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L_p-Dual Affine Surface Areas for the General L_p-Intersection Bodies

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Abstract. For $0 < p < 1$, the notions of symmetric and asymmetric L_p-intersection bodies were introduced by Haberl and Ludwig. Recently, Wang and Li defined the general L_p-intersection bodies. In this paper, associated with the L_p-dual affine surface areas, we give the extremum values of the general L_p-intersection bodies. Moreover, a Brunn-Minkowski type inequality and a monotone inequality for the L_p-dual affine surface area version of general L_p-intersection bodies are established, respectively.

1. Introduction and Main Results

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. $\mathcal{K}^n_o$ denote the set of convex bodies (containing the origin in their interiors) in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

If K is a compact star shaped (with respect to the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [4])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (respect to the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. For the set of star bodies about the origin, the set of star bodies whose centroid lie at the origin and the set of origin-symmetric star bodies in $\mathbb{R}^n$, we write $\mathcal{S}^n_o$, $\mathcal{S}^n_c$ and $\mathcal{S}^n_{os}$, respectively.

The notion of classical intersection body was introduced by Lutwak [14]. In the past three decades, the intersection bodies have received considerable attentions, see two good books [4, 21].

The L_p-intersection bodies were first introduced by Haberl and Ludwig (see [6]). For $K \in \mathcal{S}^n_o$ and $0 < p < 1$, the L_p-intersection body, I_pK, of K is the origin-symmetric star body whose radial function is defined by

$$\rho(I_pK, u)^p = \frac{1}{2}\int_K \mid u \cdot x \mid^{-p} dx = \frac{1}{2(n-p)}\int_{S^{n-1}} \mid u \cdot v \mid^{-p} \rho(K, v)^{n-p} dS(v),$$

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for all $u \in S^{n-1}$. Here $u \cdot x$ denotes the standard inner product of u and x. Regarding the investigation of L_p-intersection body, we refer to [5, 6, 37, 38].

Meanwhile, Haberl and Ludwig ([6]) defined the asymmetric L_p-intersection bodies as follows: For $K \in \mathcal{S}_o^n$ and $0 < p < 1$, the asymmetric L_p-intersection body, I_p^+K, of K is given by

$$\rho(I_p^+K, u)^p = \int_{K \cap u^+} | u \cdot x |^{-p} \, dx, \tag{1}$$

for all $u \in S^{n-1}$, where $u^+ = \{x : u \cdot x \geq 0, x \in \mathbb{R}^n\}$. They ([6]) also defined $I_p^-K = I_p^+(-K)$. From this, we see that for all $u \in S^{n-1}$,

$$\rho(I_p^-K, u)^p = \rho(I_p^+(-K), u)^p = \int_{-K \bigcap u^+} | u \cdot x |^{-p} \, dx = \int_{K \bigcap (-u)^+} | u \cdot x |^{-p} \, dx = \rho(I_p^+K, -u)^p = \rho(-I_p^+K, u)^p.$$

This yields that

$$I_p^-K = I_p^+(-K) = -I_p^+K. \tag{2}$$

Based on above asymmetric L_p-intersection bodies, Wang and Li (see [29, 30]) introduced the notion of general L_p-intersection bodies with a parameter τ as follows: For $K \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1, 1]$, the general L_p-intersection body, $I_p^\tau K \in \mathcal{S}_o^n$, of K is given by

$$\rho(I_p^\tau K, u)^p = f_1(\tau)\rho(I_p^+K, u)^p + f_2(\tau)\rho(I_p^-K, u)^p, \tag{3}$$

for all $u \in S^{n-1}$. Here

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \ \ f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{4}$$

Obviously, for $\tau = 0$, we see that $I_p^0K = I_pK$. From (4), we easily know that

$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau). \tag{5}$$

$$f_1(\tau) + f_2(\tau) = 1. \tag{6}$$

Further, by (1), (3), (5) and (6), Wang and Li ([29]) gave that for $\tau \in [-1, 1]$,

$$I_p^{-\tau}K = I_p^\tau(-K) = -I_p^\tau K. \tag{7}$$

Associated with the general L_p-intersection bodies, Wang and Li ([29]) proved the following extremal values inequality and a Brunn-Minkowski inequality.

Theorem 1.A. *For $K \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1, 1]$, then*

$$V(I_pK) \leq V(I_p^\tau K) \leq V(I_p^\pm K).$$

If K is not origin-symmetric, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.

