Filomat 33:14 (2019), 4687–4695
https://doi.org/10.2298/FIL1914687O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Results for a Class of Subordinate Functions

M. Obradović[a], Z. Peng[b]

[a]*Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000*
[b]*Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, P. R. China*

Abstract. In this article, a class of subordinate functions is introduced. The bounds of the coefficients of the functions in this class are investigated.

1. Introduction

Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let S be the subset of $\mathcal{A}$ consisting of functions f that are univalent in $\mathbb{D}$. A function $f \in S$ is called starlike if $f(\mathbb{D})$ is starlike with respect to the origin. The class of all starlike functions is denoted by S^*. A function $f \in S^*$ if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \quad (|z| < 1).$$

A function $f \in S$ is called convex if $f(\mathbb{D})$ is a convex set. The class of all convex functions is denoted by K. A function $f \in K$ if and only if

$$\operatorname{Re}[1 + \frac{zf''(z)}{f'(z)}] > 0 \quad (|z| < 1).$$

A function f analytic in $\mathbb{D}$ is said to be typically real if it has real values on the real axis and nonreal values elsewhere. Let T denote the class of all typically real functions f such that $f(0) = 0$ and $f'(0) = 1$.

Let $f(z)$ and $g(z)$ be analytic in the unit disk $\mathbb{D}$. We say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if

$$f(z) = g(\omega(z)), \quad |z| < 1$$

for some analytic function $\omega(z)$ with $|\omega(z)| \leq |z|$. If $g(z)$ is univalent, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Let

$$\mathcal{U}_f(z) := \left(\frac{z}{f(z)}\right)^2 f'(z) - 1$$

2010 *Mathematics Subject Classification*. 30C45
Keywords. Univalent, Coefficient, Subordinate
Received: 13 August 2018; Accepted: 16 September 2018
Communicated by Miodrag Mateljević
The work of the first author was supported by MNZZS Grant, No. ON174017, Serbia. The research of the corresponding author was supported by the Key Laboratory of Applied Mathematics in Hubei Province, China.
Email addresses: milutin.obradovic@gmail.com, obrad@grf.bg.ac.rs (M. Obradović), pengzg65@hotmail.com, pengzhigang@hubu.edu.cn (Z. Peng)

and let

$$\mathcal{U}(\lambda) := \left\{ f \in \mathcal{A} : \ |\mathcal{U}_f(z)| < \lambda, \ z \in \mathbb{D} \right\},$$

where $0 < \lambda \leq 1$. We put $\mathcal{U}(1) = \mathcal{U}$. It is well known that the functions in $\mathcal{U}$ are univalent[1]. Since $\mathcal{U}(\lambda) \subset \mathcal{U}$ for $0 < \lambda \leq 1$, the functions in $\mathcal{U}(\lambda)$ are also univalent when $0 < \lambda \leq 1$. Up to now, the class $\mathcal{U}$ have been studied in detail for many years[2–6].

Let $\mathcal{U}_a$, $0 \leq a \leq 1$, denote the class of functions $f \in \mathcal{A}$ such that

$$\frac{z}{f(z)} \prec 1 - 2az + az^2, \tag{1}$$

that is

$$\frac{z}{f(z)} = 1 - 2a\omega(z) + a\omega^2(z) \tag{2}$$

with $\omega(z)$ analytic in $\mathbb{D}$ and satisfying $|\omega(z)| \leq |z|$.

In the case of $a = 0$, the only function in the class $\mathcal{U}_0$ is $f(z) = z$. If $a = 1$, the condition (1) becomes

$$\frac{z}{f(z)} \prec (1 - z)^2,$$

Or equivalently,

$$\frac{f(z)}{z} \prec \frac{1}{(1 - z)^2}.$$

It is known that if $f \in S^*$, then

$$\frac{f(z)}{z} \prec \frac{1}{(1 - z)^2}.$$

Thus $S^* \subset \mathcal{U}_1$([7], p.37). In [8] M.Obradović proved that $\mathcal{U} \subset \mathcal{U}_1$. In the subsequent part of this article, we assume that $0 < a \leq 1$.

