Spectral Properties of Square Hyponormal Operators

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, we denote by $T^*$, $\ker(T)$, $r(T)$, $\sigma(T)$, $\sigma_0(T)$, $\sigma_r(T)$, respectively, the adjoint, the null space, the range, the spectrum, the approximate point spectrum and the residual spectrum of $T$. It is well-known that $\sigma(T) = \sigma_0(T) \cup \sigma_r(T)$.

An operator $T \in B(\mathcal{H})$ is self-adjoint if $T = T^*$. An operator $T \in B(\mathcal{H})$ is normal and 2-normal if $T^* T = T T^*$ and $T^* T^2 = T^2 T$, respectively. By Fuglede-Putnam Theorem, it is easily to see that $T$ is 2-normal if and only if $T^2$ is normal (see [4]). An operator $T \in B(\mathcal{H})$ is positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle = 0$, for all $x \in \mathcal{H}$. For self-adjoint operators $T, S \in B(\mathcal{H})$, $T \geq S$ means $T - S \geq 0$.

For an operator $T \in B(\mathcal{H})$, let $|T| = (T^* T)^{1/2}$ and $|T^*| = (T T^*)^{1/2}$. For $0 < p \leq 1$, $T$ is said to be $p$-hyponormal if $|T|^p \geq |T|^p$. When $p = 1$ and $p = \frac{1}{2}$, $T$ is said to be hyponormal and semi-hyponormal, respectively. Notice that $T$ is hyponormal if and only if $\|T^* x\| \leq \|T x\|$, for all $x \in \mathcal{H}$. By Corollary 1 of [3], in general, if $T$ is $p$-hyponormal (0 < $p \leq 1$), then $T^n$ is $\frac{p}{n}$-hyponormal. An operator $T \in B(\mathcal{H})$ is said to be paranormal if $\|T x\| \leq \|T^2 x\| \cdot \|x\|$, for all $x \in \mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be algebraically hyponormal if $p(T)$ is hyponormal and paranormal, for some nonconstant complex polynomial $p$, respectively.

In [7, 8], the authors showed that if $T$ is algebraically hyponormal and algebraically paranormal, then $T$ is isoloid and Weyl’s Theorem holds, respectively.
The aim of this paper is to study a bounded linear operator $T$ on a complex Hilbert space such that $T^2$ is a hyponormal operator. Firstly, notice that there exists an operator $T$ such that $T^2$ is hyponormal and $T$ is not hyponormal.

Let $\mathcal{H} = \ell^2$ and $T$ be the unilateral shift with the weights $\{a_n \geq 0\}$ such that

$$Tx := (0, a_1x_1, a_2x_2, ...) \quad \text{for} \quad x = (x_1, x_2, ...) \in \mathcal{H}.$$ 

Then $T$ is hyponormal if and only if $a_j \leq a_{j+1}$ $(j = 1, 2, ...)$, i.e., $\{a_j\}$ is a monotone increasing sequence, for $a_j = 1$ $(j \neq 2)$ and $a_2 = \frac{1}{2}$. Since the sequence $\{a_n\}$ is not increasing, the operator $T$ is not hyponormal. But since

$$T^2x = (0, 0, a_1a_2x_1, a_2a_3x_2, ...) \quad \text{and} \quad T^2x = (a_1a_2x_3, a_2a_3x_4, ...),$$

$T^2$ is hyponormal if and only if $a_2a_{j+1} \leq a_{j+2}a_{j+3}$ for $j = 1, 2, ...$. Hence, by this weights $a_j = 1$ $(j \neq 2)$ and $a_2 = \frac{1}{2}$, the operator $T^2$ is hyponormal and $T$ is not hyponormal.

In [4–6], the authors have studied spectral properties of $n$-normal operator, that is, an operator $T$ such that $T^n$ is normal, in the cases that $\sigma(T) \cap (-\sigma(T)) = \emptyset$ or $\sigma(T) \cap (-\sigma(T)) \subset \{0\}$. Since an operator $T$ such that $T^2$ is hyponormal is algebraically hyponormal, $T$ is isoloid and Weyl’s Theorem holds. Hence, we study other spectral properties of such an operator $T$ in this paper.

2. Basic properties

In the beginning, we introduce a square hyponormal operator and investigate some basic properties of this operator.

**Definition 2.1.** For an operator $T \in B(\mathcal{H})$, $T$ is said to be square hyponormal if $T^2$ is hyponormal.

The following result follows from the definition of square hyponormal operators.

**Theorem 2.2.** Let $T \in B(\mathcal{H})$ be square hyponormal. Then the following statements hold.

1. If $T$ is invertible, then so is $T^{-1}$.

2. For an even number $n = 2k \in \mathbb{N}$, $T^n$ is $\frac{1}{k}$-hyponormal.

3. If $S \in B(\mathcal{H})$ is unitary equivalent to $T$, then $S$ is square hyponormal.

4. If $T - t$ is square hyponormal for all $t > 0$, then $T$ is hyponormal.

**Proof.**

1. is clear.

