Filomat 33:15 (2019), 4837–4843
https://doi.org/10.2298/FIL1915837A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Bilateral Contraction via Simulation Function

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Abstract. In this paper we introduce the notion of a bilateral contraction that combine the ideas of Ćirić type contraction and Caristi type contraction with a help of simulation functions. We investigate the existence of a fixed point of such contractions in the framework of complete metric spaces. We present an example to clarify the statement of the given result.

1. Introduction and Preliminaries

There exists a consensus on the initiation of the metric fixed point theory that Banach's fixed point theorem [12] is the pioneer and the most significant result in this research area. From that point on, a huge number of publications appeared to extend and generalize the renowned fixed point result of Banach. On the other hand, most of the mentioned results just have used the same techniques that appeared in Banach's proof. There were also some attempts that aimed to shorten the proof of Banach's fixed point results. Among them, Caristi [16, 17] studied the curtailment of the Banach's proof and he has discovered another interesting result during this process: If a self-mapping T on a complete metric space (X, d) satisfies the inequality $d(x, Tx) \leq \vartheta(x) - \vartheta(Tx)$, then T possesses a fixed point, where, $\vartheta : X \to \mathbb{R}$ is a lower semi-continuous function and it is bounded below. Obviously, the statements are quite different. Indeed, the structures of the proofs are also distinct from each other.

In this short note, we propose a new fixed point theorem based on the new notion, bilateral contraction, that is inspired by Ćirić type contraction (an extension of a Banach's contraction) and Caristi type contraction, by using a simulation function, in the context of complete metric spaces.

In what follows, we shortly examine the auxiliary function: Simulation function. This notion was introduced by Khojasteh *et al.* [21] and it is announced in 2015:

Definition 1.1. *(See [21]) A* simulation function *is a mapping* $\mathfrak{z} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *satisfying the following conditions:*

$(\mathfrak{z}_1)$ $\mathfrak{z}(0, 0) = 0$;

$(\mathfrak{z}_2)$ $\mathfrak{z}(t, s) < s - t$ *for all* $t, s > 0$;

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25)
Keywords. bilateral contraction, Ćirić type contraction, Caristi type contraction, Ssimulation functions, fixed point
Received: 30 April 2019; Accepted: 21 June 2019
Communicated by Dragan S. Djordjević
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(ζ_3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{1}$$

In the same year, 2015, this notion is refined by Argoubi et al. *[7] by removing the first axiom* (ζ_1). *Indeed, it is derived form* (ζ_2). *From now on, we consider the simulation functions in the sense of Argoubi* et al. *[7], that is,* ζ *satisfies only* (ζ_2) *and* (ζ_3). *In the sequel, the the letter* $\mathcal{Z}$ *will denote the family of all simulation functions* $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *that satisfy* (ζ_2) *and* (ζ_3). *Notice also that due to the axiom* (ζ_2) *yields that*

$$\zeta(t,t) < 0 \text{ for all } t > 0. \tag{2}$$

Example 1.2. *(See e.g.[5, 21, 22]) Suppose that* $\phi_k : [0,\infty) \to [0,\infty)$, $k = 1,2,3$, *are continuous functions so that* $\phi_i(t) = 0$ *if, and only if,* $t = 0$. *We introduce the function* $\zeta_i : [0,\infty)\times[0,\infty) \to \mathbb{R}$, $i = 1,2,3,4,5,6$ *as*

(*i*) $\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ *for all* $t,s \in [0,\infty)$, *where* $\phi_1(t) < t \le \phi_2(t)$ *for all* $t > 0$.

(*ii*) $\zeta_2(t,s) = s - \dfrac{f(t,s)}{g(t,s)}t$ *for all* $t,s \in [0,\infty)$, *where* $f, g : [0,\infty)^2 \to (0,\infty)$ *are two continuous functions with respect to each variable such that* $f(t,s) > g(t,s)$ *for all* $t,s > 0$.