Examples.

1. Let $h(z) = \frac{z}{1-az}$, $0 < a \leq 1$. Then $h(z) \in \mathcal{U}_a$. To prove this, we need to show that

$$q_1(z) := 1 - az \prec 1 - 2az + az^2 := q(z).$$

Since the function q is univalent in $\mathbb{D}$ (we can check it directly by definition) and $q_1(0) = q(0) = 1$, it is enough to prove that $q_1(\mathbb{D}) \subset q(\mathbb{D})$([9], p.190). The boundary of $q(\mathbb{D})$ is given by

$$q(e^{i\theta}) = 1 - 2ae^{i\theta} + ae^{2i\theta} = u + iv,$$

where

$$u = 1 - 2a\cos\theta + a\cos(2\theta), \ v = -2a\sin\theta + a\sin(2\theta).$$

If we denote by $d(1, M)$ the distance between the points 1 and M, where M belongs to the boundary of $q(\mathbb{D})$, then

$$d^2(1, M) = (u - 1)^2 + v^2 = 5a^2 - 4a^2\cos\theta \geq a^2,$$

that is, $d(1, M) \geq a$, which means that $q(\mathbb{D})$ contains the disk with center 1 and with radius a, and this disk is just $q_1(\mathbb{D})$. It is clear that $h(z) = \frac{z}{1-az}$, with $0 < a \leq 1$, is univalent.

2. The function $f_a(z) \in \mathcal{U}_a$ defined by (2) with $\omega(z) = z$ is of the following form:

$$f_a(z) = \frac{z}{1 - 2az + az^2} = z + 2az^2 + (4a - 1)az^3 + 4a^2(2a - 1)z^4 + ... \tag{3}$$

We can prove directly by definition that f_a is univalent in $\mathbb{D}$.

3. Let's put $\omega(z) = z^k$ in (2), where $k \geq 2$ is an integer number, then we have the function

$$f_k(z) = \frac{z}{1 - 2az^k + az^{2k}} \in \mathcal{U}_a$$

and

$$f_k'(z) = \frac{1 + 2a(k-1)z^k - (2k-1)az^{2k}}{(1 - 2az^k + az^{2k})^2}.$$

After some elementary calculation we conclude that f_k' has zeros in $\mathbb{D}$ if $k > \left[\frac{1}{4}\left(\frac{1}{a} + 3\right)\right]$, which implies that the function f_k is not univalent for such k.

In this article we obtain additional information on the class $\mathcal{U}_a$.

2. Main results

Lemma 2.1. *Let $\omega(z)$ be a nonconstant analytic function in $\mathbb{D}$ with $\omega(0) = 0$. If $|\omega|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then there exists $m \geq 1$ such that $z_0\omega'(z_0) = m\omega(z_0)$.*

Lemma 1 is due to Jack[10].

Theorem 2.2. *Let $f \in \mathcal{U}(a)$, $0 < a \leq 1$, with $\frac{z}{f(z)} \neq 1 - a$ for every $z \in \mathbb{D}$, then $f \in \mathcal{U}_a$.*

Proof. Let $f \in \mathcal{U}(a)$, $0 < a \leq 1$, and let f satisfy the relation (2). Then $\omega(0) = 0$ and since

$$a(\omega(z) - 1)^2 = \frac{z}{f(z)} - (1 - a) \neq 0,$$

we claim that ω is analytic in $\mathbb{D}$. We want to prove that $|\omega(z)| < 1$, $z \in \mathbb{D}$. If not, then there exists a z_0, $z_0 \in \mathbb{D}$, such that $|\omega(z_0)| = 1$. If we put $\omega(z_0) = e^{i\varphi}$ for some real φ, then by using Jack's lemma we have $z_0\omega'(z_0) = me^{i\varphi}$, $m \geq 1$. So, by using these facts and (2) we have