2. Since $T^2$ is hyponormal, by Corollary 1 of [3], $T^n = T^{2k} = (T^2)^k$ is $\frac{1}{k}$-hyponormal.

3. is clear.

4. Since

$$0 \leq (T - t)^2(T - t)^2 - (T - t)^2(T - t)^2 = T^2T^2 - T^2T^2 - 2T^2T^2 + 4T^2T^2 - 2T^2T^2,$$

we obtain that

$$0 \leq \frac{1}{4t^2}(T^2 - T^2) - (T - t)^2(T - t)^2 = \frac{1}{4t^2}(T^2 - T^2) - (T - t)^2(T - t)^2$$

Letting $t \to \infty$, we have $T^2T - TT^* \geq 0$. \qed

We now consider the restriction of a square hyponormal operator to an invariant closed subspace.
\textbf{Theorem 2.3.} Let $T \in B(\mathcal{H})$ be square hyponormal and $M$ be an invariant closed subspace for $T$. Then $T_{|M}$ is square hyponormal.

\textit{Proof.} Since $M$ is an invariant closed subspace for $T$, we observe that

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{33} \end{bmatrix} : \begin{bmatrix} M \\ M^{\perp} \end{bmatrix} \to \begin{bmatrix} M \\ M^{\perp} \end{bmatrix}.$$ 

Therefore, for $D = T_{1}T_{2} + T_{2}T_{3}$, since

$$T^2 = \begin{bmatrix} T_{11}^2 & DT_{12} \\ 0 & T_{33}^2 \end{bmatrix} \quad \text{and} \quad (T^2)' = \begin{bmatrix} (T_{11}^2)' & 0 \\ D' & (T_{33}^2)' \end{bmatrix},$$

we have

$$(T^2)'T^2 - T^2(T^2)' = \begin{bmatrix} (T_{11}^2)'T_{11}^2 - T_{11}^2(T_{11}^2)' - DD' \\ D'T_{11}^2 - T_{11}^2D' \end{bmatrix} \begin{bmatrix} (T_{11}^2)'D - D(T_{11}^2)' \\ D'D + (T_{33}^2)'T_{33}^2 - T_{33}^2(T_{33}^2)' \end{bmatrix} \geq 0.$$ 

Hence we deduce that $(T_{11}^2)'I_{11}^2 - T_{11}^2(T_{11}^2)' - DD' \geq 0$ and so $(T_{11}^2)'I_{11}^2 - T_{11}^2(T_{11}^2)' \geq 0$. Therefore, $T_{|M}$ is square hyponormal. \qed

3. Spectral property

Under some additional assumptions, we study spectral properties of a square hyponormal operator in this section. Firstly, we show the following theorem.

\textbf{Theorem 3.1.} Let $T \in B(\mathcal{H})$ be square hyponormal. If $\mu(\sigma(T)) = 0$, then $T^2$ is normal, where $\mu$ is the planar Lebesgue measure.

\textit{Proof.} Since $\mu(\sigma(T)) = 0$, we have that $\mu(\sigma(T^2)) = 0$ by the spectral mapping theorem. By $T^2$ is hyponormal and Putnam’s Theorem, it holds

$$\|T^2 - T^2T^2\| \leq \frac{1}{\pi} \mu(\sigma(T^2)) = 0.$$ 

Hence, $T^2$ is normal. \qed

\textbf{Remark 3.2.} If $T$ is $p$-hyponormal and square hyponormal with $\mu(\sigma(T)) = 0$, then, by Corollary 2 of [3], $T$ is normal. But let $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. Then $S$ is square hyponormal with $\mu(\sigma(S)) = 0$ and $S$ is not normal.

If $T$ is compact, then $\mu(\sigma(T)) = 0$. Hence, we have the following corollary.

\textbf{Corollary 3.3.} If $T \in B(\mathcal{H})$ is compact square hyponormal, then $T^2$ is normal.

An operator $T \in B(\mathcal{H})$ is said to have SVEP (single-valued extension property) if for every open subset $G$ of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function $f$ on $G$ such that $(T - z)f(z) \equiv 0$ on $G$, then $f(z) \equiv 0$ on $G$. It is well known that:

1. If $\ker(T - z) \perp \ker(T - w)$ for any distinct nonzero eigenvalues $z$ and $w$, then $T$ has SVEP.
2. Let $p$ be polynomial. If $p(T)$ has SVEP, then $T$ has SVEP.

See details in [2, 11, 12]. Since it is clear that a hyponormal operator has SVEP, we have the next corollary by (2).
Corollary 3.4. Let \( T \in \mathcal{B}(\mathcal{H}) \) be square hyponormal. Then \( T \) has SVEP.