(*iii*) $\zeta_3(t,s) = s - \phi_3(s) - t$ *for all* $t,s \in [0,\infty)$.

(*iv*) *If* $\vartheta : [0,\infty) \to [0,1)$ *is a function such that* $\limsup_{t\to r^+} \vartheta(t) < 1$ *for all* $r > 0$, *and we define*

$$\zeta_4(t,s) = s\,\vartheta(s) - t \qquad \textit{for all } s,t \in [0,\infty).$$

(*v*) *If* $\eta : [0,\infty) \to [0,\infty)$ *is an upper semi-continuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$, *and we define*

$$\zeta_5(t,s) = \eta(s) - t \qquad \textit{for all } s,t \in [0,\infty).$$

(*vi*) *If* $\phi : [0,\infty) \to [0,\infty)$ *is a function such that* $\int_0^\varepsilon \phi(u)du$ *exists and* $\int_0^\varepsilon \phi(u)du > \varepsilon$, *for each* $\varepsilon > 0$, *and we define*

$$\zeta_6(t,s) = s - \int_0^t \phi(u)du \qquad \textit{for all } s,t \in [0,\infty).$$

It is clear that each function ζ_i $(i = 1,2,3,4,5,6)$ *forms a simulation function.*

We refer to [3–5, 10, 11, 18–22] for more details and examples on simulation function.

Assume that T is a self-mapping on a metric space (X,d) and $\zeta \in \mathcal{Z}$. A function T is called a $\mathcal{Z}$*-contraction* with respect to ζ [21], if

$$\zeta(d(Tx,Ty), d(x,y)) \ge 0 \qquad \text{for all } x,y \in X. \tag{3}$$

Due to (ζ_2), we have the following inequality

$$d(Tx,Ty) \ne d(x,y) \text{ for all distinct } x,y \in X. \tag{4}$$

Accordingly, we deduce that T cannot be an isometry whenever T is a $\mathcal{Z}$-contraction.

Theorem 1.3. *Every* $\mathcal{Z}$*-contraction on a complete metric space possesses a unique fixed point.*

2. Main Result

We start with the definition of a bilateral contraction:

Definition 2.1. *Let T be a self-mapping on a metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that*

$$d(x, Tx) > 0 \text{ implies } \mathfrak{z}(d(Tx, Ty), (\varphi(x) - \varphi(Tx))C_T(x, y)) \geq 0, \tag{5}$$

in which

$$C_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2}\right\}$$

for all $x, y \in X$, then T is called a bilateral contraction of Ćirić-Caristi.

This is the main result of this note.

Theorem 2.2. *Let T be a bilateral contraction of Ćirić-Caristi in the setting of a complete metric space (X, d). Then, T possesses at least one fixed point.*

Proof. The proofs consists of several steps.

Step 1. **Construction of an iterative sequence:** For this purpose, take any initial point $x \in X$ and rename as $x_0 := x$. Starting with this point, we construct an iterative sequence $\{x_n\}$ by $x_{n+1} := Tx_n = T^n x_0$ for each $n \in \mathbb{N}$. Notice that in case of having an inequality $x_k = x_{k+1} = Tx_k$, for some $k \in \mathbb{N}$, the proof is completed. Hereby, throughout the proof, we assume that $x_n \neq x_{n+1}$, for any $n \in \mathbb{N}$, that is,