Theorem 1.B. *For $K, L \in \mathcal{S}_o^n$, $0 < p < 1$ and $n - p > q > 0$, then for $\tau \in [-1, 1]$,*

$$V(I_p^\tau(K \widetilde{+}_q L))^{\frac{pq}{n(n-p)}} \leq V(I_p^\tau K)^{\frac{pq}{n(n-p)}} + V(I_p^\tau L)^{\frac{pq}{n(n-p)}},$$

with equality if and only if K and L are dilates. Here "$\widetilde{+}_q$" denotes the L_q-radial addition.

The general L_p-intersection bodies belong to a new and rapidly evolving asymmetric L_p-Brunn-Minkowski theory that has its own origin in the work of Ludwig, Haberl and Schuster (see [5–9, 16, 17]). For the further researches of asymmetric L_p-Brunn-Minkowski theory, also see [1–3, 10–13, 18–20, 22, 25–36, 39–41].

In 2010, Wang, Yuan and He ([23]) showed a type of L_p-dual affine surface area $\widetilde{\Omega}_p(K)$ of K. In 2015, Wang and Wang ([24]) made the following improvement: For $K \in \mathcal{S}^n_o$ and $p > 0$, the L_p-dual affine surface area, $\widetilde{\Omega}_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n\widetilde{V}_p(K, Q^*)V(Q)^{\frac{p}{n}} : Q \in S^n_c\}. \tag{8}$$

Here the $\widetilde{V}_p(M, N)$ denotes the L_p-dual mixed volume of $M, N \in \mathcal{S}^n_o$. When $Q \in \mathcal{S}^n_{os}$, definition (8) was given by Pei and Wang (see [19]). Now, we improve above definition (8) as follows: For $K \in \mathcal{S}^n_o$ and $p > 0$, the L_p-dual affine surface area, $\widetilde{\Omega}_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\{n\widetilde{V}_p(K, Q^*)V(Q)^{\frac{p}{n}} : Q \in S^n_o\}. \tag{9}$$

Remark 1.1. Recall that Lutwak's L_p affine surface area was defined as follows (see [15]): For $K \in \mathcal{K}^n_o$ and $p \geq 1$, the L_p affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K, Q^*)V(Q)^{\frac{p}{n}} : Q \in \ S^n_o\}. \tag{10}$$

Here, $V_p(M, N)$ denotes the L_p mixed volume of $M, N \in \mathcal{K}^n_o$ (see [15]). Compare to (9) and (10), we see that definition (9) is really the duality of definition (10).

In this paper, associated with the L_p-dual affine surface areas, we study the general L_p-intersection bodies. Firstly, combined with (9), we obtain the extremum values for the L_p-dual affine surface areas of general L_p-intersection bodies.

Theorem 1.1. *For $K \in \mathcal{S}^n_o$, $0 < p < 1$ and $\tau \in [-1, 1]$, then*

$$\widetilde{\Omega}_p(I_pK) \leq \widetilde{\Omega}_p(I^\tau_pK) \leq \widetilde{\Omega}_p(I^\pm_pK), \tag{11}$$

if K is not origin-symmetric, there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$.

Then, we establish the following L_p-dual affine surface areas version of Brunn-Minkowski inequality for the general L_p-intersection bodies.

Theorem 1.2. *For $K, L \in \mathcal{S}^n_o$, $n \geq 2$, $0 < p < 1$, $0 < q < n - p$ and $\tau \in [-1, 1]$, then*

$$\widetilde{\Omega}_p(I^\tau_p(K\widetilde{+}_qL))^{\frac{pq(n+p)}{n(n-p)^2}} \leq \widetilde{\Omega}_p(I^\tau_pK)^{\frac{pq(n+p)}{n(n-p)^2}} + \widetilde{\Omega}_p(I^\tau_pL)^{\frac{pq(n+p)}{n(n-p)^2}}, \tag{12}$$

with equality if and only if I^τ_pK and I^τ_pL are dilates.

Finally, we give a monotone inequality for the general L_p-intersection bodies.

Theorem 1.3. *For $K, L \in \mathcal{S}^n_o$, $0 < p < 1$ and $\tau \in [-1, 1]$, if $K \subseteq L$, then*

$$\widetilde{\Omega}_p(I^\tau_pK) \leq \widetilde{\Omega}_p(I^\tau_pL), \tag{13}$$

equality holds when $K = L$.

Please see the next section for the above interrelated background materials. The proofs of Theorems 1.1-1.3 will be completed in Section 3.