$$\begin{aligned}
|\mathcal{U}_f(z_0)| &= \left|\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1\right|_{z=z_0} \\
&= \left|(1 - 2a\omega(z) + a\omega^2(z)) - z(-2a\omega'(z) + 2a\omega(z)\omega'(z)) - 1\right|_{z=z_0} \\
&= \left|2a(m-1)e^{i\varphi} - a(2m-1)e^{i2\varphi}\right| \\
&\geq a\left(\left|(2m-1)e^{i2\varphi}\right| - 2\left|(m-1)e^{i\varphi}\right|\right) \\
&= a,
\end{aligned}$$

which contradicts $f \in \mathcal{U}(a)$. Thus, $|\omega(z)| < 1$, $z \in \mathbb{D}$ and by using (2) we have the statement of the theorem. □

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{U}_1$, that is

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^2},$$

then

$$\sum_{n=2}^{\infty} a_n z^{n-1} \prec \sum_{n=2}^{\infty} n z^{n-1}.$$

For $f \in \mathcal{U}_a$, $0 < a < 1$, We can get a similar conclusion.

Theorem 2.3. *Let* $f \in \mathcal{U}_a$, $0 < a \le 1$, *and* $f(z) = z + a_2z^2 + ...$ *Then we have the next relation*

$$\sum_{n=2}^{\infty} a_n z^{n-1} \prec \sum_{n=2}^{\infty} \frac{\sin(n\alpha)}{\sin\alpha} (\sqrt{a})^{n-1} z^{n-1}, \tag{4}$$

where $\alpha = \arccos(\sqrt{a})$. *In the case of* $a = 1$, $\alpha = 0$, $\sin(n\alpha)/\sin\alpha$ *should be understood as* n.

Proof. since $\alpha = \arccos(\sqrt{a}), 0 < a \le 1$, we have $\cos\alpha = \sqrt{a}$, $0 \le \alpha < \frac{\pi}{2}$. We also have that $1 - 2az + az^2 = 0$ for $z = \frac{1}{\sqrt{a}} e^{\pm i\alpha}$ and the next factorization:

$$\begin{aligned} 1 - 2az + az^2 &= a\left(z - \frac{1}{\sqrt{a}} e^{-i\alpha}\right)\left(z - \frac{1}{\sqrt{a}} e^{i\alpha}\right) \\ &= \left(1 - \sqrt{a} z e^{i\alpha}\right)\left(1 - \sqrt{a} z e^{-i\alpha}\right). \end{aligned}$$

Now, from (1) we obtain that

$$\begin{aligned} \frac{f(z)}{z} &\prec \frac{1}{1 - 2az + az^2} \\ &= \frac{1}{\left(1 - \sqrt{a} z e^{i\alpha}\right)\left(1 - \sqrt{a} z e^{-i\alpha}\right)} \\ &= \frac{1}{(2i\sin\alpha)\sqrt{a}z}\left(\frac{1}{1 - \sqrt{a} z e^{i\alpha}} - \frac{1}{1 - \sqrt{a} z e^{-i\alpha}}\right) \\ &= 1 + \sum_{n=2}^{\infty} \frac{\sin(n\alpha)}{\sin\alpha} (\sqrt{a})^{n-1} z^{n-1}, \end{aligned}$$

and therefore the relation (4) holds. □

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{U}_1$, then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^2}.$$

So

$$\frac{f(z)}{z} = \int_{|x|=1} \frac{1}{(1 - xz)^2} d\mu(x), \tag{5}$$

or equivalently,

$$f(z) = \int_{|x|=1} \frac{z}{(1 - xz)^2} d\mu(x), \tag{6}$$

where μ is a probability measure on $\partial\mathbb{D} = \{z : |z| = 1\}$([7], p.51). It follows from (6) that $|a_n| \le n$.