Let \( \mathcal{K}(\mathcal{H}) \) be the set of all compact operators on \( \mathcal{H} \). Then, for \( T \in \mathcal{B}(\mathcal{H}) \), the Weyl spectrum \( \sigma_w(T) \) and the Browder spectrum \( \sigma_b(T) \) of \( T \) are defined as belows:

\[
\sigma_w(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} \sigma(T + K) \quad \text{and} \quad \sigma_b(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H}), TK = KT} \sigma(T + K).
\]

If \( T \) has SVEP, then \( \sigma_w(T) = \sigma_b(T) \) by Corollary 3.53 of [2]. Let \( \mathcal{H} = \sigma(T) \) denote the set of all analytic function defined on an open set containing \( \sigma(T) \). Then, by Corollary 3.72 of [2], we have the following result.

Corollary 3.5. Let \( T \in \mathcal{B}(\mathcal{H}) \) be square hyponormal. Then, for \( f \in \mathcal{H}(\sigma(T)) \),

\[
\sigma_w(f(T)) = \sigma_b(f(T)) = f(\sigma_w(T)) = f(\sigma_b(T)).
\]

Next for \( T \in \mathcal{B}(\mathcal{H}) \), we set the following property:

\[
(\ast) \quad \sigma(T) \cap (-\sigma(T)) \in [0].
\]

Then we begin with the following result.

Theorem 3.6. Let \( T \in \mathcal{B}(\mathcal{H}) \) be square hyponormal with \((\ast)\) and \( M \) be an invariant subspace for \( T \). If \( \sigma(T_M) = \{z\} \), then the following assertions hold.

1. If \( z = 0 \), then \( (T_M)^2 = 0 \).
2. If \( z \neq 0 \), then \( T_M \) is hyponormal.

Proof. (1) By Theorem 2.3, \( T_M \) is square hyponormal. Since \( \sigma((T_M)^2) = \{0\} \), we have \( (T_M)^2 = 0 \) by Putnam’s theorem.

(2) Similarly, from \( \sigma((T_M)^2) = \{z^2\} \), we get \( (T_M)^2 = z^2 \) and hence

\[
0 = (T_M)^2 - z^2 = (T_M + z)(T_M - z).
\]

By the assumption \((\ast)\), \(-z \not\in \sigma(T)\) and there exists \( (T_M + z)^{-1} \). Hence, it holds \( T_M - z = 0 \). \( \square \)

Theorem 3.7. Let \( T \in \mathcal{B}(\mathcal{H}) \) be a square hyponormal operator. If \( T \) satisfies \((\ast)\), then \( \sigma(T) = [\mathbb{Z} : z \in \sigma_a(T^*)] \).

Proof. Since \( \sigma(T) = \sigma_a(T) \cup \sigma_r(T) \), we may show \( \sigma_a(T) \subset [\mathbb{Z} : z \in \sigma_a(T^*)] \).

1. If \( 0 \in \sigma_a(T) \), then \( 0 \in \sigma_a(T^2) \) and \( T^2 \) is hyponormal. Hence, it is easy to see \( 0 \in \sigma_a(T^*) \).

2. Let \( z \in \sigma_a(T) \) and \( z \neq 0 \). Then there exists a sequence \( \{x_n\} \) of unit vectors such that \( (T - z)x_n \to 0 \) as \( n \to \infty \). Thus, \( (T^2 - z^2)x_n \to 0 \) as \( n \to \infty \). Because \( T^2 \) is hyponormal, we have \( (T^2 - z^2)x_n \to 0 \) and \( (T^* - \overline{z})(T^* - z)x_n \to 0 \) as \( n \to \infty \). By the assumption \((\ast)\), \(-\overline{z} \not\in \sigma(T^*)\) which gives \( (T^* - \overline{z})x_n \to 0 \) as \( n \to \infty \) and therefore \( \overline{z} \in \sigma_a(T^*) \). It completes the proof. \( \square \)

Theorem 3.8. Let \( T \in \mathcal{B}(\mathcal{H}) \) be square hyponormal and satisfy \((\ast)\).

1. If \( z \) and \( w \) are distinct eigen-values of \( T \) and \( x, y \in \mathcal{H} \) are corresponding eigen-vectors, respectively, then \( \langle x, y \rangle = 0 \).

2. If \( z, w \) are distinct values of \( \sigma_a(T) \) and \( \{x_n\}, \{y_n\} \) are the sequences of unit vectors in \( \mathcal{H} \) such that \( (T - z)x_n \to 0 \) and \( (T - w)y_n \to 0 \) as \( n \to \infty \), then \( \lim_{n \to \infty} \langle x_n, y_n \rangle = 0 \).

