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Convenient Properties of Stratified L-Convergence Tower Spaces

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Abstract. In this paper, convenient properties of stratified L-convergence tower spaces are investigated. Firstly, it is shown that the construct of stratified L-convergence tower spaces and the construct of stratified L-Kent convergence tower spaces are strong topological universes. Secondly, the concepts of symmetric stratified L-Kent convergence tower spaces and complete stratified L-filter tower spaces are introduced and it is proved that the resulting constructs are isomorphic and they are strongly Cartesian closed.

1. Introduction

From a structural point of view, the construct (i.e., concrete category) SL-**Top** of stratified L-topological spaces and continuous maps cannot be discussed in a satisfactory way [21], e.g. it is not Cartesian closed. In the classical case, this deficiency can be overcome by considering suitable superconstructs of **Top**, the construct of topological spaces. This leads to some generalizations of topological spaces such as generalized convergence spaces introduced by Fischer [8]. So it is quite natural to apply a related approach in the lattice-valued case. In the case $L = [0,1]$, Lowen et al. [21] considered fuzzy convergence spaces as a generalization of Choquet convergence spaces [2]. The resulting construct is Cartesian closed. However, this theory relies essentially on Lowen's definition of convergence for stratified $[0,1]$-topological spaces [22], where prime prefilters play a crucial role. As Höhle [11] pointed out, this theory may turn out to be void in the case of more general lattices L. Later, Höhle suggested to develop a new convergence theory based on (stratified) L-filters [12] and on the definition of convergence in (stratified) L-topological spaces. Following this idea, there are at least two kinds of lattice-valued convergence spaces. In 2001, Jäger [14] introduced the concept of stratified L-generalized convergence spaces and showed that the resulting construct is a Cartesian closed topological construct, which contains the construct of stratified L-topological spaces as a reflective subconstruct. Later, Flores et al. [9] proposed a new kind of lattice-valued convergence spaces, which is called stratified L-convergence tower spaces in this paper. Moreover, the resulting construct is not only Cartesian closed, but also extensional. There are so many following works on lattice-valued convergence spaces (see [1, 3, 4, 10, 15–20, 24, 26, 29, 33, 40]). From the aspect of convexity theory, a categorical approach has also been applied to the fuzzy convex structures (see [30–32, 34, 36–38]). In this framework, Pang [28] proposed the concept of fuzzy convergence structures in fuzzy convex spaces and established the categorical relationship between fuzzy convex structures and fuzzy convergence structures.

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Besides Cartesian closedness, Preuss [35] proposed several categorical properties, such as extensionality and closedness of products of quotient maps. Since the terminology of "convenient topology" is adopted in [35], all these categorical properties are also called convenient properties. Actually, the construct **GConv** of generalized convergence spaces and continuous maps fulfils all these properties: (1) it is Cartesian closed; (2) it is extensional; (3) product of quotient maps is a quotient map. That is, **GConv** is a strong topological universe. In the classical case, there are some other constructs possessing these convenient properties, such as semiuniform convergence spaces and filter spaces. In the lattice-valued case, Fang introduced stratified L-semiuniform convergence spaces [5], stratified L-ordered quasiuniform limit spaces [6], lattice-valued preuniform convergence spaces [7]. Pang introduced stratified L-filter spaces [23] and stratified L-ordered filter spaces [25]. Furthermore, for L-fuzzifying convergence spaces [41], Pang [27] also studied its convenient properties. It is shown that these resulting constructs are all strong topological universes. All these motivate us to consider the convenient properties of lattice-valued convergence spaces. In this paper, we will focus on the convenient properties of stratified L-convergence tower spaces.

The paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we discuss the convenient properties of stratified L-convergence tower spaces in the sense of Flores et al. In Section 4, we study the convenient properties of stratified L-Kent convergence tower spaces. In Sections 5 and 6, we propose the notions of symmetric stratified L-Kent convergence tower spaces and complete stratified L-filter tower spaces, and investigate their convenient properties as well as their categorical relationship.

2. Preliminaries

Throughout this paper, L denotes a frame, which means that L is a complete lattice, and for all $a,\ b_i \in L\ (i \in I)$, the following distributive law is valid

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I} (a \wedge b_i).$$

The least element and the greatest element of L are denoted by $\bot$ and $\top$, respectively. Let a, b be elements in L. We say "a is wedge below b" in symbols $a \prec b$ if for every subset $D \subseteq L$, $\bigvee D \geqslant b$ implies $a \leqslant d$ for some $d \in D$. We denote $\beta(a) = \{b \mid b \prec a\}$. It is easy to see that $a \prec \bigwedge_{i\in I} b_i$ implies $a \prec b_i$ for every $i \in I$, whereas $a \prec \bigvee_{i\in I} b_i$ is equivalent to $a \prec b_i$ for some $i \in I$. Clearly, every completely distributive complete lattice is a frame, and $a = \bigvee \beta(a)$ holds for each $a \in L$, whenever L is completely distributive complete lattice.

For a nonempty set X, we can extend the lattice operations pointwise from L to L^X, the set of all L-subsets on X. The smallest element and the largest element in L^X are denoted by $\underline{\bot}$ and $\underline{\top}$, respectively. For each $a \in L$, $\underline{a}$ denotes the constant map $X \longrightarrow L$, $x \longmapsto a$.

Definition 2.1 (Höhle and Šostak [12]). *A map $\mathcal{F} : L^X \longrightarrow L$ is called a stratified L-filter on X if it satisfies*

(F1) $\mathcal{F}(\underline{\bot}) = \bot, \mathcal{F}(\underline{\top}) = \top$;

(F2) $A \leqslant B \Rightarrow \mathcal{F}(A) \leqslant \mathcal{F}(B)$;

(F3) $\mathcal{F}(A \wedge B) \geqslant \mathcal{F}(A) \wedge \mathcal{F}(B)$;

(Fs) $a \wedge \mathcal{F}(A) \leqslant \mathcal{F}(\underline{a} \wedge A)$.

The family of all stratified L-filters on X will be denoted by $\mathcal{F}_L^s(X)$. For every $x \in X$, $[x] \in \mathcal{F}_L^s(X)$ is defined by $[x](A) = A(x)$ for all $A \in L^X$.

On the set $\mathcal{F}_L^s(X)$ of all stratified L-filters on X, define $\mathcal{F} \leqslant \mathcal{G}$ by $\mathcal{F}(A) \leqslant \mathcal{G}(A)$, $\mathcal{F}_\bot^X(A) = \bigwedge_{x\in X} A(x)$ for all $A \in L^X$. Then $(\mathcal{F}_L^s(X), \leqslant)$ is a poset having the least element $\mathcal{F}_\bot^X$. For a nonempty family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of stratified L-filters, the infimum $\bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda$ is given by $(\bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda)(A) = \bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda(A)$ for all $A \in L^X$. In order to guarantee the least upper bound for a family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$, Höhle and Šostak presented the following lemma.

Lemma 2.2 (Höhle and Šostak [12]). *For a family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of stratified L-filters on X, there exists a stratified L-filter $\mathcal{F}$ such that $\mathcal{F}_\lambda \leqslant \mathcal{F}$ $(\forall \lambda \in \Lambda)$, if and only if*

$$\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) = \bot \text{ whenever } A_1 \wedge \cdots \wedge A_n = \underline{\bot},$$

for $n \in \mathbb{N}, A_1, \cdots, A_n \in L^X$, $\{\lambda_1, \cdots, \lambda_n\} \subseteq \Lambda$. *In the case of existence, the supremum* $\bigvee_{\lambda\in\Lambda} \mathcal{F}_\lambda$ *of a nonempty family* $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ *of stratified L-filters is given by*

$$\left(\bigvee_{\lambda\in\Lambda} \mathcal{F}_\lambda\right)(A) = \bigvee_{n\in\mathbb{N}} \bigvee\{\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) \mid A_1 \wedge \cdots \wedge A_n \leqslant A\}$$

for all $A \in L^X$.

Let $\varphi : X \longrightarrow Y$ be a map and $\mathcal{F}$ be a stratified L-filter on X. Define $\varphi^{\rightarrow} : L^X \longrightarrow L^Y$ and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ by $\varphi^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(B) = B \circ \varphi$ for $B \in L^Y$, respectively. Then the map $\varphi^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ defined by $\varphi^{\Rightarrow}(\mathcal{F})(A) = \mathcal{F}(\varphi^{\leftarrow}(A))$ for $A \in L^Y$, is a stratified L-filter on Y, which is called the image of $\mathcal{F}$ under φ (see [12]). In [14], Jäger also proved that given a map $\varphi : X \longrightarrow Y$ and a stratified L-filter $\mathcal{F}$ on Y, the map $\varphi^{\Leftarrow}(\mathcal{F}) : L^X \longrightarrow L$ defined by

$$\forall A \in L^X,\ \ \varphi^{\Leftarrow}(\mathcal{F})(A) = \bigvee_{\varphi^{\leftarrow}(B)\leqslant A} \mathcal{F}(B)$$

is a stratified L-filter on X if and only if $\mathcal{F}(B) = \bot$ whenever $\varphi^{\leftarrow}(B) = \underline{\bot}$ for all $B \in L^Y$. In case $\varphi^{\Leftarrow}(\mathcal{F}) \in \mathcal{F}_L^s(X)$, it is called the inverse image of $\mathcal{F}$ under φ.

Lemma 2.3 (Höhle [13] and Jäger [14]). *Let* $\mathcal{F}$ *be a stratified L-filter on Y and* $\varphi : X \longrightarrow Y$ *be a map. Then the following statements are equivalent:*

(1) $\varphi^{\Leftarrow}(\mathcal{F})$ *is a stratified L-filter.*

(2) $\mathcal{F}(B) = \bot$ *whenever* $\varphi^{\leftarrow}(B) = \underline{\bot}$ *for all* $B \in L^Y$.