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) > 0. \tag{6}$$

Step 2. **Dominating the ratio of adjacent distance by a constant κ:** There exist $\kappa \in [0, 1)$ such that

$$d(x_n, x_{n+1}) \leq \kappa d(x_n, x_{n-1}).$$

For simplicity, we assume that $\delta_n = d(x_n, x_{n-1})$. On account of (5), we get

$$\begin{aligned} 0 &\leq \mathfrak{z}(d(Tx, Ty), (\varphi(x) - \varphi(Tx))C_T(x, y)) \\ &< (\varphi(x) - \varphi(Tx))C_T(x, y) - d(Tx, Ty), \end{aligned}$$

which yields

$$\begin{aligned} \delta_{n+1} &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\ &< (\varphi(x_{n-1}) - \varphi(Tx_{n-1}))C_T(x_{n-1}, x_n) \\ &= (\varphi(x_{n-1}) - \varphi(x_n)) \max\{\delta_n, \delta_{n+1}, \tfrac{d(x_{n-1}, x_{n+1})}{2}\} \\ &\leq (\varphi(x_{n-1}) - \varphi(x_n)) \max\{\delta_n, \delta_{n+1}\}. \end{aligned} \tag{7}$$

Now, we examine these cases explicitly.

Case 1: Let $\max\{\delta_n, \delta_{n+1}\} = \delta_n$. Regarding (7), we obtain

$$\begin{aligned} \delta_{n+1} &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\ &\leq (\varphi(x_{n-1}) - \varphi(x_n))d(x_n, x_{n-1}) = (\varphi(x_{n-1}) - \varphi(x_n))\delta_n. \end{aligned}$$

Consequently, we find

$$0 < \frac{\delta_{n+1}}{\delta_n} \leq \varphi(x_{n-1}) - \varphi(x_n) \text{ for each } n \in \mathbb{N}.$$

Thus, from the inequality above, we deduce that the sequence $\{\varphi(x_n)\}$ is necessarily positive and non-increasing. Accordingly, it converges to some $\ell \geq 0$. On the other hand, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{\delta_{k+1}}{\delta_k} &\leq \sum_{k=1}^{n} \varphi(x_{k-1}) - \varphi(x_k) \\ &= (\varphi(x_0) - \varphi(x_1)) + (\varphi(x_1) - \varphi(x_2)) + \dots + (\varphi(x_{n-1}) - \varphi(x_n)) \\ &= \varphi(x_0) - \varphi(x_n) \to \varphi(x_0) - \ell < \infty, \text{ as } n \to \infty.\end{aligned}$$

It implies that

$$\sum_{n=1}^{\infty} \frac{\delta_{n+1}}{\delta_n} < \infty.$$

Accordingly, we have

$$\lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = 0. \tag{8}$$

On account of (8), for $\kappa \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\delta_{n+1}}{\delta_n} \leq \kappa, \tag{9}$$

for all $n \geq n_0$, in other words,

$$d(x_{n+1}, x_n) \leq \kappa d(x_n, x_{n-1}), \tag{10}$$

for all $n \geq n_0$.

Case 2: Let $\max\{\delta_n, \delta_{n+1}\} = \delta_{n+1}$. Taking the inequality (7) into account, we derive that

$$d(x_{n+1}, x_n) \leq (\varphi(x_{n-1}) - \varphi(x_n))d(x_{n+1}, x_n).$$

Since $\{\varphi(x_n)\}$ is non-increasing and positive sequence, and so converges to some $\ell \geq 0$ (see the similar argument in Case 1) thus we have $1 \leq (\varphi(x_{n-1}) - \varphi(x_n)) \to 0$, as $n \to \infty$ and this is a contradiction. Consequently, $\max\{\delta_n, \delta_{n+1}\} = \delta_n$, and

$$d(x_n, x_{n+1}) \leq \kappa d(x_n, x_{n-1}) = \kappa^n d(x_1, x_0). \tag{11}$$

Step 3: **Showing the constructive recursive sequence $\{x_n\}$ converges to $x_* \in X$.** From the previous step, we indicate that the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing, bounded below with the relation (11). Eventually, the $\{d(x_{n+1}, x_n)\}$ is convergent to some $L \geq 0$. Baring $\kappa < 1$ in mind, it can be easily affirmed that $L = 0$. Furthermore, for each $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_i, x_{i+1}) \leq \frac{\kappa^n}{1-\kappa} d(x_0, x_1).$$

It means that $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence and since X is complete, there exists $x_* \in X$ such that $\{x_n\}$ converges to x_*.