In the case of $0 < a < 1$, estimating the sharp bounds of the coefficients of $f \in \mathcal{U}_a$ seems to be difficult. However, we can give a rough estimation on the bounds of the coefficients of $f \in \mathcal{U}_a$ by using a result of Rogosinski.

Lemma 2.4. *[11] If* $g(z) \in T$ *and* $f(z) = a_1z + a_2z^2 + a_3z^3 + \cdots \prec g(z)$, *then* $|a_n| \le n$.

Theorem 2.5. *Let* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{U}(a)$, $0 < a \le 1$, *then* $|a_n| \le 2a(n-1)$.

Proof. Since $f(z) \in \mathcal{U}(a)$, $0 < a \le 1$, it follows that

$$\frac{1}{2a}\left(\frac{f(z)}{z} - 1\right) \prec \frac{z - \frac{1}{2}z^2}{1 - 2az + az^2} =: g(z).$$

As $g(z)$ is a univalent function with real coefficient and $g'(0) = 1$, $g(z) \in T$. So, by Lemma 2.4, we get $|a_n| \le 2a(n-1)$. □

In the following theorem we try to give the sharp estimation of $|a_2|$, $|a_3|$ and $|a_4|$ for $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{U}_a$ with $0 < a < 1$. For the proof of the theorem we need the following lemma, which is due to D. V. Prokhorov and J. Szynal.

Lemma 2.6. *[12] If $\omega(z) = c_1z + c_2z^2 + \cdots$ is analytic in $\mathbb{D}$ and satisfy the condition $|\omega(z)| < 1$ for $|z| < 1$,*

$$\Psi(\omega) = |c_3 + \mu c_1c_2 + \nu c_1^3|, \ \mu, \nu \text{ are real,}$$

then the following sharp estimate $|\Psi(\omega)| \leq \Phi(\mu, \nu)$ holds, where

$$\Phi(\mu,\nu) = \begin{cases} 1, & (\mu,\nu) \in D_1 \bigcup D_2 \bigcup \{(2,1)\} \\ |\nu|, & (\mu,\nu) \in \bigcup_{k=3}^{7} D_k \\ \frac{2}{3}(|\mu|+1)\left(\frac{|\mu|+1}{3(|\mu|+1+\nu)}\right)^{1/2}, & (\mu,\nu) \in D_8 \bigcup D_9 \\ \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\left(\frac{\mu^2-4}{3(\nu-1)}\right)^{1/2}, & (\mu,\nu) \in D_{10} \bigcup D_{11} - \{(2,1)\} \\ \frac{2}{3}(|\mu|-1)\left(\frac{|\mu|-1}{3(|\mu|-1-\nu)}\right)^{1/2}, & (\mu,\nu) \in D_{12} \end{cases} \tag{7}$$