(3) $\mathcal{F}(\top_{Y-\varphi(X)}) = \bot$, *where* $\top_{Y-\varphi(X)}(y) = \top$ *whenever* $y \in Y - \varphi(X)$, *and* $\bot$, *otherwise.*

In [14], Jäger proposed that the product $\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ of a family of stratified L-filters $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$, where for each $\lambda \in \Lambda$, X_λ is a nonempty set and $\mathcal{F}_\lambda \in \mathcal{F}_L^s(X_\lambda)$, is defined as follows:

$$\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda := \bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda) \in \mathcal{F}_L^s\left(\prod_{\lambda\in\Lambda} X_\lambda\right),$$

where for each $\lambda \in \Lambda$, $p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ is the projection map. The existence of $\bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda)$ was clarified in [3].

Lemma 2.4 (Jäger [14]). *Let* $\{X_\lambda\}_{\lambda\in\Lambda}$ *be a family of nonempty sets,* $p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ *the projection map,* $\mathcal{F}_\lambda \in \mathcal{F}_L^s(X_\lambda)$ $(\forall\lambda \in \Lambda)$ *and* $\mathcal{F} \in \mathcal{F}_L^s(\prod_{\lambda\in\Lambda} X_\lambda)$. *Then the following statements hold:*

(1) $\prod_{\lambda\in\Lambda} p_\lambda^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{F}$.

(2) $p_\lambda^{\Rightarrow}\left(\prod_{\mu\in\Lambda} \mathcal{F}_\mu\right) \geqslant \mathcal{F}_\lambda$, $\forall\lambda \in \Lambda$.

(3) $p_\lambda^{\Rightarrow}\left(\prod_{\mu\in\Lambda} p_\mu^{\Rightarrow}(\mathcal{F})\right) = p_\lambda^{\Rightarrow}(\mathcal{F})$, $\forall\lambda \in \Lambda$.

In the following, we list some related concepts about category theory.

Definition 2.5 (Preuss [35]). *A morphism* $f : A \longrightarrow B$ *in a construct* **C** *is called an isomorphism provided that there is a* **C***-morphism* $g : B \longrightarrow A$ *such that* $g \circ f = 1_A$ *and* $f \circ g = 1_B$.

Definition 2.6 (Preuss [35]). *A subconstruct* **A** *of* **C** *is called an isomorphism-closed subconstruct of* **C** *provided that for each* **C***-object C, if C is isomorphic to an* **A***-object A, then C is also an* **A***-object.*

Definition 2.7 (Preuss [35]). *A construct* **C** *is called Cartesian closed provided that the following conditions are satisfied:*

(1) *For each pair* (X, Y) *of* **C***-objects, there exists a product* $X \times Y$ *in* **C**.

(2) *For each pair of* **C***-objects X and Y, there exists a* **C***-object* Y^X *(called power object) and a* **C***-morphism* $ev_{[X,Y]} : Y^X \times X \longrightarrow Y$ *(called evaluation morphism) such that for each* **C***-object Z and each* **C***-morphism* $\varphi : Z \times X \longrightarrow Y$, *there exists a unique* **C***-morphism* $\varphi_* : Z \longrightarrow Y^X$ *such that* $ev_{[X,Y]} \circ (\varphi_* \times id_X) = \varphi$.

Recall in a topological construct **C**, a partial morphism from X to Y is a **C**-morphism $\varphi : Z \longrightarrow Y$ whose domain is a subobject of X. A topological construct **C** is called extensional provided that every **C**-object Y has a one-point extension Y^*, in the sense that every **C**-object Y can be embedded via the addition of a single point ∞_Y into a **C**-object Y^* such that for every partial morphism $\varphi : Z \longrightarrow Y$ from X to Y, the map φ^*: $X \to Y^*$ defined by

$$\varphi^*(x) = \begin{cases} \varphi(x), & \text{if } x \in Z; \\ \infty_Y, & \text{if } x \notin Z \end{cases}$$

is a **C**-morphism.

Several categorical properties for a topological construct are proposed by Preuss in the book [35], namely
(CP1) **C** is Cartesian closed;
(CP2) **C** is extensional;
(CP3) In **C** product of quotient maps is a quotient map.

According to the terminology of [35], a topological construct **C** is called
(1) strongly Cartesian closed provided that **C** fulfills (CP1) and (CP3);
(2) a topological universe provided that **C** fulfills (CP1) and (CP2);
(3) a strong topological universe provided that **C** fulfills (CP1)–(CP3).

For other notions about category theory, we refer to [35].

3. Stratified *L*-convergence tower spaces

In this section, we will show that the construct *SL*-**CTS** of stratified L-convergence tower spaces is a strong topological universe.

Definition 3.1 (Flores et al. [9]). *A stratified L-convergence tower space is a pair (X, q), where $q = \{q_\alpha \mid \alpha \in L\}$ is a nonempty family of subsets of $\mathcal{F}_L^s(X) \times X$ which satisfies*

(LCT1) *For each $x \in X$ and $\alpha \in L$, $([x], x) \in q_\alpha$ and $(\mathcal{F}_\perp^X, x) \in q_\perp$;*

(LCT2) *If $(\mathcal{F}, x) \in q_\alpha$ and $\mathcal{F} \leqslant \mathcal{G}$, then $(\mathcal{G}, x) \in q_\alpha$;*

(LCT3) *$q_\beta \subseteq q_\alpha$ whenever $\alpha \leqslant \beta$.*

For a stratified L-convergence tower space (X, q), q is called a stratified L-convergence tower structure on X.

A stratified L-convergence tower space (X, q) is called left continuous if $\bigcap_{\alpha \in U} q_\alpha = q_{\vee U}$ for any nonempty subset $U \subseteq L$.

*A map $\varphi : (X, q) \longrightarrow (Y, q')$ between two stratified L-convergence tower spaces is called continuous provided $(\mathcal{F}, x) \in q_\alpha$ implies $(\varphi^\Rightarrow(\mathcal{F}), \varphi(x)) \in q'_\alpha$ for all $\alpha \in L$. The construct of stratified L-convergence tower spaces (resp., left continuous stratified L-convergence tower spaces) and continuous maps is denoted by SL-*__CTS__ *(resp., LC-SL-*__CTS__*).*

Remark 3.2. *In [9], the authors called the structured spaces in Definition 3.1 stratified L-convergence spaces. In order to distinguish this concept from Jäger's stratified L-generalized convergence spaces, we call it stratified L-convergence tower space.*

Definition 3.3. *For a nonempty set X, let $q(X)$ denote the fibre*

$$\{q \mid q \text{ is a stratified } L\text{-convergence tower structure on } X\}$$

of X. For all q_1, $q_2 \in q(X)$, we say that $q_1 \leqslant q_2$ if the identity map $id_X : (X, q_1) \longrightarrow (X, q_2)$ is continuous. In this case, we also say (X, q_1) is finer than (X, q_2), or (X, q_2) is coarser than (X, q_1), denoted by $(X, q_1) \leqslant (X, q_2)$.

Example 3.4. *Let X be a nonempty set.*

(1)*We define $(q_{ind})_\alpha = \mathcal{F}_L^s(X) \times X$ for all $\alpha \in L$. Then $q_{ind} = \{(q_{ind})_\alpha \mid \alpha \in L\}$ is the coarsest stratified L-convergence tower structure on X, called the indiscrete stratified L-convergence tower structure on X.*

(2) *We define the discrete stratified L-convergence tower structure $q_{dis} = \{(q_{dis})_\alpha \mid \alpha \in L\}$ on X as follows: for each $\alpha \in L$,*

if $\alpha = \perp$, *then* $(q_{dis})_\alpha = \mathcal{F}_L^s(X) \times X$;
if $\alpha \neq \perp$, *then* $(\mathcal{F}, x) \in (q_{dis})_\alpha$ *iff* $\mathcal{F} \geqslant [x]$.
Then $q_{dis} = \{(q_{dis})_\alpha \mid \alpha \in L\}$ *is the finest stratified L-convergence tower structure on X.*

Next we will discuss the convenient properties of stratified L-convergence tower spaces. In [9], Flores et al. had proved that the construct *SL*-**CTS** is topological, Cartesian closed and extensional. However, many proofs are omitted. In order to keep self-contained, we recall some conclusions and give the concrete form of some special objects in *SL*-**CTS**.

Theorem 3.5 (Flores et al. [9]). *SL*-**CTS** *is topological over* **Set**.

Proof. We only give the concrete form of the initial structures. For a family of stratified L-convergence tower spaces $\{(X_\lambda, q_\lambda)\}_{\lambda\in\Lambda}$ and a family of maps $\{\varphi_\lambda : X \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$, define $q = \{q_\alpha\}_{\alpha\in L}$ as follows: for each $\alpha \in L$, $q_\alpha \subseteq \mathcal{F}_L^s(X) \times X$ and

$$(\mathcal{F}, x) \in q_\alpha \iff \forall \lambda \in \Lambda,\ (\varphi_\lambda^{\Rightarrow}(\mathcal{F}), \varphi_\lambda(x)) \in (q_\lambda)_\alpha.$$

Then q is the initial structure w.r.t. the source $\{\varphi_\lambda : X \longrightarrow (X_\lambda, q_\lambda)\}_{\lambda\in\Lambda}$. □

Definition 3.6. *Let* $\{(X_\lambda, q_\lambda)\}_{\lambda\in\Lambda}$ *be a family of stratified L-convergence tower spaces and* $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ *be the source formed by the family of the projection maps* $\{p_\lambda\}_{\lambda\in\Lambda}$. *The initial structure w.r.t.* $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ *is called the stratified product L-convergence tower structure, denoted by* $\prod_{\lambda\in\Lambda} q_\lambda$. *The pair* $(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} q_\lambda)$ *is called the product space of* $\{(X_\lambda, q_\lambda)\}_{\lambda\in\Lambda}$. *For the product space of two stratified L-convergence tower spaces* (X, q_X) *and* (Y, q_Y), *we write* $(X \times Y, q_X \times q_Y)$ *for* $(X, q_X) \times (Y, q_Y)$.

