- A set C is closed in $(\mathbb{R}, \tau)$ iff $C = K \cup B$ where K is closed in τ_e and B is a countable subset of $\mathbb{R}$.
- A subset of $(\mathbb{R}, \tau)$ is compact iff it is finite so $(\mathbb{R}, \tau)$ is not σ-compact. On the other hand, $(\mathbb{R}, \tau)$ is Menger by [23].

Now we define the bitopological space as follows: Let τ_1 be the countable complement extension topology on $\mathbb{R}$ and τ_2 be the Euclidean topology on $\mathbb{R}$.

- $(\mathbb{R}, \tau_1, \tau_2)$ is $(1,2)$-almost Menger because $(\mathbb{R}, \tau_1)$ is Menger.
- $(\mathbb{R}, \tau_1, \tau_2)$ is not $(1,2)$-almost Alster. Indeed, let $\mathcal{U}$ be the family of all τ_1-compact subsets of $\mathbb{R}$, then $\mathcal{U}$ is a τ_1-Alster cover of $\mathbb{R}$. But any family consisting of the τ_2-closures of the elements in a countable subfamily of $\mathcal{U}$ does not cover $\mathbb{R}$.

Proposition 3.10. *If (X, τ_1, τ_2) is an (i, j)-almost Menger bitopological space, then X is (i, j)-almost Lindelöf.*

Proof. Let $\mathcal{U}$ be a τ_i-open cover of X. For each $n \in \mathbb{N}$ let us take $\mathcal{U}_n = \mathcal{U}$. Then the elements of the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ are τ_i-open covers of X and since X is (i, j)-almost Menger we obtain that X is (i, j)-almost Lindelöf. □

By Proposition 3.8 and Proposition 3.10 we have (i, j)-almost Alster bitopological spaces are (i, j)-almost Lindelöf. The following example shows that the reverse implication does not hold in general.

Example 3.11. Let $\mathcal{B}$ be a base for the Euclidean topology on $\mathbb{R}$. The family

$$\mathcal{B}^\star = \mathcal{B} \cup \{\{q\} : q \in \mathbb{Q}\}$$

is a base of a topology on $\mathbb{R}$. We define $\tau^\star$, the discrete rational extension of Euclidean topology to be the topology generated by $\mathcal{B}^\star$ [29, Example 70].

- $(\mathbb{R}, \tau^\star)$ is a metrizable space. $(\mathbb{R}, \tau^\star)$ is not σ-compact but it is Lindelöf.

Now let τ_2 be the Euclidean topology on $\mathbb{R}$ and let τ_1 be the discrete rational extension of τ_2.

- $(\mathbb{R}, \tau_1, \tau_2)$ is $(1,2)$-almost Lindelöf since $(\mathbb{R}, \tau_1)$ is a Lindelöf space.
- $(\mathbb{R}, \tau_1, \tau_2)$ is not $(1,2)$-almost σ-compact. Thus $(\mathbb{R}, \tau_1, \tau_2)$ is not $(1,2)$-almost Alster by Theorem 3.7.

Now we note that a topological space (X, τ) is a P-space if any intersection of countably many open sets of X is again open. Let us recall [15] that a bitopological space (X, τ_1, τ_2) is said to be (i, j)-weakly P-space if for every countable family $\{U_n : n \in \mathbb{N}\}$ of τ_i-open subsets of X, $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} U_n) = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(U_n)$. Concerning to the question in under which conditions (i, j)-almost Lindelöf bitopological spaces are (i, j)-almost Alster we give the following proposition:

Proposition 3.12. *If (X, τ_1, τ_2) is (i, j)-almost Lindelöf and (X, τ_i) is P-space, then X is (i, j)-almost Alster.*

Proof. It is clear from every τ_i-Alster cover is a τ_i-open cover. □

Corollary 3.13. *Let (X, τ_1, τ_2) be a bitopological space with (X, τ_i) is P-space. The following statements are equivalent:*
(1) (X, τ_1, τ_2) is (i, j)-almost Alster.
(2) (X, τ_1, τ_2) is (i, j)-almost Menger.
(3) (X, τ_1, τ_2) is (i, j)-almost Lindelöf.