Proof. (1) follows from (2). So, we show (2). Since \( (T - z)x_n \to 0 \) and \( (T - w)y_n \to 0 \) as \( n \to \infty \), it holds that \( (T^2 - z^2)x_n \to 0 \) and \( (T^2 - w^2)y_n \to 0 \). Because \( T^2 \) is hyponormal, we get \( (T^2 - \overline{w^2})y_n \to 0 \). Hence,

\[
\lim_{n \to \infty} z^2 \langle x_n, y_n \rangle = \lim_{n \to \infty} z^2 \langle x_n, y_n \rangle = \lim_{n \to \infty} (T^2 - z^2) \langle x_n, y_n \rangle = \lim_{n \to \infty} \langle x_n, T^2y_n \rangle = \lim_{n \to \infty} w^2 \langle x_n, y_n \rangle.
\]

If \( z^2 = w^2 \), then \( (z + w)(z - w) = 0 \). Since \( z \neq w \), we have \( z = -w \). By \((\ast)\), this implies \( z = w = 0 \). Therefore, \( z^2 \neq w^2 \), and so \( \lim_{n \to \infty} \langle x_n, y_n \rangle = 0 \). \( \square \)
Thus, we have the following corollary.

**Corollary 3.9.** Let $T \in B(\mathcal{H})$ be square hyponormal and satisfy $(\ast)$. If $z$ and $w$ are distinct eigen-values of $T$, then $\ker(T - z) \perp \ker(T - w)$.

Let $M$ be a subspace of $\mathcal{H}$. $M$ is said to be a reducing subspace for $T$ if $T(M) \subset M$ and $T^*(M) \subset M$, that is, $M$ is an invariant subspace for $T$ and $T^*$. Then we have a following result.

**Theorem 3.10.** Let $T \in B(\mathcal{H})$ be square hyponormal and satisfy $(\ast)$. If $z$ is a non-zero eigen-value of $T$, then $\ker(T - z) = \ker(T^2 - z^2) \subset \ker(T^2 - z^2) = \ker(T^* - \overline{z})$ and hence $\ker(T - z)$ is a reducing subspace for $T$.

**Proof.** Firstly, we show that $\ker(T - z) = \ker(T^2 - z^2)$. Because it is clear that $\ker(T - z) \subset \ker(T^2 - z^2)$, we will verify that $\ker(T^2 - z^2) \subset \ker(T - z)$. Let $x \in \ker(T^2 - z^2)$, i.e., $(T^2 - z^2)x = 0$. Then $(T + z)(T - z)x = 0$. Since $z \neq 0$, by the assumption $(\ast)$, we have $-z \notin \sigma(T)$. Hence, it follows $(T - z)x = 0$ and $x \in \ker(T - z)$. Therefore, $\ker(T^2 - z^2) \subset \ker(T - z)$ and $\ker(T - z) = \ker(T^2 - z^2)$. Since $T^2$ is hyponormal, $\ker(T^2 - z^2) \subset \ker(T^2 - \overline{z})$. Evidently, $\ker(T^* - \overline{z}) \subset \ker(T^2 - z^2)$. Let $x \in \ker(T^2 - z^2)$. Because $(T^* + \overline{z})(T^* - \overline{z})x = 0$ and $T^* \overline{z}$ is invertible by the assumption $(\ast)$, we obtain that $x \in \ker(T^* - \overline{z})$. Hence, $\ker(T^2 - z^2) = \ker(T^* - \overline{z})$. Finally, by the above results, it is clear that $\ker(T - z)$ is a reducing subspace for $T$. □

The following remark is same with the corresponding in the paper of [5].

**Remark 3.11.** In general, $\ker(T)$ is not a reducing subspace for a square hyponormal operator $T$.

(1) Let $T$ be as in Example 2.3 of [1], that is, let $\mathcal{H} = \ell^2$, $\{e_j\}_{j=1}^{\infty}$ be the standard orthonormal basis of $\ell^2$ and $T$ be defined by

$$Te_j = \begin{cases} e_1 & (j = 1) \\ e_{j+1} & (j = 2k) \\ 0 & (j = 2k + 1). \end{cases}$$

Then $T$ is a square hyponormal operator and satisfies $(\ast)$. Since $e_3 \in \ker(T)$ and $TT^*e_3 = e_3 \neq 0$, $\ker(T)$ does not reduce $T$. Let $P$ be the orthogonal projection to the first coordinate. Since $T^2 = P$, it is clear that $\ker(T) \nsubseteq \ker(T^2) = \ker(P)$.

(2) We give an easy example. Let $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C}^2$. Since $S^2 = 0$ and $\sigma(S) = \{0\}$, $S$ is square hyponormal and satisfies $(\ast)$. Let $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $x \in \ker(S)$ and $SS^*x = x \neq 0$. Hence, $\ker(S)$ does not reduce $S$ and $\ker(S) \nsubseteq \ker(S^2) = \mathbb{C}^2$.