Since the construct *SL*-**CTS** is topological over **Set**, there exists a unique final structure w.r.t. a sink $\{\varphi_\lambda : (X_\lambda, q_\lambda) \longrightarrow X\}_{\lambda\in\Lambda}$ in *SL*-**CTS**. Now we explore the concrete form of the final structures.

Proposition 3.7. *Let X be a nonempty set,* $\{(X_\lambda, q_\lambda)\}_{\lambda\in\Lambda}$ *be a family of stratified L-convergence tower spaces and* $\{\varphi_\lambda : X_\lambda \longrightarrow X\}_{\lambda\in\Lambda}$ *be a family of maps. Then define* $q = \{q_\alpha \mid \alpha \in L\}$ *as follows. For each* $\alpha \in L$, $q_\alpha \subseteq \mathcal{F}_L^s(X) \times X$ *and*
(1) $q_\perp = \mathcal{F}_L^s(X) \times X$;
(2) *If* $\alpha \neq \perp$, $(\mathcal{F}, x) \in q_\alpha \iff \mathcal{F} \geqslant [x]$ *or* $\exists \lambda \in \Lambda$ *and* $(\mathcal{G}_\lambda, x_\lambda) \in (q_\lambda)_\alpha$ *such that* $\varphi_\lambda^{\Rightarrow}(\mathcal{G}_\lambda) \leqslant \mathcal{F}$ *and* $\varphi_\lambda(x_\lambda) = x$.
Further, if $X = \bigcup_{\lambda\in\Lambda} \varphi_\lambda(X_\lambda)$, *then it follows that* $q = \{q_\alpha \mid \alpha \in L\}$ *has the following form:*
(1) $q_\perp = \mathcal{F}_L^s(X) \times X$;
(2) *If* $\alpha \neq \perp$, $(\mathcal{F}, x) \in q_\alpha \iff \exists \lambda \in \Lambda$ *and* $(\mathcal{G}_\lambda, x_\lambda) \in (q_\lambda)_\alpha$ *such that* $\varphi_\lambda^{\Rightarrow}(\mathcal{G}_\lambda) \leqslant \mathcal{F}$ *and* $\varphi_\lambda(x_\lambda) = x$.

Proof. Firstly, it is easy to check that (X, q) is a stratified L-convergence tower space.

Secondly, let (Y, q') be a stratified L-convergence tower space and $\varphi : X \longrightarrow Y$ be a map. It suffices to prove that the continuity of $\varphi \circ \varphi_\lambda$ for all $\lambda \in \Lambda$ implies the continuity of φ. For each $\alpha \in L$, take any $(\mathcal{F}, x) \in q_\alpha$, we need only show that $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\alpha$.

If $\alpha = \perp$, then it follows from $(\mathcal{F}_\perp^Y, \varphi(x)) \in q'_\perp$ and $\varphi^{\Rightarrow}(\mathcal{F}) \geqslant \mathcal{F}_\perp^Y$ that $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\perp$.

If $\alpha \neq \perp$, then we divide into two cases:

Case 1: $\mathcal{F} \geqslant [x]$, then $\varphi^{\Rightarrow}(\mathcal{F}) \geqslant \varphi^{\Rightarrow}([x]) = [\varphi(x)]$. Since $([\varphi(x)], \varphi(x)) \in q'_\alpha$, by (LCT2), we have $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\alpha$.

Case 2: $\mathcal{F} \ngeqslant [x]$, then there exist $\lambda \in \Lambda$ and $(\mathcal{G}_\lambda, x_\lambda) \in (q_\lambda)_\alpha$ such that $\varphi_\lambda^{\Rightarrow}(\mathcal{G}_\lambda) \leqslant \mathcal{F}$ and $\varphi_\lambda(x_\lambda) = x$. Since $\varphi \circ \varphi_\lambda$ is continuous, we have $(\varphi^{\Rightarrow}(\varphi_\lambda^{\Rightarrow}(\mathcal{G}_\lambda)), \varphi(\varphi_\lambda(x_\lambda))) \in q'_\alpha$. This implies $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\alpha$.

As a consequence, $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\alpha$ for all $(\mathcal{F}, x) \in q_\alpha$. This proves the continuity of φ.

Further, if $X = \bigcup_{\lambda\in\Lambda} \varphi_\lambda(X_\lambda)$, then for each $x \in X$ with $\mathcal{F} \geqslant [x]$, there exist $\lambda_0 \in \Lambda$ and $x_{\lambda_0} \in X_{\lambda_0}$ such that $x = \varphi_{\lambda_0}(x_{\lambda_0})$. Let $\mathcal{G}_{\lambda_0} = [x_{\lambda_0}]$. Then $(\mathcal{G}_{\lambda_0}, x_{\lambda_0}) \in (q_{\lambda_0})_\alpha$ and $\varphi_{\lambda_0}^{\Rightarrow}(\mathcal{G}_{\lambda_0}) = \varphi_{\lambda_0}^{\Rightarrow}([x_{\lambda_0}]) = [\varphi_{\lambda_0}(x_{\lambda_0})] = [x] \leqslant \mathcal{F}$, as desired. □

Definition 3.8. *Let* (X, q) *be a stratified L-convergence tower space and* $\varphi : X \longrightarrow Y$ *be a surjective map. Then the final structure* q' *w.r.t. the sink* $\{\varphi : (X, q) \longrightarrow Y\}$ *is called the quotient structure of* (X, q) *and the map* φ *is called the quotient map. The pair* (Y, q') *is called the quotient space.*

Proposition 3.9. *If (Y, q') is a quotient space of (X, q) w.r.t. the quotient map $\varphi : X \longrightarrow Y$. Then for each $\alpha \in L$, it follows that*
$(\mathcal{G}, y) \in q'_\alpha \Longleftrightarrow \exists(\mathcal{F}, x) \in q_\alpha$ *such that* $\varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}$ *and* $\varphi(x) = y$.

Proof. By Proposition 3.7 and Definition 3.8, it suffices to prove that for each $(\mathcal{G}, y) \in q'_\perp = \mathcal{F}_L^s(Y) \times Y$, there exist $\lambda \in \Lambda$ and $(\mathcal{F}, x) \in (q_\lambda)_\perp$ such that $\varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}$ and $\varphi(x) = y$.

Since $\varphi : X \longrightarrow Y$ is surjective, there exists $x \in X$ such that $\varphi(x) = y$. By (LCT1), we know $(\mathcal{F}_\perp^X, x) \in q_\perp$. Further, for each $A \in L^Y$, it follows that

$$\varphi^{\Rightarrow}(\mathcal{F}_\perp^X)(A) = \mathcal{F}_\perp^X(\varphi^{\leftarrow}(A)) = \bigwedge_{x \in X} A(\varphi(x)) = \bigwedge_{y \in Y} A(y) = \mathcal{F}_\perp^Y(A),$$

where the third equality holds since φ is surjective. This implies that $\varphi^{\Rightarrow}(\mathcal{F}_\perp^X) = \mathcal{F}_\perp^Y$. Since $\mathcal{F}_\perp^Y$ is the least stratified L-filter on Y, we have $\varphi^{\Rightarrow}(\mathcal{F}_\perp^X) = \mathcal{F}_\perp^Y \leqslant \mathcal{G}$. As a consequence, for each $(\mathcal{G}, y) \in q'_\perp$, there exists $(\mathcal{F}_\perp^X, x) \in q_\perp$ such that $\varphi^{\Rightarrow}(\mathcal{F}_\perp^X) \leqslant \mathcal{G}$ and $\varphi(x) = y$, as desired. $\square$

Proposition 3.10 (Flores et al [9]). *SL-**CTS** is Cartesian closed.*

Proposition 3.11 (Flores et al [9]). *SL-**CTS** is extensional.*

The proofs of Propositions 3.10 and 3.11 can be found in [9]. We only give the concrete form of one-point extensions in SL-**CTS**. Given a stratified L-convergence tower space (X, q), denote $X^* = X \cup \{\infty\}$ and let $i_X : X \longrightarrow X^*$ be the inclusion map, where $\infty \notin X$. Define $q^* = \{q_\alpha^* \mid \alpha \in L\}$ as follows: $(\mathcal{F}, x) \in q_\alpha^*$ iff it satisfies either of the following conditions:
(1) $x = \infty$;
(2) $x \neq \infty$, $\mathcal{F}_X$ fails to exist;
(3) $x \neq \infty$, $(\mathcal{F}_X, x) \in q_\alpha$ if $\mathcal{F}_X$ exists,
where $\mathcal{F}_X = i_X^{\Leftarrow}(\mathcal{F})$ provided that the latter is a stratified L-filter.

In order to show the productivity of quotient maps in SL-**CTS**, the following lemma is necessary.