We have now in a position to give the promised relation with the (i, j)-almost Alster property and productively almost Lindelöf bitopological spaces.

Theorem 3.14. *Let (X, τ_1, τ_2) be an (i, j)-almost Alster and (Y, σ_1, σ_2) be an (i, j)-almost Lindelöf bitopological spaces. Then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is (i, j)-almost Lindelöf.*

Proof. Let $\mathcal{W}$ be a $\tau_i{\times}\sigma_i$-open cover of $X{\times}Y$. Without loss of generality we may assume that $\mathcal{W}$ is closed under finite unions.

For each τ_i-compact $C \subseteq X$ and each $y \in Y$ we can choose $W(C,y) \in \mathcal{W}$ such that $C{\times}\{y\} \subset W(C,y)$. On the other side there are τ_i-neighbourhood $U(C,y)$ and σ_i-neighbourhood $V(C,y)$ of C and y respectively, such that $C{\times}\{y\} \subset U(C,y){\times}V(C,y) \subset W(C,y)$ by the Wallace Theorem [13, Theorem 5.12].

The set $\{V(C,y) : y \in Y\}$ is a σ_i-open cover of Y. By the hypothesis there exists a countable subset $Y(C)$ of Y such that $\bigcup_{y\in Y(C)} Cl_{\sigma_j}(V(C,y)) = Y$.

Let $U(C) = \bigcap_{y\in Y(C)} U(C,y)$. Then for each $y \in Y(C)$ we have $C{\times}\{y\} \subset U(C){\times}V(C,y) \subset W(C,y)$. $U(C)$ is a τ_i-G_δ set containing C and the set $\{U(C) : C \subset X, \tau_i\text{-compact}\}$ is a τ_i-Alster cover of X. By the hypothesis there exists a countable family of τ_i-compact sets $\{C_n : n \in \mathbb{N}\}$ such that $\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U(C_n)) = X$.

We now have:

$$\begin{aligned} X\times Y &= \bigcup_{n\in\mathbb{N}}\Big(Cl_{\tau_j}\big(U(C_n)\big)\times \bigcup_{y\in Y(C_n)} Cl_{\sigma_j}\big(V(C_n,y)\big)\Big)\\ &= \bigcup_{n\in\mathbb{N}}\Big(\bigcup_{y\in Y(C_n)} Cl_{\tau_j}\big(U(C_n)\big)\times Cl_{\sigma_j}\big(V(C_n,y)\big)\Big)\\ &\subseteq \bigcup_{n\in\mathbb{N}}\bigcup_{y\in Y(C_n)} Cl_{\tau_j\times\sigma_j}\big(W(C_n,y)\big) \end{aligned}$$

and thus $(X{\times}Y, \tau_1{\times}\sigma_1, \tau_2{\times}\sigma_2)$ is (i,j)-almost Lindelöf. □

This last result shows that (i,j)-almost Alster bitopological spaces are productively (i,j)-almost Lindelöf. Now we will consider the bitopological spaces that are productively (i,j)-weakly Lindelöf.

Definition 3.15. A bitopological space (X,τ_1,τ_2) is said to be (i,j)-weakly Alster, if for every τ_i-Alster cover $\mathcal{U}$ of X there is a countable subset $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathrm{Cl}_{\tau_j}(\cup\mathcal{V}) = X$.

Obviously (i,j)-almost Alster bitopological spaces are (i,j)-weakly Alster.

Theorem 3.16. *Let (X,τ_1,τ_2) be an (i,j)-weakly Alster and (Y,σ_1,σ_2) be an (i,j)-weakly Lindelöf bitopological spaces then $(X\times Y, \tau_1\times\sigma_1, \tau_2\times\sigma_2)$ is (i,j)-weakly Lindelöf.*