and

$$\begin{aligned}
D_1 &= \{(\mu,\nu) : |\mu| \leq \tfrac{1}{2}, \ -1 \leq \nu \leq 1\} \\
D_2 &= \{(\mu,\nu) : \tfrac{1}{2} \leq |\mu| \leq 2, \ \tfrac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leq \nu \leq 1\} \\
D_3 &= \{(\mu,\nu) : |\mu| \leq \tfrac{1}{2}, \ \nu \leq -1\} \\
D_4 &= \{(\mu,\nu) : |\mu| \geq \tfrac{1}{2}, \ \nu \leq -\tfrac{2}{3}(|\mu|+1)\} \\
D_5 &= \{(\mu,\nu) : |\mu| \leq 2, \ \nu \geq 1\} \\
D_6 &= \{(\mu,\nu) : 2 \leq |\mu| \leq 4, \ \nu \geq \tfrac{1}{12}(\mu^2+8)\} \\
D_7 &= \{(\mu,\nu) : |\mu| \geq 4, \ \nu \geq \tfrac{2}{3}(|\mu|-1)\} \\
D_8 &= \{(\mu,\nu) : \tfrac{1}{2} \leq |\mu| \leq 2, \ -\tfrac{2}{3}(|\mu|+1) \leq \nu \leq \tfrac{4}{27}(|\mu|+1)^3 - (|\mu|+1)\} \\
D_9 &= \{(\mu,\nu) : |\mu| \geq 2, \ -\tfrac{2}{3}(|\mu|+1) \leq \nu \leq \tfrac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\} \\
D_{10} &= \{(\mu,\nu) : 2 \leq |\mu| \leq 4, \ \tfrac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \leq \nu \leq \tfrac{1}{12}(\mu^2+8)\} \\
D_{11} &= \{(\mu,\nu) : |\mu| \geq 4, \ \tfrac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \leq \nu \leq \tfrac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4}\} \\
D_{12} &= \{(\mu,\nu) : |\mu| \geq 4, \ \tfrac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4} \leq \nu \leq \tfrac{2}{3}(|\mu|-1)\}
\end{aligned}$$

Theorem 2.7. *Let $f(z) = z + a_2z^2 + a_3z^3 + ... \in \mathcal{U}_a$, $0 < a \leq 1$. Then we have*
(i) $|a_2| \leq 2a$;
(ii) $|a_3| \leq \begin{cases} 2a, & 0 < a \leq \frac{3}{4} \\ (4a-1)a, & \frac{3}{4} \leq a \leq 1 \end{cases}$
(iii) $|a_4| \leq 2a\Phi_1(a)$,
where

$$\Phi_1(a) = \begin{cases} 1, & 0 < a \leq a_1 \\ \frac{8a}{3}\sqrt{\frac{2}{3(2a+1)}}, & a_1 \leq a \leq a_2 \\ \frac{1}{3}(64a^4 - 64a^3 + 4a^2 + 6a)\sqrt{\frac{16a^2-8a-3}{3(4a^2-2a-1)}}, & a_2 \leq a \leq a_3 \\ 2a(2a-1), & a_3 \leq a \leq 1 \end{cases}$$

and $a_1 = \frac{27+\sqrt{4185}}{128}$, $a_3 = \frac{2+\sqrt{22}}{8}$ *and* $a_2 = 0.83085...$ *is the root of the equation*

$$32a^3 - 16a^2 - 10a + 1 = 0.$$

In cases (i), (ii) and (iii) (first and last line) the results are the best possible.

Proof. If we put $\omega(z) = c_1z + c_2z^2 + ...$, then from relation (4) we have

$$\sum_{n=2}^{\infty} a_n z^{n-1} = \sum_{n=2}^{\infty} \frac{\sin(n\alpha)}{\sin\alpha}(\sqrt{a})^{n-1}\left(c_1z + c_2z^2 + ...\right)^{n-1}, \tag{8}$$

where $\alpha = \arccos(\sqrt{a})$. By using the fact $\cos\alpha = \sqrt{a}$ and the next formulas

$$\frac{\sin(2\alpha)}{\sin\alpha} = 2\cos\alpha, \ \frac{\sin(3\alpha)}{\sin\alpha} = 4\cos^2\alpha - 1, \ \frac{\sin(4\alpha)}{\sin\alpha} = 4\cos\alpha(2cos^2\alpha - 1),$$

and by comparing the coefficients in (8), we can get

$$\begin{cases} a_2 = & 2ac_1, \\ a_3 = & 2ac_2 + (4a-1)ac_1^2, \\ a_4 = & 2a(c_3 + (4a-1)c_1c_2 + 2a(2a-1)c_1^3). \end{cases} \tag{9}$$

(i) From (9) we have $|a_2| = 2a|c_1| \le 2a$, since $|c_1| \le 1$. The function f_a given in (3) shows that the result is the best possible.