Lemma 3.12 (Pang [23]). *Let $\{\varphi_\lambda : X_\lambda \longrightarrow Y_\lambda\}$ be a family of surjective maps and $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ be a family of stratified L-filters with $\mathcal{F}_\lambda \in \mathcal{F}_L^s(X_\lambda)$. Then*

$$\left(\prod_{\lambda \in \Lambda} \varphi_\lambda\right)^{\Rightarrow} \left(\prod_{\lambda \in \Lambda} \mathcal{F}_\lambda\right) = \prod_{\lambda \in \Lambda} \varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda).$$

Proposition 3.13. *If $\{\varphi_\lambda : (X_\lambda, q_\lambda) \longrightarrow (Y_\lambda, q'_\lambda)\}$ is a family of quotient maps in SL-**CTS**, then the product map*

$$\prod_{\lambda \in \Lambda} \varphi_\lambda : \left(\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} q_\lambda\right) \longrightarrow \left(\prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} q'_\lambda\right)$$

*is a quotient map in SL-**CTS**.*

Proof. Define $\varphi := \prod_{\lambda \in \Lambda} \varphi_\lambda$, $(X, q) := (\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} q_\lambda)$ and $(Y, q') := \left(\prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} q'_\lambda\right)$.

Let

$$\begin{array}{ccc} (X, q) & \xrightarrow{\varphi} & (Y, q') \\ {\scriptstyle p_\lambda}\big\downarrow & & {\scriptstyle p'_\lambda}\big\downarrow \\ (X_\lambda, q_\lambda) & \xrightarrow{\varphi_\lambda} & (Y_\lambda, q'_\lambda) \end{array}$$

be the product commutation diagram in **Set**.

In order to show that φ is a quotient map, it suffices to prove:

(1) φ is surjective.
(2) (Y, q') is the the quotient space of (X, q) w.r.t. φ.

For (1), since $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ are all surjective, it follows that φ is surjective.

For (2), since (Y, q') is the product of $\{(Y_\lambda, q'_\lambda)\}_{\lambda\in\Lambda}$, by Definition 3.6, we have $q' = \{q'_\alpha \mid \alpha \in L\}$, where $q'_\alpha \subseteq \mathcal{F}^s_L(Y) \times Y$ satisfies

$$(\mathcal{G}, y) \in q'_\alpha \Longleftrightarrow \forall \lambda \in \Lambda,\ (p'^{\Rightarrow}_\lambda(\mathcal{G}), p'_\lambda(y)) \in (q'_\lambda)_\alpha.$$

Suppose that (Y, q'') is the quotient space of (X, q). By Proposition 3.9, we have $q'' = \{q''_\alpha \mid \alpha \in L\}$, where $q''_\alpha \subseteq \mathcal{F}^s_L(Y) \times Y$ satisfies that for each $(\mathcal{G}, y) \in q''_\alpha$, there exists $(\mathcal{F}, x) \in q_\alpha$ such that $\varphi(x) = y$ and $\varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}$. It suffices to prove that $q' = q''$, i.e., $q'_\alpha = q''_\alpha$ for all $a \in L$.

On one hand, take any $(\mathcal{G}, y) \in q''_\alpha$. Then there exists $(\mathcal{F}, x) \in q_\alpha$ such that $\varphi(x) = y$ and $\varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}$. Since (X, q) is the product of $\{(X_\lambda, q_\lambda)\}_{\lambda\in\Lambda}$, we know $(p^{\Rightarrow}_\lambda(\mathcal{F}), p_\lambda(x)) \in (q_\lambda)_\alpha$ for all $\lambda \in \Lambda$. By the continuity of φ_λ, it follows that $(\varphi^{\Rightarrow}_\lambda(p^{\Rightarrow}_\lambda(\mathcal{F})), \varphi_\lambda(p_\lambda(x))) \in (q'_\lambda)_\alpha$. Thus we have $(p'^{\Rightarrow}_\lambda(\varphi^{\Rightarrow}(\mathcal{F})), p'_\lambda(\varphi(x))) \in (q'_\lambda)_\alpha$, as the diagram is commutative. This implies that $(p'^{\Rightarrow}_\lambda(\mathcal{G}), p'_\lambda(y)) \in (q'_\lambda)_\alpha$ for all $\lambda \in \Lambda$. Thus $(\mathcal{G}, y) \in q'_\alpha$. By the arbitrariness of $(\mathcal{G}, y)$, we have $q''_\alpha \subseteq q'_\alpha$.

On the other hand, take any $(\mathcal{G}, y) \in q'_\alpha$. Then $(p'^{\Rightarrow}_\lambda(\mathcal{G}), p'_\lambda(y)) \in (q'_\lambda)_\alpha$ for all $\lambda \in \Lambda$. Since (Y_λ, q'_λ) is the quotient space of (X_λ, q_λ), there exists $(\mathcal{F}_\lambda, x_\lambda) \in (q_\lambda)_\alpha$ such that $\varphi_\lambda(x_\lambda) = p'_\lambda(y)$ and $\varphi^{\Rightarrow}_\lambda(\mathcal{F}_\lambda) \leqslant p'^{\Rightarrow}_\lambda(\mathcal{G})$. Let $\mathcal{F} = \prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ and $x = (x_\lambda)_{\lambda\in\Lambda}$. By Lemma 2.4, we know $p^{\Rightarrow}_\lambda(\mathcal{F}) \geqslant \mathcal{F}_\lambda$. Then it follows from $(\mathcal{F}_\lambda, x_\lambda) \in (q_\lambda)_\alpha$ that $(p^{\Rightarrow}_\lambda(\mathcal{F}), p_\lambda(x)) \in (q_\lambda)_\alpha$ for all $\lambda \in \Lambda$. This implies that $(\mathcal{F}, x) \in q_\alpha$. Then by Lemmas 2.4 and 3.12, we have

$$\varphi^{\Rightarrow}(\mathcal{F}) = \left(\prod_{\lambda\in\Lambda} \varphi_\lambda\right)^{\Rightarrow} \left(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\right) = \prod_{\lambda\in\Lambda} \varphi^{\Rightarrow}_\lambda(\mathcal{F}_\lambda) \leqslant \prod_{\lambda\in\Lambda} p'^{\Rightarrow}_\lambda(\mathcal{G}) \leqslant \mathcal{G}$$

and $\varphi(x) = \prod_{\lambda\in\Lambda} \varphi_\lambda((x_\lambda)_{\lambda\in\Lambda}) = \prod_{\lambda\in\Lambda} p'_\lambda(y) = y$. Thus, there exists $(\mathcal{F}, x) \in q_\alpha$ such that $\varphi(x) = y$ and $\varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}$. This means $(\mathcal{G}, y) \in q''_\alpha$. By the arbitrariness of $(\mathcal{G}, y)$, we have $q'_\alpha \subseteq q''_\alpha$, as desired. □

Proposition 3.14. *SL*-**CTS** *satisfies* (CP3)*, i.e., in SL*-**CTS** *product of quotient maps is a quotient map.*

By Propositions 3.10, 3.11 and 3.14, we obtain the main result in this section.

Theorem 3.15. *SL*-**CTS** *is a strong topological universe.*

4. Stratified *L*-Kent convergence tower spaces

In this section, we will show that the construct of stratified *L*-Kent convergence tower spaces is a strong topological universe. Firstly, we list some definitions and lemmas in preparations for the proofs.

Definition 4.1 (Flores et al. [9]). *A stratified L-convergence tower space is called a stratified L-Kent convergence tower space if it satisfies: for each $\alpha \in L$,*

(LKCT) *If $(\mathcal{F}, x) \in q_\alpha$, then $(\mathcal{F} \wedge [x], x) \in q_\alpha$.*

The full subconstruct of SL-**CTS***, consisting of stratified L-Kent convergence tower spaces, is denoted by SL*-**KCTS***.*

Lemma 4.2 (Flores et al. [9]). *SL*-**KCTS** *is both bireflective and bicoreflective in SL*-**CTS***.*

Proof. In [9], the authors only claimed this conclusion and did not give the corresponding proofs. For the use in the sequel, we give the *SL*-**KCTS**-bicoreflector. Let (X, q) be a stratified *L*-convergence tower space and define $cq = \{cq_\alpha \mid \alpha \in L\}$ as follows: For each $\alpha \in L$,

$$(\mathcal{F}, x) \in cq_\alpha \Longleftrightarrow (\mathcal{F} \wedge [x], x) \in q_\alpha.$$

Then $id_X : (X, cq) \longrightarrow (X, q)$ is the *SL*-**KCTS**-bicoreflector. □

Lemma 4.3 (Preuss [35]). *Let* **C** *be a topological construct. Then*

(1) *If* **D** *is a bicoreflective (full and isomorphism-closed) subconstruct of* **C** *which is closed under the formation of finite products in* **C**, *then* **D** *fulfills* (CP1) *whenever* **C** *fulfills* (CP1). *The power objects in* **D** *arise from the corresponding power objects in* **C** *by applying the bicoreflector.*

(2) *If* **D** *is a bicoreflective (full and isomorphism-closed) subconstruct of* **C** *which is closed under the formation of subspaces in* **C**, *then* **D** *fulfills* (CP2) *whenever* **C** *fulfills* (CP2). *The one-point extensions in* **D** *arise from the corresponding one-point extensions in* **C** *by applying the bicoreflector.*

(3) *If* **D** *is a bicoreflective (full and isomorphism-closed) subconstruct of* **C** *which is closed under the formation of products in* **C**, *then* **D** *fulfills* (CP3) *whenever* **C** *fulfills* (CP3). *The quotient objects are formed as in* **C**.

Definition 4.4. *A continuous map $\varphi : (X, q) \longrightarrow (Y, q')$ between stratified L-convergence tower spaces is called an isomorphism provided that $\varphi : X \longrightarrow Y$ is bijective and that its inverse map $\psi : (Y, q') \longrightarrow (X, q)$ is continuous. We say that a stratified L-convergence tower space (X, q) is isomorphic to a stratified L-convergence tower space (Y, q') if there exists an isomorphism between them.*

Since the morphisms in SL-**CTS** are continuous maps between stratified L-convergence tower spaces, it is easy to see that isomorphisms in Definition 4.4 are precisely isomorphisms in SL-**CTS**.