Proof. Let $\mathcal{W}$ be a $\tau_i{\times}\sigma_i$-open cover of $X{\times}Y$. For each τ_i-compact $A \subseteq X$ and each $y \in Y$, the set $A{\times}\{y\}$ is $\tau_i{\times}\sigma_i$-compact. Then there exists $W(A,y) \in \mathcal{W}$ and a τ_i-neighbourhood $U(A,y)$ of A and a σ_i-neighbourhood $V(A,y)$ of y, such that

$$A{\times}\{y\} \subset U(A,y){\times}V(A,y) \subset W(A,y).$$

The set $\{V(A,y) : y \in Y\}$ is a σ_i-open cover of Y. Since Y is an (i,j)-weakly Lindelöf bitopological space, there exists a countable subset $Y(A)$ of Y such that

$$Cl_{\sigma_j}\Big(\bigcup_{y\in Y(A)} V(A,y)\Big) = Y. \tag{1}$$

Let $U(A) = \bigcap_{y\in Y(A)} U(A,y)$. The set $\{U(A) : A \subset X, \tau_i\text{-compact}\}$ is a τ_i-Alster cover of X and since X is (i,j)-weakly Alster there exists a countable family of τ_i-compact sets $\{A_n : n \in \mathbb{N}\}$ such that

$$Cl_{\tau_j}\Big(\bigcup_{n\in\mathbb{N}} U(A_n)\Big) = X. \tag{2}$$

Now we will show that the union of the countable subfamily $\{W(A_n, y) : y \in Y(A_n), n \in \mathbb{N}\}$ of $\mathcal{W}$ is $\tau_j\times\sigma_j$-dense in $X\times Y$. Let $(a, b) \in X\times Y$ and S is a τ_j-open neighbourhood of a and T is a σ_j-open neighbourhood of b. By the equality (2) there exists $n_a \in \mathbb{N}$ such that $S \cap U(A_{n_a}) \neq \emptyset$. Applying the equality (1), we find $y_b \in Y(A_{n_a})$ such that $T \cap V(A_{n_a}, y_b) \neq \emptyset$. Then

$$\begin{aligned}\emptyset \neq \Big(S \cap U(A_{n_a})\Big) \times \Big(T \cap V(A_{n_a}, y_b)\Big) &= \Big(S \times T\Big) \cap \Big(U(A_{n_a}) \times V(A_{n_a}, y_b)\Big)\\ &\subset \Big(S \times T\Big) \cap W(A_{n_a}, y_b)\end{aligned}$$

which shows that $(X\times Y, \tau_1\times\sigma_1, \tau_2\times\sigma_2)$ is (i, j)-weakly Lindelöf. □

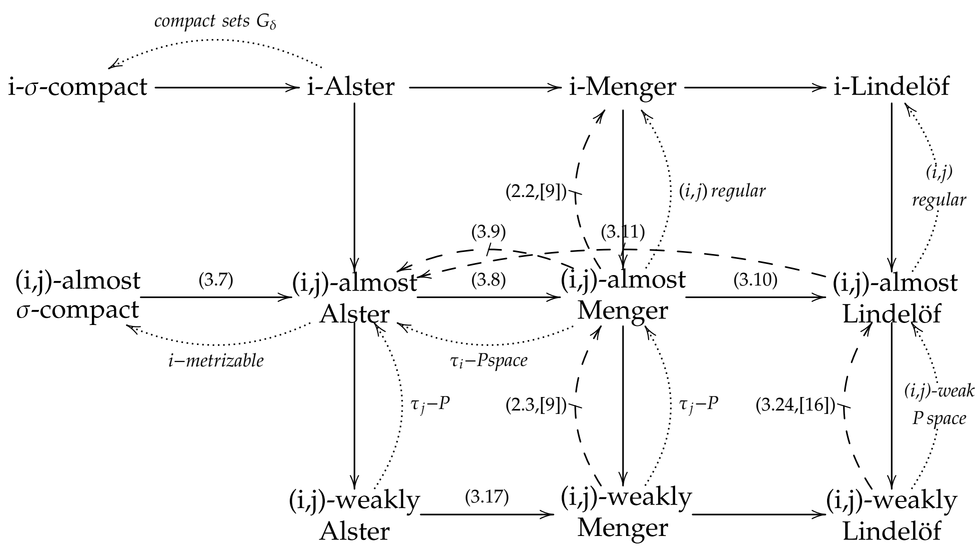
Finally we obtain that (i, j)-weakly Alster bitopological spaces are productively (i, j)-weakly Lindelöf.