Step 4: **Showing that the limit x_* of the constructive recursive sequence $\{x_n\}$ is the desired fixed point of T.**

Employing (5), if $d(x_*, Tx_*) > 0$, we find that

$$\begin{aligned}0 &\leq \mathfrak{z}(d(Tx_*, Tx_n), (\varphi(x_*) - \varphi(Tx_*))C_T(x_*, x_n)) \\ &= (\varphi(x_*) - \varphi(Tx_*))C_T(x_*, x_n) - d(Tx_*, Tx_n).\end{aligned}$$

On account the triangle inequality together with the inequality above, we derive that

$$\begin{aligned}
d(x_*, Tx_*) &\le d(x_*, x_{n+1}) + d(x_{n+1}, Tx_*) \\
&= d(x_*, x_{n+1}) + d(Tx_n, Tx_*) \\
&= d(x_*, x_{n+1}) + (\varphi(x_n) - \varphi(Tx_n))C_T(x_n, x_*) \\
&= d(x_*, x_{n+1}) + (\varphi(x_n) - \varphi(x_{n+1}))\max\left\{\begin{array}{c} d(x_n, x_*), d(x_n, Tx_n), d(x_*, Tx_*), \\ \frac{d(x_n, Tx_*) + d(Tx_n, x_*)}{2} \end{array}\right\} \\
&= d(x_*, x_{n+1}) + (\varphi(x_n) - \varphi(x_{n+1}))\max\left\{\begin{array}{c} d(x_n, x_*), d(x_n, Tx_n), d(x_*, Tx_*), \\ \frac{d(x_n, Tx_*) + d(x_{n+1}, x_*)}{2} \end{array}\right\}
\end{aligned} \tag{12}$$

Since the sequences $\{\varphi(x_n)\}$ tends to $r \ge 0$, for sufficiently large $n \in \mathbb{N}$, we have

$$d(x_*, Tx_*) \le \lim_{n\to\infty}(d(x_*, x_{n+1}) + (\varphi(x_n) - \varphi(x_{n+1})))d(x_*, Tx_*) = 0.$$

Consequently, we obtain $d(x_*, Tx_*) = 0$, that is, $Tx_* = x_*$.
□

From Theorem 2.2, we get the corresponding result for complete metric spaces. The following example shows that the Theorem 2.2 is not a consequence of Banach's contraction principle.

Example 2.3. *Let $X = \{\alpha_0, \alpha_1, \alpha_2\}$ endowed with the following metric:*

$$d(\alpha_0, \alpha_1) = 1, d(\alpha_2, \alpha_0) = 1, d(\alpha_1, \alpha_2) = \tfrac{3}{2}$$
and
$$d(a, a) = 0, \ \forall\, a \in X \ \ d(a, b) = d(b, a) \ \ \forall\, a, b \in X.$$

Let $T(\alpha_0) = \alpha_0, T(\alpha_1) = \alpha_2, T(\alpha_2) = \alpha_0$. Define $\varphi : X \to [0, \infty)$ as $\varphi(\alpha_2) = 2, \varphi(\alpha_0) = 0, \varphi(\alpha_1) = 4$. Thus for all $x \in X$ such that $d(x, Tx) > 0$, (in this example, $x \neq \alpha_0$), we have

$$\begin{aligned}
0 &\le \mathfrak{z}(d(T\alpha_1, T\alpha_2), (\varphi(\alpha_1) - \varphi(T(\alpha_1)))C_T(\alpha_2, \alpha_1)), \\
0 &\le \mathfrak{z}(d(T\alpha_2, T\alpha_1), (\varphi(\alpha_2) - \varphi(T(\alpha_2)))C_T(\alpha_2, \alpha_1)), \\
0 &\le \mathfrak{z}(d(T\alpha_1, T\alpha_0), (\varphi(\alpha_1) - \varphi(T(\alpha_1)))C_T(\alpha_1, \alpha_0)), \\
0 &\le \mathfrak{z}(d(T\alpha_2, T\alpha_0), (\varphi(\alpha_2) - \varphi(T(\alpha_2)))C_T(\alpha_2, \alpha_0)).
\end{aligned}$$