2. Notation and Background Material

In order to complete the proofs of Theorems 1.1-1.3, we will require the following notions.

If E is a nonempty subset and contains the origin in $\mathbb{R}^n$, then the polar set, E^*, of E is defined by (see [4])

$$E^* = \{x \in \mathbb{R} : x \cdot y \leq 1,\ y \in E\}.$$

For $K, L \in \mathcal{S}^n_o$, $p > 0$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-radial combination, $\lambda \circ K\widetilde{+}_p\mu \circ L$, of K and L is given by (see [5])

$$\rho(\lambda \circ K\widetilde{+}_p\mu \circ L, \cdot)^p = \lambda\rho(K, \cdot)^p + \mu\rho(L, \cdot)^p, \tag{14}$$

where $\lambda \circ K$ denotes the L_p-radial scalar multiplication and we easily obtain $\lambda \circ K = \lambda^{\frac{1}{p}} K$.

In (14), if $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero) and $n > p > 0$, the L_p-radial Blaschke combination, $\lambda \otimes K \pm_p \mu \otimes L$, of K and L is given by

$$\rho(\lambda \otimes K \pm_p \mu \otimes L, \cdot)^{n-p} = \rho(\lambda \circ K \widetilde{+}_{n-p} \mu \circ L, \cdot)^{n-p} = \lambda \rho(K, \cdot)^{n-p} + \mu \rho(L, \cdot)^{n-p}.$$

Associated with the L_p-radial combinations of star bodies, the L_p-dual mixed volumes were given as follows: For $K, L \in \mathcal{S}_o^n$, $p > 0$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_p(K, L)$, of K and L is given by (see [5, 38])

$$\frac{n}{p} \widetilde{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_p \varepsilon \circ L) - V(K)}{\varepsilon}.$$

From above definition, the integral representation of L_p-dual mixed volume can be given by (see [5])

$$\widetilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(L, u)^p dS(u), \tag{15}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From (15), we easily know that

$$\widetilde{V}_p(K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u).$$

3. Proofs of Theorems

In this section, we will prove Theorems 1.1-1.3. To complete the proof of Theorem 1.1, we require the following lemmas.

Lemma 3.1 ([22]). *If $K, L \in \mathcal{S}_o^n$, $0 < p < \frac{n}{2}$ and $\lambda, \mu \geq 0$ (not both zero), then for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(\lambda \circ K \widetilde{+}_p \mu \circ L, Q^*)^{\frac{p}{n-p}} \leq \lambda \widetilde{V}_p(K, Q^*)^{\frac{p}{n-p}} + \mu \widetilde{V}_p(L, Q^*)^{\frac{p}{n-p}},$$

with equality if and only if K and L are dilates.

Lemma 3.2. *If $K, L \in \mathcal{S}_o^n$, $0 < p < \frac{n}{2}$ and $\lambda, \mu \geq 0$ (not both zero), then*

$$\widetilde{\Omega}_p(\lambda \circ K \widetilde{+}_p \mu \circ L)^{\frac{p(n+p)}{n(n-p)}} \leq \lambda \widetilde{\Omega}_p(K)^{\frac{p(n+p)}{n(n-p)}} + \mu \widetilde{\Omega}_p(L)^{\frac{p(n+p)}{n(n-p)}}, \tag{16}$$

with equality if and only if K and L are dilates.

Proof. Since $0 < p < \frac{n}{2}$, thus $\frac{p}{n-p} > 0$. Combined with Lemma 3.1 and (9), we have

$$\begin{aligned}
\widetilde{\Omega}_p(\lambda \circ K \widetilde{+}_p \mu \circ L)^{\frac{p(n+p)}{n(n-p)}} &= \left[\sup\{ n^{\frac{n+p}{n}} \widetilde{V}_p(\lambda \circ K \widetilde{+}_p \mu \circ L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n \} \right]^{\frac{p}{n-p}} \\
&= \sup \left\{ n^{\frac{p(n+p)}{n(n-p)}} \widetilde{V}_p(\lambda \circ K \widetilde{+}_p \mu \circ L, Q^*)^{\frac{p}{n-p}} V(Q)^{\frac{p^2}{n(n-p)}} : Q \in \mathcal{S}_o^n \right\} \\
&\leq \sup \left\{ n^{\frac{p(n+p)}{n(n-p)}} [\lambda \widetilde{V}_p(K, Q^*)^{\frac{p}{n-p}} + \mu \widetilde{V}_p(L, Q^*)^{\frac{p}{n-p}}] V(Q)^{\frac{p^2}{n(n-p)}} : Q \in \mathcal{S}_o^n \right\} \\
&\leq \lambda \Big[\sup\{ n^{\frac{n+p}{n}} \widetilde{V}_p(K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n \} \Big]^{\frac{p}{n-p}} \\
&\quad + \mu \Big[\sup\{ n^{\frac{n+p}{n}} \widetilde{V}_p(L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n \} \Big]^{\frac{p}{n-p}} \\
&= \lambda \widetilde{\Omega}_p(K)^{\frac{p(n+p)}{n(n-p)}} + \mu \widetilde{\Omega}_p(L)^{\frac{p(n+p)}{n(n-p)}}.
\end{aligned}$$