(ii) Since for the function ω we have that $|c_2| \le 1 - |c_1|^2$, then from (9) we obtain

$$\begin{aligned} |a_3| &\le 2a|c_2| + |4a-1|a|c_1|^2 \\ &\le 2a(1-|c_1|^2) + |4a-1|a|c_1|^2 \\ &= 2a + a(|4a-1|-2)|c_1|^2 \end{aligned}$$

and the result depends of the sign of $|4a-1|-2$. Namely, if $|4a-1|-2 \le 0$, or equivalently, $0 < a \le \frac{3}{4}$, then $|a_3| \le 2a$. If $\frac{3}{4} \le a \le 1$, then $|4a-1|-2 \ge 0$, and $|a_3| \le (4a-1)a$, since $|c_1| \le 1$.

For the function
$$f_2(z) = \frac{z}{1-2az^2+az^4}$$
(see the example 3 with $\omega(z) = z^2$) we have

$$f_2(z) = z + 2az^3 + ...,$$

which means that our result is the best possible for the first case. For the second case see the function f_a given by (3).

(iii) From (9) we have

$$|a_4| = 2a|c_3 + (4a-1)c_1c_2 + 2a(2a-1)c_1^3| := 2a\Psi(\omega), \tag{10}$$

where
$$\Psi(\omega) = |c_3 + \mu c_1c_2 + \nu c_1^3|, \ \mu = 4a-1, \ \nu = 2a(2a-1).$$

Now, let $\Phi_1(a) = \Phi(\mu,\nu)$ with $\mu = 4a-1$, $\nu = 2a(2a-1)$.

If $0 \le a < \frac{1}{8}$, then $(\mu,\nu) \in D_2$. If $\frac{1}{8} \le a < \frac{3}{8}$, then $(\mu,\nu) \in D_1$. If $\frac{3}{8} \le a \le a_1 := \frac{27+\sqrt{4185}}{128}$, then $(\mu,\nu) \in D_2$. By Lemma 2.6,
$$\Phi_1(a) = \Phi(\mu,\nu) = 1$$

for $0 \le a \le a_1 = \frac{27+\sqrt{4185}}{128}$.

If $a_1 \le a \le \frac{3}{4}$, then $(\mu,\nu) \in D_8$. If $\frac{3}{4} \le a \le a_2$, where a_2 is the biggest root of the equation

$$32a^3 - 16a^2 - 10a + 1 = 0,$$

then $(\mu,\nu) \in D_9$. So, by Lemma 2.6,

$$\Phi_1(a) = \Phi(u,v) = \frac{2}{3}(|\mu|+1)\sqrt{\frac{|\mu|+1}{3(|\mu|+1+\nu)}} = \frac{8a}{3}\sqrt{\frac{2}{3(2a+1)}}.$$

for $a_1 \le a \le a_2$.

If $a_2 \le a \le \frac{2+\sqrt{22}}{8}$, then $(\mu,\nu) \in D_{10}$, and by Lemma 2.6,

$$\begin{aligned}\Phi_1(a) = \Phi(u,v) &= \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\sqrt{\frac{\mu^2-4}{3(\nu-1)}}\\ &= \frac{1}{3}(64a^4-64a^3+4a^2+6a)\sqrt{\frac{16a^2-8a-3}{3(4a^2-2a-1)}}.\end{aligned}$$

If $\frac{2+\sqrt{22}}{8} \le a \le 1$, then $(\mu,\nu) \in D_6$. By Lemma 2.6,

$$\Phi_1(a) = \Phi(u,v) = \nu = 2a(2a-1).$$

For the function

$$f_3(z) = \frac{z}{1-2az^3+az^6}$$

(see the example 3 with $\omega(z) = z^3$) we obtain

$$f_3(z) = z + 2az^4 + ...,$$

which means that our result is the best possible for the first case. For the last case see the function f_a given by (3). □