Theorem 3.17. *If a bitopological space (X, τ_1, τ_2) is (i, j)-weakly Alster, then X is (i, j)-weakly Menger.*

Proof. Similar to the proof of Proposition 3.8. □

By the definitions it is clear that (i, j)-weakly Menger and (i, j)-almost Lindelöf bitopological spaces are (i, j)-weakly Lindelöf.

The following figure shows the relationships among the properties we have discussed. In [2] it is proved that Alster spaces are Menger. An example for the reverse implication is given in [12] " A Hurewicz (and hence Menger) subspace of the real line which is not σ-compact (and hence not Alster) [2]."It is noted that every space with the Menger property is Lindelöf. However the space $\mathbb{P}$ of irrationals is Lindelöf but not Menger. The solid arrow indicates the implication holds with the given number. A dashed arrow means this implication does not hold with the given example number. A dotted arrow indicates that the implication holds under the indicated hypothesis.



4. Properties of (i, j)-Almost (Weakly) Alster Bitopological Spaces

Throughout this section we consider some topological properties of (i, j)-almost (weakly) Alster bitopological spaces.

We note that a subset F of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-almost Alster (resp. weakly Alster) if F is (i, j)-almost Alster (resp. weakly Alster) as a subspace of X, i.e., F is (i, j)-almost Alster (weakly Alster) with respect to the induced bitopology from the bitopology of X.

Proposition 4.1. *Every τ_i-closed and τ_j-open subset of an (i, j)-almost Alster bitopological space (X, τ_1, τ_2) is (i, j)-almost Alster.*

Proof. Let F be a τ_i-closed and τ_j-open subset of X. We will show that $(F, \tau_{1_F}, \tau_{2_F})$ is (i, j)-almost Alster. Let $\mathcal{U}$ be a τ_{i_F}-Alster cover of F. Then $\mathcal{V} = \{U \cup (X \setminus F) : U \in \mathcal{U}\}$ is a τ_i-Alster cover of X. Since (X, τ_1, τ_2) is (i, j)-almost Alster there exists a countable subfamily $\{U_n : n \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}\big(U_n \cup (X \setminus F)\big) = X$.

On the other side, F is τ_j-open by the hypothesis then we obtain

$$\Big(\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n)\Big) \cup (X \setminus F) = X.$$

Now take the intersection of both sides by F, we have $\bigcup_{n\in\mathbb{N}} Cl_{\tau_{j_F}}(U_n) = F$.

This proves that $(F, \tau_{1_F}, \tau_{2_F})$ is (i, j)-almost Alster. □

Proposition 4.2. *Let (X, τ_1, τ_2) be a bitopological space and A be a τ_j-dense subset of X. If A is (i, j)-weakly Alster, then X is (i, j)-weakly Alster.*

Proof. Let $\mathcal{U}$ be a τ_i-Alster cover of X. Then $\mathcal{U}_A = \{U \cap A : U \in \mathcal{U}\}$ is a τ_{i_A}-Alster cover of A. Since A is (i, j)-weakly Alster there is a countable subfamily $\{U_n \cap A : n \in \mathbb{N}\}$ of $\mathcal{U}_A$ such that $A = Cl_{\tau_{j_A}}(\bigcup_{n\in\mathbb{N}}(U_n \cap A))$. Now we have $A \subseteq Cl_{\tau_j}(\bigcup_{n\in\mathbb{N}}(U_n \cap A)) \subseteq Cl_{\tau_j}(\bigcup_{n\in\mathbb{N}} U_n)$ and by the fact that A is a τ_j-dense subset of X, we obtain $X = Cl_{\tau_j}(A) \subseteq Cl_{\tau_j}(\bigcup_{n\in\mathbb{N}} U_n)$. □

Theorem 4.3. *If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are (i, j)-almost Alster bitopological spaces then $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is (i, j)-almost Alster.*

Proof. Let $\mathcal{W}$ be a $\tau_i{\times}\sigma_i$-Alster cover of $X{\times}Y$.