For an isolated point $\lambda$ of $\sigma(T)$, the Riesz idempotent for $\lambda$ is defined by

$$E_\lambda(\lambda) = \frac{1}{2\pi i} \int_{\partial D} (z - T)^{-1} \, dz,$$

where $D$ is a closed disk centered at $\lambda$ which contains no other points of $\sigma(T)$. For an operator $T \in \mathcal{L}(\mathcal{H})$, the quasinilpotent part of $T$ is defined by

$$\mathcal{H}_0(T) := \{x \in \mathcal{H} : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Then $\mathcal{H}_0(T)$ is a linear (not necessarily closed) subspace of $\mathcal{H}$. It is known that if $T$ has SVEP, then

$$\mathcal{H}_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n \to \infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\} = E_\lambda(\lambda)\mathcal{H}.$$
for all \( \lambda \in \mathbb{C} \). In general, \( \ker(T - \lambda)^m \subset \mathcal{H}_0(T - \lambda) \) and \( \mathcal{H}_0(T - \lambda) \) is not closed. However, if \( \lambda \) is an isolated point of \( \sigma(T) \), then \( E_T(\{\lambda\})\mathcal{H} = \mathcal{H}_0(T - \lambda) \) and \( \mathcal{H}_0(T - \lambda) \) is closed. Also, if \( T \) is normal and \( T = \int_{\sigma(T)} \lambda dF(\lambda) \) is the spectral decomposition of \( T \), then

\[
\mathcal{H}_0(T - \lambda) = E_T(\{\lambda\})\mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*.
\]

In 2012, J. T. Yuan and G. X. Ji ([12, Lemma 5.2]) proved following Lemma.

**Lemma 3.12.** Let \( T \in \mathcal{B}(\mathcal{H}) \), \( m \) be a positive integer and \( \lambda \) be an isolated point of \( \sigma(T) \).
(i) The following assertions are equivalent:
(a) \( E_T(\{\lambda\})\mathcal{H} = \ker(T - \lambda)^m \),
(b) \( \ker(E_T(\{\lambda\})) = (T - \lambda)^m\mathcal{H} \).

In this case, \( \lambda \) is a pole of the resolvent of \( T \) and the order of \( \lambda \) is not greater than \( m \).
(ii) If \( \lambda \) is a pole of the resolvent of \( T \) and the order of \( \lambda \) is \( m \), then the following assertions are equivalent:
(a) \( E_T(\{\lambda\}) \) is self-adjoint.
(b) \( \ker((T - \lambda)^m) \subset \ker((T - \lambda)^m)^* \).
(c) \( \ker((T - \lambda)^m) = \ker((T - \lambda)^m)^* \).

By this lemma, we prove the following theorem.

**Theorem 3.13.** Let \( T \in \mathcal{B}(\mathcal{H}) \) be square hyponormal and satisfy (\ast). Let \( \lambda \) be an isolated point of spectrum of \( T \). Then the following statements hold.
(i) If \( \lambda = 0 \), then \( \mathcal{H}_0(T) = \ker(T^2) = \ker(T^{2^2}) \), \( E_T(\{0\}) \) is self-adjoint and the order of pole \( \lambda \) is not greater than \( 2 \).
(ii) If \( \lambda \neq 0 \), then \( \mathcal{H}_0(T - \lambda) = \ker(T - \lambda) = \ker((T - \lambda)^*) \), \( E_T(\{\lambda\}) \) is self-adjoint and the order of pole \( \lambda \) is \( 1 \).

Proof. (i) Assume that \( \lambda = 0 \). Since \( \sigma(T^2) = \{z^2 : z \in \sigma(T)\} \), it follows that \( 0 \) is an isolated point of spectrum of \( T^2 \). We prove that \( \mathcal{H}_0(T) = \mathcal{H}_0(T^2) \). Let \( x \in \mathcal{H}_0(T) \). Then \( \|T^n x\|^{\frac{1}{n}} \to 0 \) and \( \|T^{2^n} x\|^{\frac{1}{2^n}} \to 0 \). Hence, \( x \in \mathcal{H}_0(T^2) \). Conversely, let \( x \in \mathcal{H}_0(T^2) \). Then \( \|T^{2^n} x\|^{\frac{1}{2^n}} \to 0 \) and so \( \|T^{2^n} x\|^{\frac{1}{2^n}} \to 0 \). From

\[
\frac{\|T^{2^n+1} x\|^{\frac{1}{2^n+1}}}{\|T\|^{\frac{1}{2^n}}} \leq \left( \frac{\|T^{2^n} x\|^{\frac{1}{2^n}}}{\|T\|^{\frac{1}{2^n}}} \right)^{\frac{1}{2^n}} \to 0 \ (n \to \infty),
\]

it follows that \( x \in \mathcal{H}_0(T) \). Therefore, \( \mathcal{H}_0(T) = \mathcal{H}_0(T^2) \). Since \( T^2 \) is hyponormal, we observe that \( E_{T^2}(\{0\})\mathcal{H} = \mathcal{H}_0(T^2) = \ker(T^2) \) by Stampfli [10]. So,

\[
E_T(\{0\})\mathcal{H} = \mathcal{H}_0(T) = \mathcal{H}_0(T^2) = E_{T^2}(\{0\})\mathcal{H} = \ker(T^2) = \ker(T^2).
\]