Thus

$$\widetilde{\Omega}_p(\lambda\circ K\widetilde{+}_p\mu\circ L)^{\frac{p(n+p)}{n(n-p)}}\leq\lambda\widetilde{\Omega}_p(K)^{\frac{p(n+p)}{n(n-p)}}+\mu\widetilde{\Omega}_p(L)^{\frac{p(n+p)}{n(n-p)}}.$$

This yields (16). According to the equality condition of Lemma 3.1, we see that equality holds in (16) if and only if K and L are dilates. □

Lemma 3.3 ([29]). *If $K\in\mathcal{S}_o^n$ and $0<p<1$, then $I_p^+K=I_p^-K$ if and only if K is origin-symmetric.*

Lemma 3.4 ([29]). *If $K\in\mathcal{S}_o^n$, $0<p<1$, $\tau\in[-1,1]$ and $\tau\neq 0$, then $I_p^\tau K=I_p^{-\tau}K$ if and only if K is origin-symmetric.*

Lemma 3.5. *If $K\in\mathcal{S}_o^n$ and $p>0$, then*

$$\widetilde{\Omega}_p(-K)=\widetilde{\Omega}_p(K). \tag{17}$$

Proof. From definition (9) and (15), we have

$$\begin{aligned}
n^{-\frac{p}{n}}\widetilde{\Omega}_p(-K)^{\frac{n+p}{n}} &= \sup\{n\widetilde{V}_p(-K,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathcal{S}_o^n\}\\
&= \sup\left\{\left[\int_{S^{n-1}}\rho_{-K}^{n-p}(u)\rho_{Q^*}^p(u)du\right]V(Q)^{\frac{p}{n}}:Q\in\mathcal{S}_o^n\right\}\\
&= \sup\left\{\left[\int_{S^{n-1}}\rho_{K}^{n-p}(-u)\rho_{-Q^*}^p(-u)du\right]V(Q)^{\frac{p}{n}}:Q\in\mathcal{S}_o^n\right\}\\
&= \sup\{n\widetilde{V}_p(K,-Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathcal{S}_o^n\}\\
&= \sup\{n\widetilde{V}_p(K,(-Q)^*)V(-Q)^{\frac{p}{n}}:-Q\in\mathcal{S}_o^n\}\\
&= n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}}.
\end{aligned}$$

This yields (17). □

Proof of Theorem 1.1. For $K\in\mathcal{S}_o^n$, $0<p<1$, and $\tau\in[-1,1]$. By (3) , (14) and (16), we get

$$\begin{aligned}
\widetilde{\Omega}_p(I_p^\tau K)^{\frac{p(n+p)}{n(n-p)}} &= \widetilde{\Omega}_p(f_1(\tau)\circ I_p^+K\widetilde{+}_p f_2(\tau)\circ I_p^-K)^{\frac{p(n+p)}{n(n-p)}}\\
&\leq f_1(\tau)\widetilde{\Omega}_p(I_p^+K)^{\frac{p(n+p)}{n(n-p)}}+f_2(\tau)\widetilde{\Omega}_p(I_p^-K)^{\frac{p(n+p)}{n(n-p)}}.
\end{aligned} \tag{18}$$

From (2) and (17), we know

$$\widetilde{\Omega}_p(I_p^-K)=\widetilde{\Omega}_p(-I_p^+K)=\widetilde{\Omega}_p(I_p^+K). \tag{19}$$

Combined with (18), (19) and (6), we easily get

$$\widetilde{\Omega}_p(I_p^\tau K)\leq\widetilde{\Omega}_p(I_p^\pm K).$$

This gives the right side of inequality (11).