For each τ_i-compact set $A \subseteq X$ and σ_i-compact set $B \subseteq Y$ there exits $W(A, B) \in \mathcal{W}$ such that $A{\times}B \subset W(A, B)$.

Since $W(A, B)$ is a $\tau_i{\times}\sigma_i$-G_δ set, $W(A, B)$ can be written as $W(A, B) = \bigcap_{n\in\mathbb{N}} W_n(A, B)$ where for each $n \in \mathbb{N}$, $W_n(A, B)$ is a $\tau_i{\times}\sigma_i$-open subset of $X{\times}Y$.

For every $n \in \mathbb{N}$, there exists a τ_i-neighbourhood $U_n(A, B)$ and a σ_i-neighbourhood $V_n(A, B)$ of A and B respectively, such that

$$A{\times}B \subset U_n(A, B){\times}V_n(A, B) \subset W_n(A, B).$$

Let $U(A, B) = \bigcap_{n\in\mathbb{N}} U_n(A, B)$ and $V(A, B) = \bigcap_{n\in\mathbb{N}} V_n(A, B)$ then, $U(A, B)$ is a τ_i-G_δ set containing A and $V(A, B)$ is a σ_i-G_δ set containing B.

For every τ_i-compact set $A \subseteq X$, the set $\{V(A, B) : B \subset Y, \sigma_i\text{-compact}\}$ is a σ_i-Alster cover of Y. By hypothesis there is a countable family $\mathcal{A}$ consisting of σ_i-compact subsets of Y such that $\bigcup_{B\in\mathcal{A}} Cl_{\sigma_j}\big(V(A, B)\big) = Y$.

On the other hand, let $U(A) = \bigcap_{B\in\mathcal{A}} U(A, B)$ then $U(A)$ is a τ_i-G_δ set containing A, so that for every $B \in \mathcal{A}$ we have $A{\times}B \subset U(A){\times}V(A, B) \subset W(A, B)$.

The set $\{U(A) : A \subset X, \tau_i\text{-compact}\}$ is an τ_i-Alster cover of X. Thus for a countable set $\{A_n : n \in \mathbb{N}\}$ consisting of τ_i-compact subsets of X we obtain $\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}\big(U(A_n)\big) = X$.

Then we have:

$$\begin{aligned} X \times Y &= \bigcup_{n\in\mathbb{N}} \Big(Cl_{\tau_j}\big(U(A_n)\big) \times \bigcup_{B\in\mathcal{A}_n} Cl_{\sigma_j}\big(V(A_n, B)\big)\Big) \\ &= \bigcup_{n\in\mathbb{N}} \Big(\bigcup_{B\in\mathcal{A}_n} Cl_{\tau_j}\big(U(A_n)\big) \times Cl_{\sigma_j}\big(V(A_n, B)\big)\Big) \\ &\subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{B\in\mathcal{A}_n} Cl_{\tau_j\times\sigma_j}\big(W(A_n, B)\big) \end{aligned}$$

thus the product bitopological space $(X{\times}Y, \tau_1{\times}\sigma_1, \tau_2{\times}\sigma_2)$ is (i, j)-almost Alster. □

In [8], for $k \in \mathbb{N}$, the power bitopological space X^k of a bitopological space (X, τ_1, τ_2) is defined as $(X^k, {\tau_1}^k, {\tau_2}^k)$. The proof of Theorem 4.3 leads to the following:

Corollary 4.4. *Let (X, τ_1, τ_2) be an (i, j)-almost Alster bitopological space and $n \in \mathbb{N}$. Then the bitopological space $(X^n, {\tau_1}^n, {\tau_2}^n)$ is (i, j)-almost Alster.*

Theorem 4.5. *If (X, τ_1, τ_2) and (Y, σ_1, σ_2) are (i, j)-weakly Alster bitopological spaces then their product $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is (i, j)-weakly Alster.*

Proof. The proof uses the same techniques as the proof of Theorem 3.16 except that we are having a σ_i-compact subset $B \subseteq Y$ instead of $y \in Y$ and obtain $A \times B \subset U(A, B) \times V(A, B) \subset W(A, B)$.