Now, \( 0 \) is a pole of the resolvent of \( T \), the order of \( T \) is not greater than \( 2 \) and \( E_T(\{0\}) \) is self-adjoint by Lemma 3.12.

(ii) Next we assume that \( \lambda \neq 0 \). Then \( \lambda^2 \) is an isolated point of \( \sigma(T^2) \) by Lemma 2.1 of [5]. We will prove \( \mathcal{H}_0(T - \lambda) = \mathcal{H}_0(T^2 - \lambda^2) \). Let \( x \in \mathcal{H}_0(T - \lambda) \). Then \( \|(T - \lambda)^n x\|^{\frac{1}{n}} \to 0 \) and

\[
\|((T^2 - \lambda^2)^n x)\|^{\frac{1}{n}} \leq \|((T + \lambda)^n (T - \lambda)^n x)\|^{\frac{1}{n}} \leq \|T + \lambda\|\|(T - \lambda)^n x\|^{\frac{1}{n}} \to 0,
\]

which implies \( \mathcal{H}_0(T - \lambda) \subset \mathcal{H}_0(T^2 - \lambda^2) \). Conversely, let \( x \in \mathcal{H}_0(T^2 - \lambda^2) \). Since \( T + \lambda \) is invertible by the assumption (\ast), we have

\[
\|(T - \lambda)^n x\|^{\frac{1}{n}} = \|(T + \lambda)^{-n} (T + \lambda)^n (T - \lambda)^n x\|^{\frac{1}{n}} \leq \|(T + \lambda)^{-1}\|^n \|(T^2 - \lambda^2)^n x\|^{\frac{1}{n}} \leq \|(T + \lambda)^{-1}\|^n \|T^2 - \lambda^2\|^n \|x\|^{\frac{1}{n}} \to 0.
\]
Hence, $\mathcal{H}_0(T - \lambda) \supset \mathcal{H}_0(T^2 - \lambda^2)$ and $\mathcal{H}_0(T - \lambda) = \mathcal{H}_0(T^2 - \lambda^2)$. Because $T^2$ is hyponormal, it follows that $E_T((\lambda^2))\mathcal{H} = \mathcal{H}_0(T^2 - \lambda^2) = \ker(T^2 - \lambda^2)$ by Stampfli [10]. Hence

$$E_T((\lambda))\mathcal{H} = \mathcal{H}_0(T - \lambda) = \mathcal{H}_0(T^2 - \lambda^2) = E_T((\lambda))\mathcal{H} = \ker(T^2 - \lambda^2) = \ker(T^2 - \lambda^2).$$

Since $(T + \lambda)^*\mathcal{H}$ is invertible, we get

$$E_T((\lambda))\mathcal{H} = \ker(T - \lambda) = \ker((T - \lambda)^*).$$

Thus, $\lambda$ is a pole of the resolvent of $T$, the order of $\lambda$ is not greater than 2 and $E_T((\lambda))\mathcal{H}$ is self-adjoint by Lemma 3.12.

Let $D$ be a bounded open subset of $C$ and $L^2(D, \mathcal{H})$ be the Hilbert space of measurable function $f : D \rightarrow \mathcal{H}$ such that

$$\|f\| = \left( \int_D \|f(z)\|^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty,$$

where $\mu$ is the planar Lebesgue measure. Let $W^2(D, \mathcal{H})$ be the Sobolev space with respect to $\partial_0$ and of order 2 whose derivatives $\partial f$ and $\partial^2 f$ in the sense of distributions belong to $L^2(D, \mathcal{H})$. The norm $\|f\|_{W^2}$ is given by

$$\|f\|_{W^2} = \left( \|f\|^2 + \|\partial f\|^2 + \|\partial^2 f\|^2 \right)^{\frac{1}{2}} \quad \text{for } f \in L^2(D, \mathcal{H}).$$

In [4], Alzuraqi and Patel proved the following.

Proposition 3.14. (Alzraqi and Patel [4], Theorem 2.37) Let $D$ be an arbitrary bounded disk in $C$. If $T \in B(\mathcal{H})$ is 2-normal with the assumption $\sigma(T) \cap (-\sigma(T)) = \emptyset$, then the operator

$$z - T : W^2(D, \mathcal{H}) \rightarrow L^2(D, \mathcal{H})$$

is one to one for every $z \in C$.