Thus the mapping T satisfies our condition and also has a fixed point. Note that $d(T\alpha_1, T\alpha_0) = d(\alpha_1, \alpha_0)$. Thus, it does not satisfy conditions of Banach contraction principle.

3. Immediate Consequence

In this section, we shall list some immediate consequence of our main result. We shall skip the proof of these corollaries since they can be easily observed by verbatim of the proof of Theorem 2.2.

Corollary 3.1. *Let T be self mapping on a complete metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that $d(x, Tx) > 0$ implies*

$$\mathfrak{z}\left(d(Tx, Ty), (\varphi(x) - \varphi(Tx))\max\{d(x, y), d(x, Tx), d(y, Ty)\}\right) \ge 0,$$

for all $x, y \in X$, then, T possesses at least a fixed point.

Corollary 3.2. *Let T be self mapping on a complete metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that $d(x, Tx) > 0$ implies*

$$\mathfrak{z}\left(d(Tx, Ty), (\varphi(x) - \varphi(Tx)) \max\{d(x, Tx), d(y, Ty)\}\right) \geq 0, \tag{13}$$

for all $x, y \in X$, then, T possesses at least a fixed point.

Corollary 3.3. *Let T be self mapping on a complete metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that $d(x, Tx) > 0$ implies*

$$\mathfrak{z}\left(d(Tx, Ty), (\varphi(x) - \varphi(Tx)) \max\left\{d(x, y), \frac{d(y, Tx) + d(x, Ty)}{2}\right\}\right) \geq 0, \tag{14}$$

for all $x, y \in X$, then, T possesses at least a fixed point.

Corollary 3.4. *Let T be self mapping on a complete metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that $d(x, Tx) > 0$ implies*

$$\mathfrak{z}\left(d(Tx, Ty), (\varphi(x) - \varphi(Tx)) d(x, y)\right) \geq 0, \tag{15}$$

for all $x, y \in X$, then, T possesses at least a fixed point.

Corollary 3.5. *Let T be self mapping on a complete metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that $d(x, Tx) > 0$ implies*

$$\mathfrak{z}\left(d(Tx, Ty), (\varphi(x) - \varphi(Tx)) \frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}\right) \geq 0, \tag{16}$$

for all $x, y \in X$, then, T possesses at least a fixed point.

Corollary 3.6. *Let T be self mapping on a complete metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that $d(x, Tx) > 0$ implies*

$$\mathfrak{z}\left(d(Tx, Ty), (\varphi(x) - \varphi(Tx)) \frac{d(y, Ty) + d(x, Tx)}{2}\right) \geq 0, \tag{17}$$

for all $x, y \in X$, then, T possesses at least a fixed point.

Corollary 3.7. *Let T be self mapping on a complete metric space (X, d). If there exists $\mathfrak{z} \in \mathcal{Z}$ and $\varphi : X \to [0, \infty)$ such that $d(x, Tx) > 0$ implies*

$$\mathfrak{z}\left(d(Tx, Ty), (\varphi(x) - \varphi(Tx)) \frac{d(x, Tx) + d(x, Ty) + d(y, Tx)}{3}\right) \geq 0, \tag{18}$$

for all $x, y \in X$, then, T possesses at least a fixed point.

Remark 3.8. *It is clear that the list of the corollaries can be extended. We aim to list only the immediate consequences. As a continuation of this work, it will be interesting to investigate the characterizations of the given results of this paper in the setting of b-metric space see e.g. [1]–[15].*

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this group No. RG-1440-025.

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