Lemma 4.5. *The construct SL-***KCTS** *is an isomorphism-closed subconstruct of SL-***CTS**.

Proof. Let $\varphi : (X, q) \longrightarrow (Y, q')$ be an isomorphism in SL-**CTS** and (X, q) be a stratified L-Kent convergence tower space. It suffices to show that (Y, q') satisfies (LKCT). Now let ψ denote the inverse map of φ. By the continuity of φ and ψ, we have

$$\begin{aligned}
(\mathcal{G}, y) \in q'_\alpha \quad &\Longrightarrow \quad (\psi^{\Rightarrow}(\mathcal{G}), \psi(y)) \in q_\alpha \\
&\Longleftrightarrow \quad (\psi^{\Rightarrow}(\mathcal{G}) \wedge [\psi(y)], \psi(y)) \in q_\alpha \\
&\Longleftrightarrow \quad (\psi^{\Rightarrow}(\mathcal{G} \wedge [y]), \psi(y)) \in q_\alpha \\
&\Longrightarrow \quad (\varphi^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{G} \wedge [y])), \varphi(\psi(y))) \in q'_\alpha \\
&\Longleftrightarrow \quad (\mathcal{G} \wedge [y], y) \in q'_\alpha.
\end{aligned}$$

This means $(\mathcal{G}, y) \in q'_\alpha$ implies $(\mathcal{G} \wedge [y], y) \in q'_\alpha$, as desired. □

Proposition 4.6. *SL-***KCTS** *is Cartesian closed.*

Proof. Since SL-**KCTS** is bireflective in SL-**CTS**, the initial structures in SL-**KCTS** are formed as in SL-**CTS**. Thus SL-**KCTS** is closed under the formation of finite products in SL-**CTS**. By Proposition 3.10 and Lemma 4.3 (1), we obtain that the construct SL-**KCTS** satisfies (CP1), i.e., SL-**KCTS** is Cartesian closed. Next we give the concrete form of power objects in SL-**KCTS**. Let (X, q) and (Y, q') be stratified L-Kent convergence tower spaces and let $C[X, Y]$ be the set of all continuous maps from (X, q) to (Y, q'), and denote the evaluation map $ev : C[X, Y] \times X \longrightarrow Y$ by $ev(\varphi, x) = \varphi(x)$ for each $(\varphi, x) \in C[X, Y] \times X$. Let $c = \{c_\alpha \mid \alpha \in L\}$ denote the power structure in SL-**CTS** w.r.t. (X, q) and (Y, q') (see [9]). Then for each $\alpha \in L$ and $(\Phi, \varphi) \in \mathcal{F}^s_L(C[X, Y]) \times C[X, Y]$, we have

$$(\Phi, \varphi) \in c_\alpha \Longleftrightarrow (\mathcal{F}, x) \in q_\beta \textit{ implies } (ev^{\Rightarrow}(\Phi \times \mathcal{F}), \varphi(x)) \in q'_\beta \textit{ for all } \beta \leqslant \alpha.$$

By Lemma 4.3 (1), the power object in SL-**KCTS** arise from $(C[X, Y], c)$ by applying the SL-**KCTS**-bicoreflector. Hence, let $(C[X, Y], Kc)$ denote the power object w.r.t. (X, q) and (Y, q') in SL-**KCTS**. Then by Lemma 4.2, for each $\alpha \in L$, we have

$$(\Phi, \varphi) \in Kc_\alpha \Longleftrightarrow (\mathcal{F}, x) \in q_\beta \textit{ implies } (ev^{\Rightarrow}((\Phi \wedge [\varphi]) \times \mathcal{F}), \varphi(x)) \in q'_\beta \textit{ for all } \beta \leqslant \alpha,$$

which provides the concrete form of Kc. □

Proposition 4.7. *SL-***KCTS** *is extensional.*

Proof. Since SL-**KCTS** is bireflective in SL-**CTS**, it follows that SL-**KCTS** is closed under the formation of subspaces in SL-**CTS**. By Proposition 3.11 and Lemma 4.3 (2), we know that the construct SL-**KCTS** satisfies (CP2). Next we give the concrete form of one-point extensions in SL-**KCTS**. Given a stratified L-Kent convergence tower space (X, q), denote the one-point extension of (X, q) in SL-**CTS** and SL-**KCTS** by (X^*, q^*) and (X^*, Kq^*), respectively. By Lemma 4.3 (2), (X^*, Kq^*) arises from (X^*, q^*) by applying the SL-**KCTS**-bicoreflector. This means that for each $\alpha \in L$,

$$(\mathcal{F}, x) \in Kq^*_\alpha \Longleftrightarrow (\mathcal{F} \wedge [x], x) \in q^*_\alpha.$$

By the definition of q^*_α, for each $(\mathcal{F}, x) \in Kq^*_\alpha$, there are the following cases:

(1) $x = \infty$;

(2) $x \neq \infty$, $(\mathcal{F} \wedge [x])_X = i_X^{\Leftarrow}(\mathcal{F} \wedge [x])$ fails to exist;

(3) $x \neq \infty$, $(\mathcal{F} \wedge [x])_X = i_X^{\Leftarrow}(\mathcal{F} \wedge [x])$ exists and $((\mathcal{F} \wedge [x])_X, x) \in q_\alpha$.

Further, for each $A \in L^X$ with $i_X^{\leftarrow}(A) = \underline{\perp}$, $(\mathcal{F} \wedge [x])(A) = \mathcal{F}(A) \wedge A(x) = \perp$ whenever $x \neq \infty$. By Lemma 2.3, $(\mathcal{F} \wedge [x])_X$ exists whenever $x \neq \infty$. Therefore, we have

$$(\mathcal{F}, x) \in Kq^*_\alpha \Longleftrightarrow x = \infty \; or \; ((\mathcal{F} \wedge [x])_X, x) \in q_\alpha \; for \; x \neq \infty,$$

which provides the concrete form of Kq^*. □

Proposition 4.8. *SL-**KCTS** satisfies (CP3), i.e., in SL-**KCTS**, the product of quotient maps is a quotient map.*

Proof. Since SL-**KCTS** is both bireflective and bicoreflective in SL-**CTS**, it follows that SL-**KCTS** is closed under the formation of products and quotient objects. By Proposition 3.14 and Lemma 4.3 (3), we obtain that SL-**KCTS** satisfies (CP3). Further, the quotient spaces in SL-**KCTS** are formed as in SL-**CTS**. That is, for a stratified L-Kent convergence tower space (X, q) and a surjective map $\varphi : X \longrightarrow Y$, the quotient space (Y, q') of (X, q) in SL-**KCTS** has the following form. That is,

$$(\mathcal{G}, y) \in q'_\alpha \Longleftrightarrow \exists (\mathcal{F}, x) \in q_\alpha \; s.t. \; \varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G} \; and \; \varphi(x) = y,$$

for each $\alpha \in L$. □

By Propositions 4.6–4.8, we obtain

Theorem 4.9. *SL-**KCTS** is a strong topological universe.*

5. Symmetric stratified L-Kent convergence tower spaces

In this section, we will introduce the notion of symmetric stratified L-Kent convergence tower spaces and discuss its convenient properties.

Definition 5.1. *A stratified L-Kent convergence tower space (X, q) is called symmetric provided that for each $\alpha \in L$,*

(SLKCT) *If $(\mathcal{F}, x) \in q_\alpha$ and $\mathcal{F} \leqslant [y]$, then $(\mathcal{F}, y) \in q_\alpha$.*

*The full construct of SL-**KCTS**, consisting of symmetric stratified L-Kent convergence tower spaces, is denoted by SSL-**KCTS**.*

In order to show the relations between symmetric stratified L-Kent convergence tower spaces and stratified L-Kent convergence tower spaces, we first give the following lemma.

Lemma 5.2. *Let (X, q) be a stratified L-Kent convergence tower space and define $sq = \{sq_\alpha \mid \alpha \in L\}$ as follows: for each $\alpha \in L$,*

$$(\mathcal{F}, x) \in sq_\alpha \Longleftrightarrow \exists y \in X, \; s.t. \; (\mathcal{F} \wedge [x], y) \in q_\alpha.$$

Then (X, sq) is a symmetric stratified L-Kent convergence tower space.

Proof. (LCT1) For each $\alpha \in L$, $([x], x) \in sq_\alpha$ is obvious. Since $(\mathcal{F}_\perp^X, x) \in q_\perp$, it follows that $(\mathcal{F}_\perp^X \wedge [x], x) \in q_\perp$. This implies $(\mathcal{F}_\perp^X, x) \in sq_\perp$.

(LCT2) and (LKCT) are straightforward.

(LCT3) Take any $\alpha \leqslant \beta$ with $(\mathcal{F}, x) \in sq_\beta$. Then there exists $y \in X$ such that $(\mathcal{F} \wedge [x], y) \in q_\beta \subseteq q_\alpha$. This implies $(\mathcal{F}, x) \in sq_\alpha$.

(SLKCT) Take $(\mathcal{F}, x) \in sq_\alpha$ and $\mathcal{F} \leqslant [y]$. Then there exists $z \in X$ such that $(\mathcal{F} \wedge [x], z) \in q_\alpha$. Since $\mathcal{F} \wedge [y] = \mathcal{F} \geqslant \mathcal{F} \wedge [x]$, we have $(\mathcal{F} \wedge [y], z) \in q_\alpha$. This implies $(\mathcal{F}, y) \in sq_\alpha$. □

Proposition 5.3. *SSL-***KCTS** *is bireflective in SL-***KCTS***.*

Proof. Let (X, q) be a stratified L-Kent convergence tower space and define $sq = \{sq_\alpha \mid \alpha \in L\}$ as follows: for each $\alpha \in L$,

$$(\mathcal{F}, x) \in sq_\alpha \Longleftrightarrow \exists y \in X,\ s.t.\ (\mathcal{F} \wedge [x], y) \in q_\alpha.$$

By Lemma 5.2, we know (X, sq) is symmetric stratified L-Kent convergence tower space. Further, we claim that $id_X : (X, q) \longrightarrow (X, sq)$ is the *SSL*-**KCTS**-bireflector.