We would like to prove this result as follows.

Theorem 3.15. Let $D$ be an arbitrary bounded disk in $C$ and $T \in B(\mathcal{H})$ be square hyponormal with $(\ast)$. Then the operator

$$z - T : W^2(D, \mathcal{H}) \rightarrow L^2(D, \mathcal{H})$$

is one to one for every $z \in C$.

Proof. Let $f \in W^2(D, \mathcal{H})$, $S = z - T$ and $Sf = 0$. We show $f = 0$. Then

$$\|f\|_{W^2} = \|f\|_{L^2,D}^2 + \|\partial f\|_{L^2,D}^2 + \|\partial^2 f\|_{L^2,D}^2$$

$$= \int_D \|f(z)\|^2 \, d\mu(z) + \int_D \|\partial f(z)\|^2 \, d\mu(z) + \int_D \|\partial^2 f(z)\|^2 \, d\mu(z) < \infty,$$

and

$$\|Sf\|_{W^2} = \|(z - T)f\|_{W^2}$$

$$= \|(z - T)f\|_{L^2,D}^2 + \|\partial((z - T)f)\|_{L^2,D}^2 + \|\partial^2((z - T)f)\|_{L^2,D}^2$$

$$= \|(z - T)f\|_{L^2,D}^2 + \|(z - T)\partial f\|_{L^2,D}^2 + \|(z - T)^2 f\|_{L^2,D}^2 = 0.$$
Hence,
\[ \|(z - T)^i f\|_{2,D}^2 = \int_D \|(z - T)^i f(z)\|^2 d\mu(z) = 0 \quad (i = 0, 1, 2). \]

Let \( i \) be \( i = 0, 1, 2 \). Since \((z - T)^i f(z) = 0\) for \( z \in D \), if \( z \in D \setminus \sigma(T) \), then \((z - T)^i f(z) = 0\) because \( z - T \) is invertible. This implies
\[ \|(z - T)^i f\|_{2,D\setminus\sigma(T)}^2 = \int_{D\setminus\sigma(T)} \|(z - T)^i f(z)\|^2 d\mu(z) = 0. \]

Since
\[ \|(z^2 - T^2)^i f\|_{2,D}^2 = \int_D \|(z^2 - T^2)^i f(z)\|^2 d\mu(z) \]
\[ \leq \left( \sup_{z \in D} |z + T| \right)^2 \int_D \|(z - T)^i f(z)\|^2 d\mu(z) \]
\[ = \left( \sup_{z \in D} |z + T| \right)^2 \|(z - T)^i f\|_{2,D}^2 = 0, \]
we have \((z^2 - T^2)^i f(z) = 0\) for \( z \in D \). Because \( T^2 \) is hyponormal, then
\[ \int_D \|(z^2 - T^2)^i f(z)\|^2 d\mu(z) = \|(z^2 - T^2)^i f\|_{2,D}^2 \leq \|(z^2 - T^2)^i f\|_{2,D}^2 = 0. \]
So,
\[ 0 = (z^2 - T^2)^i f(z) = (z + T)^i(z - T)^i f(z) \quad \text{for} \quad z \in D. \]

If \( z \in D \cap (\sigma(T) \setminus (-\sigma(T))) \), then \( z + T \) and \((z + T)^i \) are invertible. Hence, \((z - T)^i f(z) = 0\) for \( z \in D \cap (\sigma(T) \setminus (-\sigma(T))) \). Since \( D \) is bounded, \( \|(z^2 - T^2)^i f\|_{2,D}^2 < \infty \) and the planar Lebesgue measure of \( \sigma(T) \cap (-\sigma(T)) \) is 0, we have
\[ \|(z - T)^i f\|_{2,D}^2 = \int_{\sigma(T)} \|(z - T)^i f(z)\|^2 d\mu(z) \]
\[ + \int_{\sigma(T)^c} \|(z - T)^i f(z)\|^2 d\mu(z) \]
\[ = 0 + \max_{z \in D} \|(z - T)^i f(z)\|^2 \int_{\sigma(T)^c} \|f(z)\|^2 d\mu(z) = 0. \]

By [9, Proposition 2.1], we obtain \( \|(I - P)f\|_{2,D} = 0 \). Thus, \( f(z) = (Pf)(z) \) for \( z \in D \). From \( Sf = 0 \), we have \((Sf)(z) = (z - T)f(z) = (z - T)(Pf)(z) = 0\) for \( z \in D \).

Since \( T \) has the single-valued extension property by Corollary 3.4 and \( Pf \) is analytic, it follows that \( 0 = (Pf)(z) = f(z) \) for \( z \in D \). Hence, \( f = 0 \) and \( S \) is one to one.