Definition 2.8. *Suppose $f(z)$ is analytic in $\mathbb{D}$ and $\frac{f(z)}{z} \neq 0$. The logarithmic coefficients γ_n of f are defined by*

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n, |z| < 1.$$

Theorem 2.9. *If $f \in \mathcal{U}_a$, $0 < a \le 1$, and $\gamma_n(n = 1,2,3,\cdots)$ are its logarithmic coefficients, then*

$$\begin{aligned}\sum_{n=1}^{\infty}|\gamma_n|^2 &\le \sum_{n=1}^{\infty}\frac{1}{n^2}a^n\cos^2(n\alpha)\\ &= -\mathrm{Re}\int_0^a \frac{\ln(1-t)+\ln\left(1-(2a-1+2\sqrt{a(1-a)}i)t\right)}{2t}dt,\end{aligned}$$

where $\alpha = \arccos\sqrt{a} \in [0,\frac{\pi}{2}]$. In particular, if $a = 1$, then

$$\sum_{n=1}^{\infty}|\gamma_n|^2 \le \sum_{n=1}^{\infty}\frac{1}{n^2} = -\int_0^1\frac{\ln(1-t)}{t}dt = \frac{\pi^2}{6}.$$

Proof. Since $f \in \mathcal{U}_a$, $0 < a \le 1$, it follows from (1) that

$$\begin{aligned}\frac{f(z)}{z} &\prec \frac{1}{1-2az+az^2}\\ &= \frac{1}{\left(1-\sqrt{a}ze^{i\alpha}\right)\left(1-\sqrt{a}ze^{-i\alpha}\right)},\end{aligned}$$

where $\alpha = \arccos\sqrt{a}$. Thus

$$\begin{aligned}\ln\frac{f(z)}{z} &\prec -\ln(1-\sqrt{a}ze^{i\alpha})-\ln(1-\sqrt{a}ze^{-i\alpha})\\ &= 2\left(\sqrt{a}\cos\alpha z+\frac{1}{2}(\sqrt{a})^2\cos(2\alpha)z^2+\frac{1}{3}(\sqrt{a})^3\cos(3\alpha)z^3+\cdots\right).\end{aligned}$$

By Rogosinski's Theorem([9], p.192) we have

$$\begin{aligned}\sum_{n=1}^{\infty}|\gamma_n|^2 &\le \sum_{n=1}^{\infty}\frac{1}{n^2}a^n\cos^2(n\alpha)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{a^n}{n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}a^n\cos(2n\alpha)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{a^n}{n^2}+\frac{1}{2}\mathrm{Re}\sum_{n=1}^{\infty}\frac{1}{n^2}(ae^{i2\alpha})^n\\ &= -\frac{1}{2}\Big(\int_0^a\frac{\ln(1-t)}{t}dt+\mathrm{Re}\int_0^{ae^{i2\alpha}}\frac{\ln(1-t)}{t}dt\Big)\\ &= -\mathrm{Re}\int_0^a\frac{\ln(1-t)+\ln\left(1-(2a-1+2\sqrt{a(1-a)}i)t\right)}{2t}dt.\end{aligned}$$

□

Theorem 2.10. *If $f\in\mathcal{U}_1$, and $\gamma_n(n=1,2,3,\cdots)$ are its logarithmic coefficients, then $|\gamma_n|\le 1$. And the inequality is sharp for all n.*

Proof. Since $f\in\mathcal{U}_1$, it follows from (1) that

$$\frac{1}{2}\ln\frac{f(z)}{z} \quad \prec \quad -\ln(1-z).$$

Noting that $-\ln(1-z)\in K$, by Rogosinski's Theorem([9], p.195), we have $|\gamma_n|\le 1$. For any given n, the equality holds for the function

$$f(z)=\frac{z}{(1-z^n)^2},$$

which is in the class $\mathcal{U}_1$. □

Remark 2.11. *By using the same methods as in Th.2.7, it is possible to prove that the logarithmic coefficients of $f\in\mathcal{U}_a$ satisfy $|\gamma_1|\le a, |\gamma_2|\le a, |\gamma_3|\le a$. All these results are the best possible as the functions $f_k\in\mathcal{U}_a$ defined by*

$$f_1(z)=\frac{z}{1-2az+az^2},\, f_2(z)=\frac{z}{1-2az^2+az^4},\, f_3(z)=\frac{z}{1-2az^3+az^6}$$

show.