According to the equality condition of inequality (16), equality holds in the right side inequality of (11) if and only if I_p^+K and I_p^-K are dilates. Since $I_p^+K=-I_p^-K$, this means $I_p^+K=I_p^-K$. Thus from Lemma 3.3, it follows that if K is not origin-symmetric, then equality holds in the right-hand side inequality of (11) if and only if $\tau=\pm1$.

On the other hand, by (14), (3) and (5), we have

$$\begin{aligned}
&\rho(I_p^\tau K,\cdot)^p+\rho(I_p^{-\tau}K,\cdot)^p\\
&=f_1(\tau)\rho(I_p^+K,\cdot)^p+f_2(\tau)\rho(I_p^-K,\cdot)^p+f_1(-\tau)\rho(I_p^+K,\cdot)^p+f_2(-\tau)\rho(I_p^-K,\cdot)^p\\
&=f_1(\tau)\rho(I_p^+K,\cdot)^p+f_2(\tau)\rho(I_p^-K,\cdot)^p+f_2(\tau)\rho(I_p^+K,\cdot)^p+f_1(\tau)\rho(I_p^-K,\cdot)^p\\
&=\rho(I_p^+K,\cdot)^p+\rho(I_p^-K,\cdot)^p,
\end{aligned}$$

i.e.,

$$\frac{1}{2}\rho(I_p^\tau K,\cdot)^p + \frac{1}{2}\rho(I_p^{-\tau} K,\cdot)^p = \frac{1}{2}\rho(I_p^+ K,\cdot)^p + \frac{1}{2}\rho(I_p^- K,\cdot)^p.$$

Thus, by (3) we get

$$\rho(I_p K,\cdot)^p = \frac{1}{2}\rho(I_p^\tau K,\cdot)^p + \frac{1}{2}\rho(I_p^{-\tau} K,\cdot)^p,$$

i.e.,

$$I_p K = \frac{1}{2}\circ I_p^\tau K \widetilde{+}_p \frac{1}{2}\circ I_p^{-\tau} K.$$

This together with (16) gives

$$\begin{aligned}\widetilde{\Omega}_p(I_p K)^{\frac{p(n+p)}{n(n-p)}} &= \widetilde{\Omega}_p\Big(\frac{1}{2}\circ I_p^\tau K \widetilde{+}_p \frac{1}{2}\circ I_p^{-\tau} K\Big)^{\frac{p(n+p)}{n(n-p)}}\\ &\le \frac{1}{2}\widetilde{\Omega}_p(I_p^\tau K)^{\frac{p(n+p)}{n(n-p)}} + \frac{1}{2}\widetilde{\Omega}_p(I_p^{-\tau} K)^{\frac{p(n+p)}{n(n-p)}}.\end{aligned}$$

Similar to the proof of (19), by (7) and (17) we have

$$\widetilde{\Omega}_p(I_p^\tau K) = \widetilde{\Omega}_p(-I_p^{-\tau} K) = \widetilde{\Omega}_p(I_p^{-\tau} K).$$

Thus

$$\widetilde{\Omega}_p(I_p K) \le \widetilde{\Omega}_p(I_p^\tau K).$$

From this, we get the left side of inequality (11).

According to the equality condition of (16), we know that equality holds in the left side inequality of (11) if and only if $I_p^\tau K = I_p^{-\tau} K$. By Lemma 3.4, this implies that if K is not origin-symmetric, then equality holds in the left-hand side inequality of (11) if and only if $\tau = 0$. □

Lemma 3.6 ([22]). *If $K, L \in \mathcal{S}_o^n$, $n \ge 2$, $0 < p < 1$, $0 < q < n - p$ and $\tau \in [-1, 1]$, then for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_p(I_p^\tau(K\widetilde{+}_q L), Q)^{\frac{pq}{(n-p)^2}} \le \widetilde{V}_p(I_p^\tau K, Q)^{\frac{pq}{(n-p)^2}} + \widetilde{V}_p(I_p^\tau L, Q)^{\frac{pq}{(n-p)^2}}, \tag{20}$$

with equality if and only if $I_p^\tau K$ and $I_p^\tau L$ are dilates.