For this it suffices to prove:

(1) $id_X : (X, q) \longrightarrow (X, sq)$ is continuous.
(2) For each symmetric stratified L-Kent convergence tower space (Y, q') and each map $\varphi : X \longrightarrow Y$, the continuity of $\varphi : (X, q) \longrightarrow (Y, q')$ implies the continuity of $\varphi : (X, sq) \longrightarrow (Y, q')$.

For (1), take any $(\mathcal{F}, x) \in q_\alpha$. By (LKCT), we have $(\mathcal{F} \wedge [x], x) \in q_\alpha$. This implies $(\mathcal{F}, x) \in sq_\alpha$. Hence the continuity of $id_X : (X, q) \longrightarrow (X, sq)$ is proved.

For (2), take any $(\mathcal{F}, x) \in sq_\alpha$. Then there exists $y \in X$ such that $(\mathcal{F} \wedge [x], y) \in q_\alpha$. By the continuity of $\varphi : (X, q) \longrightarrow (Y, q')$, it follows that $(\varphi^\Rightarrow(\mathcal{F}) \wedge [\varphi(x)], \varphi(y)) \in q'_\alpha$. Since (Y, q') is symmetric and $\varphi^\Rightarrow(\mathcal{F}) \wedge [\varphi(x)] \leqslant [\varphi(x)]$, we have $(\varphi^\Rightarrow(\mathcal{F}) \wedge [\varphi(x)], \varphi(x)) \in q'_\alpha$. This implies $(\varphi^\Rightarrow(\mathcal{F}), \varphi(x)) \in q'_\alpha$. Therefore, $\varphi : (X, sq) \longrightarrow (Y, q')$ is continuous. □

Corollary 5.4. *SSL-***KCTS** *is closed under the formation of products in SL-***KCTS***.*

Next we will show the construct *SSL*-**KCTS** satisfies (CP1) and (CP3). For this we first give the following lemmas.

Lemma 5.5 (Preuss [35]). *Let* **C** *be a topological construct. Then*

(1) *If* **D** *is a bireflective (full and isomorphism-closed) subconstruct of* **C** *which is closed under the formation of power objects in* **C***, then* **D** *fulfills* (CP1) *whenever* **C** *fulfills* (CP1).

(2) *If* **D** *is a bireflective (full and isomorphism-closed) subconstruct of* **C***, then the quotient objects in* **D** *arise from the quotient objects in* **C** *by applying the bireflector.*

Lemma 5.6. *SSL-***KCTS** *is an isomorphism-closed subconstruct of SL-***KCTS***.*

Proof. Let $\varphi : (X, q) \longrightarrow (Y, q')$ be an isomorphism in *SL*-**KCTS** and (X, q) be a symmetric stratified L-Kent convergence tower space. It suffices to show that (Y, q') satisfies (SLKCT). Now let ψ denote the inverse map of φ. By the continuity of φ and ψ, we have

$$\begin{aligned}
(\mathcal{G}, y) \in q'_\alpha \text{ and } \mathcal{G} \leqslant [z] &\Longrightarrow (\psi^\Rightarrow(\mathcal{G}), \psi(y)) \in q_\alpha \text{ and } \psi^\Rightarrow(\mathcal{G}) \leqslant [\psi(z)] \\
&\Longrightarrow (\psi^\Rightarrow(\mathcal{G}), \psi(z)) \in q_\alpha \\
&\Longrightarrow (\varphi^\Rightarrow(\psi^\Rightarrow(\mathcal{G})), \varphi(\psi(z))) \in q'_\alpha \\
&\Longrightarrow (\mathcal{G}, z) \in q'_\alpha.
\end{aligned}$$

This shows that if $(\mathcal{G}, y) \in q'_\alpha$ *and* $\mathcal{G} \leqslant [z]$, then $(\mathcal{G}, z) \in q'_\alpha$, as desired. □

Lemma 5.7 (Jäger [14]). *Let* $\psi : X \longrightarrow Y$ *be a map and* $x \in X$*. Then* $ev^\Rightarrow([\psi] \times [x]) = [\psi(x)]$*.*

Proposition 5.8. *SSL-**KCTS** is Cartesian closed.*

Proof. By Proposition 4.6 and Lemma 5.5 (1), it suffices to show that SSL-**KCTS** is closed under the formation of power objects in SL-**KCTS**. Let (X, q) and (Y, q') be two symmetric stratified L-Kent convergence spaces. By Proposition 4.6, the corresponding power object $(C[X, Y], Kc)$ in SL-**KCTS** is defined as follows: for each $\alpha \in L$,

$$(\Phi, \varphi) \in Kc_\alpha \Longleftrightarrow (\mathcal{F}, x) \in q_\beta \textit{ implies } (ev^{\Rightarrow}((\Phi \wedge [\varphi]) \times \mathcal{F}), \varphi(x)) \in q'_\beta \textit{ for all } \beta \leqslant \alpha.$$

We need only show that $(C[X, Y], Kc)$ is symmetric.

Suppose that $(\Phi, \varphi) \in Kc_\alpha$ and $\psi \in C[X, Y]$ such that $\Phi \leqslant [\psi]$. Take any $\beta \leqslant \alpha$ with $(\mathcal{F}, x) \in q_\beta$. Then by (LKCT), it follows that $(\mathcal{F} \wedge [x], x) \in q_\beta$. Since $(\Phi, \varphi) \in Kc_\alpha$, we have $(ev^{\Rightarrow}((\Phi \wedge [\varphi]) \times (\mathcal{F} \wedge [x])), \varphi(x)) \in q'_\beta$. By Lemma 5.7 and $\Phi \leqslant [\psi]$, it follows that

$$ev^{\Rightarrow}((\Phi \wedge [\varphi]) \times (\mathcal{F} \wedge [x])) \leqslant ev^{\Rightarrow}(([\psi] \wedge [\varphi]) \times (\mathcal{F} \wedge [x])) \leqslant ev^{\Rightarrow}([\psi] \times [x]) \leqslant [\psi(x)].$$

Since (Y, q') is symmetric, we have $(ev^{\Rightarrow}((\Phi \wedge [\varphi]) \times (\mathcal{F} \wedge [x])), \psi(x)) \in q'_\beta$. This implies

$$(ev^{\Rightarrow}((\Phi \wedge [\psi]) \times \mathcal{F}), \psi(x)) = (ev^{\Rightarrow}(\Phi \times \mathcal{F}), \psi(x)) \in q'_\beta.$$

As a consequence, for each $\beta \leqslant \alpha$, $(\mathcal{F}, x) \in q_\beta$ implies $(ev^{\Rightarrow}((\Phi \wedge [\psi]) \times \mathcal{F}), \psi(x)) \in q'_\beta$. This means $(\Phi, \psi) \in Kc_\alpha$, as desired. □

Proposition 5.9. *Let (X, q) be a symmetric stratified L-Kent convergence tower space and $\varphi : X \longrightarrow Y$ be a surjective map. Define $Kq' = \{Kq'_\alpha \mid \alpha \in L\}$ as follows: for each $\alpha \in L$,*

$$(\mathcal{G}, y) \in Kq'_\alpha \Longleftrightarrow \exists z \in Y \textit{ and } (\mathcal{F}, x) \in q_\alpha \textit{ s.t. } \varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G} \wedge [y] \textit{ and } \varphi(x) = z.$$

*Then (Y, Kq') is the quotient space of (X, q) in SSL-**KCTS** w.r.t. the map φ.*

Proof. Suppose that (Y, q') and (Y, Kq') denote the quotient space of (X, q) in SL-**KCTS** and SSL-**KCTS**, respectively. Then for each $\alpha \in L$,

$$(\mathcal{G}, y) \in q'_\alpha \Longleftrightarrow \exists (\mathcal{F}, x) \in q_\alpha \textit{ s.t. } \varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G} \textit{ and } \varphi(x) = y.$$

By Proposition 5.3 and Lemma 5.5 (1), we know (Y, Kq') arises from (Y, q') by applying the SSL-**KCTS**-bireflector. Thus it follows that

$$\begin{aligned} (\mathcal{G}, y) \in Kq'_\alpha &\Longleftrightarrow \exists z \in Y \textit{ s.t. } (\mathcal{G} \wedge [y], z) \in q'_\alpha \\ &\Longleftrightarrow \exists z \in Y \textit{ and } (\mathcal{F}, x) \in q_\alpha \textit{ s.t. } \varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G} \wedge [y] \textit{ and } \varphi(x) = z, \end{aligned}$$

as desired. □

Proposition 5.10. *If $\{\varphi_\lambda : (X_\lambda, q_\lambda) \longrightarrow (Y_\lambda, q'_\lambda)\}$ is a family of quotient maps in SSL-**KCTS**, then the product map*

$$\prod_{\lambda \in \Lambda} \varphi_\lambda : \left(\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} q_\lambda \right) \longrightarrow \left(\prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} q'_\lambda \right)$$

*is a quotient map in SSL-**KCTS**.*

Proof. Define $\varphi := \prod_{\lambda \in \Lambda} \varphi_\lambda$, $(X, q) := (\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} q_\lambda)$ and $(Y, q') := \left(\prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} q'_\lambda\right)$.