By the hypothesis that X and Y are (i, j)-weakly Alster bitopological spaces we show that the union of the countable subfamily $\{W(A_n, B) : B \in \mathcal{A}_n, n \in \mathbb{N}\}$ of $\mathcal{W}$ is $\tau_j \times \sigma_j$-dense in $X \times Y$. Thus $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ is (i, j)-weakly Alster. □

Corollary 4.6. *Let (X, τ_1, τ_2) be an (i, j)-weakly Alster bitopological space and $n \in \mathbb{N}$. Then the bitopological space $(X^n, {\tau_1}^n, {\tau_2}^n)$ is (i, j)-weakly Alster.*

We recall that a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be double continuous (shortly d-continuous) if the induced functions $f : (X, \tau_i) \to (Y, \sigma_i)$ are continuous for $(i = 1, 2)$.

Proposition 4.7. *Every d-continuous image of an (i, j)-almost Alster bitopological space is (i, j)-almost Alster.*

Proof. Let (X, τ_1, τ_2) be an (i, j)-almost Alster bitopological space and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a d-continuous surjection. We will show that Y is (i, j)-almost Alster.

Let $\mathcal{U}$ be a σ_i-Alster cover of Y. It is easy to verify that $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is a τ_i-Alster cover of X. Since (X, τ_1, τ_2) is (i, j)-almost Alster there exists a countable subfamily $\{V_n : n \in \mathbb{N}\}$ of $\mathcal{V}$ such that $\bigcup_{n \in \mathbb{N}} Cl_{\tau_j}(V_n) = X$.

On the other hand, for every $n \in \mathbb{N}$ there exists $U_n \in \mathcal{U}$ satisfying $V_n = f^{-1}(U_n)$. As f is surjective and $\tau_j - \sigma_j$-continuous we have the followings:

$$Y = \bigcup_{n \in \mathbb{N}} f\Big(Cl_{\tau_j}(V_n)\Big) \subseteq \bigcup_{n \in \mathbb{N}} Cl_{\sigma_j}\Big(f(V_n)\Big) \subseteq \bigcup_{n \in \mathbb{N}} Cl_{\sigma_j}(U_n).$$

This means that (Y, σ_1, σ_2) is (i, j)-almost Alster. □

5. Weak Alster Properties and Selection Principles

In this section we characterize the (i, j)-almost (weakly) Alster property in terms of selection principles.

Let (X, τ_1, τ_2) be a bitopological space. The following classes of covers of X will be at the center of this investigation. We will follow the similar notations as used in the papers [4, 19].

$\mathcal{G}^{\tau_i}$: The family of all covers $\mathcal{U}$ of X for which each element of $\mathcal{U}$ is a τ_i-G_δ set.

$\mathcal{G}^{\tau_i}_{\mathcal{A}}$: The family of all τ_i-Alster covers of X.

$\mathcal{G}^{\tau_i}_{\Omega}$: The family of all covers $\mathcal{U} \in \mathcal{G}^{\tau_i}$ such that every finite subset of X is contained by an element of $\mathcal{U}$.

$\mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})$: The family, consisting of sets $\mathcal{U}$ with each element of $\mathcal{U}$ is τ_i-G_δ subset of X and $\{\mathrm{Cl}_{\tau_j}(U) : U \in \mathcal{U}\}$ covers X.

$\mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i}_{\Omega})$: The family of all sets $\mathcal{U} \in \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})$ such that for each finite subset $F \subseteq X$ there is a $U_F \subseteq \mathcal{U}$ such that $F \subseteq \mathrm{Cl}_{\tau_j}(U_F)$.

Now we give the following characterization of (i, j)-almost Alster property in terms of selection principle S_1.

Theorem 5.1. *Let (X, τ_1, τ_2) be a bitopological space. The followings are equivalent.*

1. *X is (i, j)-almost Alster;*

2. *X satisfies the selection principle* $S_1\Big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})\Big)$;
3. *X satisfies the selection principle* $S_1\Big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i}_{\Omega})\Big)$.