Theorem 2.12. *If $f\in\mathcal{U}_a$, $0<a\le 1$, then $\mathrm{Re}\frac{f(z)}{z}>0$ in the disc*

$$|z|<\begin{cases}1, & 0<a\le\frac{2}{3}\\ \sqrt{\frac{1}{a}-\frac{1}{2}}, & \frac{2}{3}\le a\le 1.\end{cases}$$

Proof. By using the definition (1) of the class $\mathcal{U}_a$ it is enough to find $z\in\mathbb{D}$ such that

$$\mathrm{Re}(1-2az+az^2)>0. \tag{11}$$

If we put $z=re^{i\theta},\ 0<r<1$, then we have

$$\mathrm{Re}(1-2az+az^2)=2ar^2\cos^2\theta-2ar\cos\theta+1-ar^2:=g(t),$$

where

$$g(t) = 2ar^2t^2 - 2art + 1 - ar^2, \ -1 \le t \le 1$$

(we put $\cos\theta = t$).
The function g has its minimum for $t_0 = \frac{1}{2r}$. If $t_0 \in (0, 1)$, then $r > \frac{1}{2}$ and

$$g(t) \ge g(t_0) = -\frac{a}{2} + 1 - ar^2 > 0$$

if $r < \sqrt{\frac{1}{a} - \frac{1}{2}}$. We note that $\frac{1}{a} - \frac{1}{2} \le 1$ if $\frac{2}{3} \le a \le 1$. For $0 < r \le \frac{1}{2}$ we have that $t_0 \ge 1$ and since $g(-1) > 0$, $g(1) > 0$, we also have that the condition (11) is satisfied. $\square$

References

[1] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, Proceedings of the American Mathematical Society 33 (1972) 392-394.
[2] M. Obradović, S. Ponnusamy, Product of univalent functions, Mathematical and Computer Modelling 57 (2013) 793-799.
[3] M. Obradović, S. Ponnusamy, New criteria and distortion theorems for univalent functions. Complex Variables, Theory and Application: An International Journal 44 (2001) 173-191.
[4] M. Obradović, s. Ponnusamy, Univalence and starlikeness of certain transforms defined by convolution, Journal of Mathematical Analysis and Applications 336 (2007) 758-767.
[5] M. Obradović, S. Ponnusamy, Radius of univalence of certain combination of univalent and analytic functions, Bulletin of the Malaysian Mathematical Sciences Society 35 (2012) 325-334.
[6] R. Fournier, S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, Complex Variables and Elliptic Equations: An International Journal 52 (2007) 1-8.
[7] D. J. Hallenbeck and T. H. MacGregor, Linear problems and convexity techniques in geometric function theory, Pitman Advanced Publishing Program, Boston, 1984.
[8] M. Obradović, Starlikeness and certain class of rational functions, Mathematische Nachrichten 175 (1995) 263-268.
[9] P. L. Duren, Univalent functions, Springer-Verlag, New York, 1983.
[10] I. S. Jack, Functions Starlike and Convex of Order α, Journal of the London Mathematical Society s2-3 (1971) 469-474.
[11] W. Rogosinski, On the coefficients of subordinate functions, Proceedings of the London Mathematical Society 48 (1943) 48-82.
[12] D. V. Prokhorov, J. Szynal, Inverse coefficients for (α, β)-convex functions, Annales Universitatis Mariae Curie-Skłodowska 35 (1981) 125-143.