Let

$$\begin{array}{ccc} (X, q) & \xrightarrow{\varphi} & (Y, q') \\ {\scriptstyle p_\lambda}\downarrow & & {\scriptstyle p'_\lambda}\downarrow \\ (X_\lambda, q_\lambda) & \xrightarrow{\varphi_\lambda} & (Y_\lambda, q'_\lambda) \end{array}$$

be the product commutation diagram in **Set**.

It suffices to prove that (Y, q') is the quotient space of (X, q) in SSL-**KCTS** w.r.t. the map φ. Since $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ are all surjective, we know φ is surjective. Suppose that (Y, Kq') is the quotient space of (X, q) in SSL-**KCTS** w.r.t. the map φ. Then by Proposition 5.9, for each $\alpha \in L$,

$$(\mathcal{G}, y) \in Kq'_\alpha \Longleftrightarrow \exists z \in Y \text{ and } (\mathcal{F}, x) \in q_\alpha \text{ s.t. } \varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G} \wedge [y] \text{ and } \varphi(x) = z.$$

Since (Y, q') is the product space of $\{(Y_\lambda, q'_\lambda)\}_{\lambda\in\Lambda}$, by Corollary 5.4, we know for each $\alpha \in L$,

$$(\mathcal{G}, y) \in q'_\alpha \Longleftrightarrow (p'^{\Rightarrow}_\lambda(\mathcal{G}), p'_\lambda(y)) \in (q'_\lambda)_\alpha \text{ for all } \lambda \in \Lambda.$$

We need only show that $Kq'_\alpha = q'_\alpha$.

On one hand, take any $(\mathcal{G}, y) \in Kq'_\alpha$. Then there exist $z \in Y$ and $(\mathcal{F}, x) \in q_\alpha$ such that $\varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G} \wedge [y]$ and $\varphi(x) = z$. It follows from $(\mathcal{F}, x) \in q_\alpha$ that $(p^{\Rightarrow}_\lambda(\mathcal{F}), p_\lambda(x)) \in (q_\lambda)_\alpha$ for all $\lambda \in \Lambda$. By the continuity of φ_λ, we have $(\varphi^{\Rightarrow}_\lambda(p^{\Rightarrow}_\lambda(\mathcal{F})), \varphi_\lambda(p_\lambda(x))) \in (q'_\lambda)_\alpha$. Then it follows that $(p'^{\Rightarrow}_\lambda(\varphi^{\Rightarrow}(\mathcal{F})), p'_\lambda(\varphi(x))) \in (q'_\lambda)_\alpha$ for all $\lambda \in \Lambda$, as the diagram is commutative. This implies $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\alpha$. Thus we have $(\mathcal{G} \wedge [y], z) \in q'_\alpha$. Since $\mathcal{G} \wedge [y] \leqslant [y]$, by (SLKCT), we obtain $(\mathcal{G} \wedge [y], y) \in q'_\alpha$. Then it follows that $(\mathcal{G}, y) \in q'_\alpha$. This proves $Kq'_\alpha \subseteq q'_\alpha$.

On the other hand, take any $(\mathcal{G}, y) \in q'_\alpha$. Then $(p'^{\Rightarrow}_\lambda(\mathcal{G}), p'_\lambda(y)) \in (q'_\lambda)_\alpha$ for all $\lambda \in \Lambda$. Since (Y_λ, q'_λ) is the quotient space of (X_λ, q_λ) w.r.t. φ_λ, there exist $z_\lambda \in Y_\lambda$ and $(\mathcal{F}_\lambda, x_\lambda) \in (q_\lambda)_\alpha$ such that $\varphi^{\Rightarrow}_\lambda(\mathcal{F}_\lambda) \leqslant p'^{\Rightarrow}_\lambda(\mathcal{G}) \wedge [p'_\lambda(y)]$ and $\varphi_\lambda(x_\lambda) = z_\lambda$. Let $\mathcal{F} = \prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$, $x = (x_\lambda)_{\lambda\in\Lambda}$ and $z = (z_\lambda)_{\lambda\in\Lambda}$. By Lemma 2.4, we know $p^{\Rightarrow}_\lambda(\mathcal{F}) \geqslant \mathcal{F}_\lambda$. Then it follows from $(\mathcal{F}_\lambda, x_\lambda) \in (q_\lambda)_\alpha$ that $(p^{\Rightarrow}_\lambda(\mathcal{F}), p_\lambda(x)) \in (q_\lambda)_\alpha$ for all $\lambda \in \Lambda$. This implies that $(\mathcal{F}, x) \in q_\alpha$. Further, by Lemmas 2.4 and 3.12, we have

$$\varphi^{\Rightarrow}(\mathcal{F}) = \left(\prod_{\lambda\in\Lambda} \varphi_\lambda\right)^{\Rightarrow} \left(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\right) = \prod_{\lambda\in\Lambda} \varphi^{\Rightarrow}_\lambda(\mathcal{F}_\lambda) \leqslant \prod_{\lambda\in\Lambda} p'^{\Rightarrow}_\lambda(\mathcal{G} \wedge [y]) \leqslant \mathcal{G} \wedge [y]$$

and $\varphi(x) = \prod_{\lambda\in\Lambda} \varphi_\lambda((x_\lambda)_{\lambda\in\Lambda}) = (z_\lambda)_{\lambda\in\Lambda} = z$. This implies $(\mathcal{G}, y) \in Kq'_\alpha$. So we have $q'_\alpha \subseteq Kq'_\alpha$, as desired. □

By Propositions 5.8 and 5.10, we obtain

Theorem 5.11. *SSL-***KCTS** *is strongly Cartesian closed.*

6. Complete stratified *L*-filter tower spaces

In this section, we will introduce the concept of complete stratified L-filter tower spaces and show that the resulting construct is isomorphic to SSL-**KCTS**.

Definition 6.1 (Yang and Li [39]). *A stratified L-filter tower space is a pair* $(X, \mathbb{F})$*, where* $\mathbb{F} = \{\mathbb{F}_\alpha \mid \alpha \in L\}$ *is a nonempty family of subsets of* $\mathcal{F}^s_L(X)$ *which satisfies*

(LFT1) *For each* $x \in X$ *and* $\alpha \in L$, $[x] \in \mathbb{F}_\alpha$*;*

(LFT2) *If* $\mathcal{F} \in \mathbb{F}_\alpha$ *and* $\mathcal{F} \leqslant \mathcal{G}$*, then* $\mathcal{G} \in \mathbb{F}_\alpha$*;*

(LT1) $\mathbb{F}_\beta \subseteq \mathbb{F}_\alpha$ *whenever* $\alpha \leqslant \beta$*;*

(LT2) $\mathbb{F}_\perp = \mathcal{F}^s_L(X)$.

A map $\varphi : (X, \mathbb{F}) \longrightarrow (Y, \mathbb{F}')$ *between two stratified L-filter tower spaces is called Cauchy continuous provided that* $\mathcal{F} \in \mathbb{F}_\alpha$ *implies* $\varphi^{\Rightarrow}(\mathcal{F}) \in \mathbb{F}'_\alpha$ *for all* $\alpha \in L$*. The construct of stratified L-filter tower spaces and Cauchy continuous maps is denoted by SL-***FTS***.*

Definition 6.2. *A stratified L-filter tower space* $(X, \mathbb{F})$ *is called complete provided that for each* $\alpha \in L$*,*

(CLFT) *If* $\mathcal{F} \in \mathbb{F}_\alpha$*, then there exists* $x \in X$ *s.t.* $\mathcal{F} \wedge [x] \in \mathbb{F}_\alpha$*.*

*The full subconstruct of SL-***FTS***, consisting of complete stratified L-filter tower spaces, is denoted by SL-***CFTS***.*

In order to show the relations between SL-**CFTS** and SL-**FTS**, we first give the following lemma.

Lemma 6.3. *Let $(X, \mathbb{F})$ be a stratified L-filter tower space and define $c\mathbb{F} = \{c\mathbb{F}_\alpha \mid \alpha \in L\}$ as follows: for each $\alpha \in L$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$,*

$$\mathcal{F} \in c\mathbb{F}_\alpha \Longleftrightarrow \exists x \in X,\ s.t.\ \mathcal{F} \wedge [x] \in \mathbb{F}_\alpha.$$

Then $(X, c\mathbb{F})$ is a complete stratified L-filter tower space.

Proof. (LFT1), (LFT2), (LT1) and (LT2) are straightforward.

(CLFT) Take any $\mathcal{F} \in c\mathbb{F}_\alpha$. Then there exists $x \in X$ such that $\mathcal{F} \wedge [x] \in \mathbb{F}_\alpha$. Further, $(\mathcal{F} \wedge [x]) \wedge [x] \in \mathbb{F}_\alpha$. This implies there exists $x \in X$ such that $(\mathcal{F} \wedge [x]) \wedge [x] \in \mathbb{F}_\alpha$. Thus $\mathcal{F} \wedge [x] \in c\mathbb{F}_\alpha$. $\square$

Theorem 6.4. *SL-**CFTS** is bicoreflective in SL-**FTS**.*

Proof. Let $(X, \mathbb{F})$ be a stratified L-filter tower space and define $c\mathbb{F} = \{c\mathbb{F}_\alpha \mid \alpha \in L\}$ as follows:

$$\mathcal{F} \in c\mathbb{F}_\alpha \Longleftrightarrow \exists x \in X,\ s.t.\ \mathcal{F} \wedge [x] \in \mathbb{F}_\alpha.$$

By Lemma 6.3, we know $(X, c\mathbb{F})$ is a complete stratified L-filter tower space. Further, we claim that $id_X : (X, c\mathbb{F}) \longrightarrow (X, \mathbb{F})$ is the SL-**CFTS**-bicoreflector. For this, it suffices to prove that

(1) $id_X : (X, c\mathbb{F}) \longrightarrow (X, \mathbb{F})$ is Cauchy continuous.