Proof. (1) ⇒ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-Alster covers of X. Define

$$\mathcal{U} = \Big\{ \bigcap_{n\in\mathbb{N}} U_n : (\forall n)(U_n \in \mathcal{U}_n)\Big\}.$$

Then clearly $\mathcal{U}$ is a τ_i-Alster cover of X. Since X is (i, j)-almost Alster there exists a countable subfamily $\{V_n : n \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(V_n) = X$.
Now for each $n \in \mathbb{N}$ put $V_n = \bigcap_{k\in\mathbb{N}} U^n_k$ $(U^n_k \in \mathcal{U}_k, \forall k \in \mathbb{N})$. Then for each $n \in \mathbb{N}$ we have $V_n \subset U^n_n \in \mathcal{U}_n$ and we obtain,

$$\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(V_n) \subset \bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U^n_n) = X$$

thus X satisfies $S_1\Big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, Cl_{\tau_j}(\mathcal{G}^{\tau_i})\Big)$.
(2) ⇒ (1) Let $\mathcal{U}$ be a τ_i-Alster cover of X. Then $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of τ_i-Alster covers of X where $\mathcal{U}_n = \mathcal{U}$ for each $n \in \mathbb{N}$. By (2) we can choose $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} Cl_{\tau_j}(U_n) = X$, thus X is (i, j)-almost Alster.
(2) ⇒ (3) If X satisfies the selection principle $S_1\Big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})\Big)$ then for every $n \in \mathbb{N}$, X^n satisfies $S_1\Big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i})\Big)$; since the finite power of an (i, j)-almost Alster space is (i, j)-almost Alster by Corollary 4.4.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-Alster covers of X. Let $\{Y_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$, where Y_n is infinite for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and each $k \in Y_n$ set $\mathcal{V}_k = \{(U)^n : U \in \mathcal{U}_k\}$. Then $(\mathcal{V}_k : k \in Y_n)$ is a sequence of τ^n_i-Alster cover of X^n for every $n \in \mathbb{N}$. Since X^n satisfies $S_1\Big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, Cl_{\tau_j}(\mathcal{G}^{\tau_i})\Big)$ for each $k \in Y_n$ there exists $V_k \in \mathcal{V}_k$ such that

$$\bigcup_{k\in Y_n} Cl_{\tau^n_j}(V_k) = X^n.$$

On the other hand, for each $n \in \mathbb{N}$ and each $k \in Y_n$ there exists $U_k \in \mathcal{U}_k$ such that $V_k = (U_k)^n$.

Now we will show $\{U_n : n \in \mathbb{N}\} \in Cl_{\tau_j}(\mathcal{G}^{\tau_i}_{\Omega})$. Consider a finite subset $F = \{x_1, x_2, \ldots, x_m\}$ of X. Now we consider F as a point in X^m like $z = (x_1, x_2, \ldots, x_m) \in X^m$. Then there exists $k \in Y_m$ such that $z \in \mathrm{Cl}_{\tau^m_j}(V_k)$. In this case $x_i \in \mathrm{Cl}_{\tau_j}(U_k)$ for every $i = 1, 2, \ldots, m$ and $F \subset \mathrm{Cl}_{\tau_j}(U_k)$. Thus X satisfies $S_1\Big(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathrm{Cl}_{\tau_j}(\mathcal{G}^{\tau_i}_{\Omega})\Big)$.
(3) ⇒ (2) Clear since $Cl_{\tau_j}(\mathcal{G}^{\tau_i}_{\Omega}) \subset Cl_{\tau_j}(\mathcal{G}^{\tau_i})$. □

We end this section by characterizing the (i, j)-weakly Alster property in terms of selection principles. Now we need the following notation.

$\mathcal{D}_{\tau_j}(\mathcal{G}^{\tau_i})$: The collection of sets $\mathcal{U}$ where each element of $\mathcal{U}$ is τ_i-G_δ sets and $\bigcup\mathcal{U}$ is dense in (X, τ_j).

Finally we note the following:

Theorem 5.2. *For a bitopological space* (X, τ_1, τ_2) *the followings are equivalent.*

1. *X is (i, j)-weakly Alster;*
2. *X satisfies the selection principle* $\mathsf{S}_1(\mathcal{G}^{\tau_i}_{\mathcal{A}}, \mathcal{D}_{\tau_j}(\mathcal{G}^{\tau_i}))$.

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