Proof of Theorem 1.2. For $K, L \in \mathcal{S}_o^n$, $n \ge 2$, $0 < p < 1$, $0 < q < n - p$ and $\tau \in [-1, 1]$, thus $\frac{(n-p)^2}{pq} > 1$, from (9) and (20), we get

$$\begin{aligned}\widetilde{\Omega}_p(I_p^\tau(K\widetilde{+}_q L))^{\frac{pq(n+p)}{n(n-p)^2}} &= \Big[\sup\{n^{\frac{n+p}{n}}\widetilde{V}_p(I_p^\tau(K\widetilde{+}_q L), Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}\Big]^{\frac{pq}{(n-p)^2}}\\ &= \sup\Big\{n^{\frac{pq(n+p)}{n(n-p)^2}}\widetilde{V}_p(I_p^\tau(K\widetilde{+}_q L), Q^*)^{\frac{pq}{(n-p)^2}}V(Q)^{\frac{p^2q}{n(n-p)^2}} : Q \in \mathcal{S}_o^n\Big\}\\ &\le \sup\Big\{n^{\frac{pq(n+p)}{n(n-p)^2}}[\widetilde{V}_p(I_p^\tau K, Q^*)^{\frac{pq}{(n-p)^2}} + \widetilde{V}_p(I_p^\tau L, Q^*)^{\frac{pq}{(n-p)^2}}]V(Q)^{\frac{p^2q}{n(n-p)^2}} : Q \in \mathcal{S}_o^n\Big\}\\ &\le \Big[\sup\{n^{\frac{n+p}{n}}\widetilde{V}_p(I_p^\tau K, Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}\Big]^{\frac{pq}{(n-p)^2}}\\ &\quad + \Big[\sup\{n^{\frac{n+p}{n}}\widetilde{V}_p(I_p^\tau L, Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}\Big]^{\frac{pq}{(n-p)^2}}\\ &= \widetilde{\Omega}_p(I_p^\tau K)^{\frac{pq(n+p)}{n(n-p)^2}} + \widetilde{\Omega}_p(I_p^\tau L)^{\frac{pq(n+p)}{n(n-p)^2}}.\end{aligned}$$

This yields inequality (12).

According to the equality condition of (20), we see that equality holds in (12) if and only if $I_p^\tau K$ and $I_p^\tau L$ are dilates. □

Taking q for $n-q$ in Theorem 1.2, we obtain a Brunn-Minkowski type inequality for the L_p-dual affine surface areas of general L_p-intersection bodies under the L_q-radial Blaschke addition.

Corollary 3.1. *If $K, L \in \mathcal{S}_o^n$, $n \geq 2$, $0 < p < 1$, $n > q > p > 0$ and $\tau \in [-1, 1]$, then*

$$\widetilde{\Omega}_p(I_p^\tau(K \pm_q L))^{\frac{p(n-q)(n+p)}{n(n-p)^2}} \leq \widetilde{\Omega}_p(I_p^\tau K)^{\frac{p(n-q)(n+p)}{n(n-p)^2}} + \widetilde{\Omega}_p(I_p^\tau L)^{\frac{p(n-q)(n+p)}{n(n-p)^2}},$$

with equality if and only if $I_p^\tau K$ and $I_p^\tau L$ are dilates.

Proof of Theorem 1.3. For $K, L \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1, 1]$. If $K \subseteq L$, then

$$\rho(K, \cdot) \leq \rho(L, \cdot), \tag{21}$$

with equality if and only if $K = L$.

From (1), (2), (3) and (21), we have

$$\rho(I_p^\tau K, \cdot) \leq \rho(I_p^\tau L, \cdot). \tag{22}$$

By (15) and (22), we easily get for any $Q \in \mathcal{S}_o^n$,

$$\widetilde{V}_p(I_p^\tau K, Q) \leq \widetilde{V}_p(I_p^\tau L, Q). \tag{23}$$

And $\widetilde{V}_p(I_p^\tau K, Q) = \widetilde{V}_p(I_p^\tau L, Q)$ if and only if $I_p^\tau K = I_p^\tau L$.

By (9) and (23), we obtain

$$\begin{aligned}
\widetilde{\Omega}_p(I_p^\tau K)^{\frac{n+p}{n}} &= \sup\{n^{\frac{n+p}{n}} \widetilde{V}_p(I_p^\tau K, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\} \\
&\leq \sup\{n^{\frac{n+p}{n}} \widetilde{V}_p(I_p^\tau L, Q^*) V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\} \\
&= \widetilde{\Omega}_p(I_p^\tau L)^{\frac{n+p}{n}}.
\end{aligned}$$

This gives (13).

According to the equality conditions of (21) and (23), we see that equality holds in (13) when $K = L$. □

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