(2) For each complete stratified L-filter tower space $(Y, \mathbb{F}')$ and a map $\varphi : Y \longrightarrow X$, the Cauchy continuity of $\varphi : (Y, \mathbb{F}') \longrightarrow (X, \mathbb{F})$ implies the Cauchy continuity of $\varphi : (Y, \mathbb{F}') \longrightarrow (X, c\mathbb{F})$.

For (1), take any $\mathcal{F} \in c\mathbb{F}_\alpha$, then there exists $x \in X$ such that $\mathcal{F} \wedge [x] \in \mathbb{F}_\alpha$. So $\mathcal{F} \in \mathbb{F}_\alpha$. This means the Cauchy continuity of $id_X : (X, c\mathbb{F}) \longrightarrow (X, \mathbb{F})$.

For (2), take any $\mathcal{G} \in \mathbb{F}'_\alpha$. By (CLFT), there exists $y \in Y$ such that $\mathcal{G} \wedge [y] \in \mathbb{F}'_\alpha$. Since $\varphi : (Y, \mathbb{F}') \longrightarrow (X, \mathbb{F})$ is Cauchy continuous, it follows that $\varphi^{\Rightarrow}(\mathcal{G}) \wedge [\varphi(y)] = \varphi^{\Rightarrow}(\mathcal{F} \wedge [y]) \in \mathbb{F}_\alpha$. This means there exists $\varphi(y) \in X$ such that $\varphi^{\Rightarrow}(\mathcal{G}) \wedge [\varphi(y)] \in \mathbb{F}_\alpha$. Hence, we have $\varphi^{\Rightarrow}(\mathcal{G}) \in c\mathbb{F}_\alpha$. This proves the Cauchy continuity of $\varphi : (Y, \mathbb{F}') \longrightarrow (X, c\mathbb{F})$. $\square$

Next we discuss the relations between SSL-**KCTS** and SL-**CFTS**.

Proposition 6.5. *Let $(X, \mathbb{F})$ be a complete stratified L-filter tower space and define $q_{\mathbb{F}} = \{(q_{\mathbb{F}})_\alpha \mid \alpha \in L\}$ as follows:*

$$(\mathcal{F}, x) \in (q_{\mathbb{F}})_\alpha \Longleftrightarrow \mathcal{F} \wedge [x] \in \mathbb{F}_\alpha.$$

Then $(X, q_{\mathbb{F}})$ is a symmetric stratified L-Kent convergence tower space.

Proof. (LCT1), (LCT2) and (LKCT) are obvious.

(LCT3) Take any $\alpha, \beta \in L$ such that $\alpha \leqslant \beta$. Then for each $(\mathcal{F}, x) \in (q_{\mathbb{F}})_\beta$, it follows that $\mathcal{F} \wedge [x] \in \mathbb{F}_\beta \subseteq \mathbb{F}_\alpha$. So we have $(\mathcal{F}, x) \in (q_{\mathbb{F}})_\alpha$. This proves $(q_{\mathbb{F}})_\beta \subseteq (q_{\mathbb{F}})_\alpha$.

(SLKCT) Take any $(\mathcal{F}, x) \in (q_{\mathbb{F}})_\alpha$ with $\mathcal{F} \leqslant [y]$. Then $\mathcal{F} \wedge [x] \in \mathbb{F}_\alpha$. By (LFT2), we know $\mathcal{F} \wedge [y] = \mathcal{F} \in \mathbb{F}_\alpha$. This implies $(\mathcal{F}, y) \in (q_{\mathbb{F}})_\alpha$. $\square$

Proposition 6.6. *Let (X, q) be a symmetric stratified L-Kent convergence tower space and define $\mathbb{F}_q = \{(\mathbb{F}_q)_\alpha \mid \alpha \in L\}$ as follows:*

$$\mathcal{F} \in (\mathbb{F}_q)_\alpha \Longleftrightarrow \exists x \in X,\ s.t.\ (\mathcal{F}, x) \in q_\alpha.$$

Then $(X, \mathbb{F}_q)$ is complete stratified L-filter tower space.

Proof. (LFT1), (LFT2) and (LT1) are straightforward.

(LT2) By (LCT1), we know $(\mathcal{F}_\perp^X, x) \in q_\perp$. Thus for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, it follows from $\mathcal{F} \geqslant \mathcal{F}_\perp^X$ that $(\mathcal{F}, x) \in q_\perp$. This means $\mathcal{F} \in (\mathbb{F}_q)_\perp$. By the arbitrariness of $\mathcal{F}$, we obtain $(\mathbb{F}_q)_\perp = \mathcal{F}_L^s(X)$.

(CLFT) If $\mathcal{F} \in (\mathbb{F}_q)_\alpha$, then there exists $x \in X$ such that $(\mathcal{F}, x) \in q_\alpha$. By (LKCT), we know $(\mathcal{F} \wedge [x], x) \in q_\alpha$. This shows $\mathcal{F} \wedge [x] \in (\mathbb{F}_q)_\alpha$. $\square$

Proposition 6.7. (1) *If $\varphi : (X, \mathbb{F}) \longrightarrow (Y, \mathbb{F}')$ is Cauchy continuous, then $\varphi : (X, q_{\mathbb{F}}) \longrightarrow (Y, q_{\mathbb{F}'})$ is continuous.*

(2) *If $\varphi : (X, q) \longrightarrow (Y, q')$ is continuous, then $\varphi : (X, \mathbb{F}_q) \longrightarrow (Y, \mathbb{F}_{q'})$ is Cauchy continuous.*

Proof. (1) For each $\alpha \in L$, take $(\mathcal{F}, x) \in (q_{\mathbb{F}})_\alpha$. Then $\mathcal{F} \wedge [x] \in \mathbb{F}_\alpha$. By the Cauchy continuity of $\varphi : (X, \mathbb{F}) \longrightarrow (Y, \mathbb{F}')$, it follows that

$$\varphi^{\Rightarrow}(\mathcal{F}) \wedge [\varphi(x)] = \varphi^{\Rightarrow}(\mathcal{F} \wedge [x]) \in \mathbb{F}'_\alpha.$$

This means $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in (q_{\mathbb{F}'})_\alpha$. Thus, $\varphi : (X, q_{\mathbb{F}}) \longrightarrow (Y, q_{\mathbb{F}'})$ is continuous.

(2) For each $\alpha \in L$, take any $\mathcal{F} \in (\mathbb{F}_q)_\alpha$. Then there exists $x \in X$ such that $(\mathcal{F}, x) \in q_\alpha$. By the continuity of $\varphi : (X, q) \longrightarrow (Y, q')$, it follows that $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\alpha$. This implies there exists $\varphi(x) \in Y$ such that $(\varphi^{\Rightarrow}(\mathcal{F}), \varphi(x)) \in q'_\alpha$. Thus, we have $\varphi^{\Rightarrow}(\mathcal{F}) \in (\mathbb{F}_{q'})_\alpha$. This proves the Cauchy continuity of $\varphi : (X, \mathbb{F}_q) \longrightarrow (Y, \mathbb{F}_{q'})$. □

Proposition 6.8. *Suppose that* $(X, \mathbb{F})$ *is a complete stratified L-filter tower space and* (X, q) *is a symmetric stratified L-Kent convergence tower space. Then* $\mathbb{F}_{q_{\mathbb{F}}} = \mathbb{F}$ *and* $q_{\mathbb{F}_q} = q$.

Proof. (1) For each $\alpha \in L$, take $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then

$$\begin{aligned} \mathcal{F} \in (\mathbb{F}_{q_{\mathbb{F}}})_\alpha &\iff \exists x \in X,\ s.t.\ (\mathcal{F}, x) \in (q_{\mathbb{F}})_\alpha \\ &\iff \exists x \in X,\ s.t.\ \mathcal{F} \wedge [x] \in \mathbb{F}_\alpha \\ &\iff \mathcal{F} \in \mathbb{F}_\alpha, \end{aligned}$$

where the last "$\iff$"holds since $(X, \mathbb{F})$ is complete.

(2) For each $\alpha \in L$, take $(\mathcal{F}, x) \in \mathcal{F}_L^s(X) \times X$. Then

$$\begin{aligned} (\mathcal{F}, x) \in (q_{\mathbb{F}_q})_\alpha &\iff \mathcal{F} \wedge [x] \in (\mathbb{F}_q)_\alpha \\ &\iff \exists y \in X,\ s.t.\ (\mathcal{F} \wedge [x], y) \in q_\alpha \\ &\iff (\mathcal{F}, x) \in q_\alpha, \end{aligned}$$

where the last "$\iff$" holds since (X, q) satisfies (LKCT) and (SLKCT). □

By Propositions 6.5–6.8, we obtain

Theorem 6.9. *SL*-**CFTS** *is isomorphic to SSL*-**KCTS**.

By Theorems 5.11 and 6.9, we have

Corollary 6.10. *SL*-**CFTS** *is strongly Cartesian closed.*

7. Conclusions

In this paper, we have discussed the convenient properties of stratified L-convergence tower spaces and stratified L-Kent convergence tower spaces. Besides Cartesian-closedness, we show the resulting constructs are both extensional and closed under the formation of products of quotient maps. That is to say, they are strong topological universes in the sense of Preuss [35]. Moreover, we introduced symmetric stratified L-Kent convergence tower spaces and complete stratified L-filter tower spaces based on the preceding works in [9, 39] and showed that they are isomorphic. As the future work, we will study the further convenient properties of stratified L-generalized convergence structures in the sense of Jäger [14].

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