An operator \( T \in B(H) \) is said to be polaroid if every isolated point of the spectrum of \( T \) is a pole of the resolvent. In [1], Aiena showed that if \( T \) is algebraically paranormal on a Banach space, then the following results hold.

(1) \( T \) is polaroid (Theorem 1.3).
(2) If \( T \) is quasinilpotent, then \( T \) is nilpotent (Lemma 1.2).

Hence, it is clear that if \( T \in B(H) \) is square hyponormal, then \( T \) is polaroid.
4. nth hyponormal operators

We now introduce and study nth hyponormal operators.

**Definition 4.1.** For \( n \in \mathbb{N} \) and an operator \( T \in \mathcal{B}(\mathcal{H}) \), \( T \) is said to be nth hyponormal if \( T^n \) is hyponormal.

As Theorem 2.3, we can verify the following result.

**Theorem 4.2.** Let \( n \in \mathbb{N} \), \( T \in \mathcal{B}(\mathcal{H}) \) be nth hyponormal and \( M \) be an invariant closed subspace for \( T \). Then \( T|_M \) is nth hyponormal.

For an nth hyponormal operator \( T \in \mathcal{B}(\mathcal{H}) \), we consider the following property:

\[
\sigma(T) \cap \left( \bigcup_{j=1}^{n-1} e^{\frac{2\pi i j}{n}} \sigma(T) \right) \subset \{0\}.
\]

**Theorem 4.3.** Let \( n \in \mathbb{N} \), \( T \in \mathcal{B}(\mathcal{H}) \) be nth hyponormal with \((**)\) and \( M \) be an invariant subspace for \( T \). If \( \sigma(T|_M) = \{z\} \), then the following assertions hold.
1. If \( z = 0 \), then \( (T|_M)^n = 0 \).
2. If \( z \neq 0 \), then \( T|_M = z \).

**Proof.** (1) By Theorem 4.2, \( T|_M \) is nth hyponormal. Since \( \sigma((T|_M)^n) = \{0\} \), by Putnam’s theorem, we conclude that \( (T|_M)^n = 0 \).
(2) Because \( \sigma((T|_M)^n) = \{z^n\} \), then \( (T|_M)^n = z^n \) and so

\[
0 = (T|_M)^n - z^n = (T|_M - e^{\frac{2\pi i}{n}} z)(T|_M - e^{\frac{2\pi i 2}{n}} z) \cdots (T|_M - e^{\frac{2\pi i (n-1)}{n}} z)(T|_M - z).
\]

From \( z \neq 0 \) and \((**)\), there exists \( (T|_M - e^{\frac{2\pi i j}{n}} z)^{-1} \), for every \( j = 1, \ldots, n-1 \), and thus \( T|_M - z = 0 \).

**Theorem 4.4.** Let \( n \in \mathbb{N} \) and \( T \in \mathcal{B}(\mathcal{H}) \) be an nth hyponormal operator. If \( T \) satisfies \((**)\), then \( \sigma(T) = \{z \in \sigma_a(T^n)\} \).

**Proof.** Because \( \sigma(T) = \sigma_a(T) \cup \sigma_r(T) \), we verify that \( \sigma_a(T) \subset \{z \in \sigma_a(T^n)\} \).
1. If \( 0 \in \sigma_a(T) \), then \( 0 \in \sigma_a(T^n) \) and, because \( T^n \) is hyponormal, we can get \( 0 \in \sigma_a(T^n) \).
2. For \( z \in \sigma_a(T) \) and \( z \neq 0 \), there exists a sequence \( \{x_m\} \) of unit vectors such that \( (T - z)x_m \to 0 \) as \( m \to \infty \). We observe that \( (T^n - z^n)x_m = (T^{n-1} - z^{n-1})(T - z)x_m \to 0 \) as \( m \to \infty \) and \( T^n \) is hyponormal, which gives \( (T^n - z^n)x_m \to 0 \) as \( m \to \infty \). By the hypothesis \((**)\) and \( z \) is non-zero, all operators \( (T^n - e^{\frac{2\pi i j}{n}} z), (T^n - e^{\frac{2\pi i j}{n}} z), \ldots, (T^n - e^{\frac{2\pi i (n-1)}{n}} z) \) are invertible. Hence, by \( T^n - z^n = (T^n - e^{\frac{2\pi i j}{n}} z)(T^n - e^{\frac{2\pi i j}{n}} z) \cdots (T^n - e^{\frac{2\pi i (n-1)}{n}} z) \), we find that \( (T^n - z^n)x_m \to 0 \) as \( m \to \infty \), that is, \( z \in \sigma_a(T^n) \), which completes the proof.

**Theorem 4.5.** Let \( n \in \mathbb{N} \) and \( T \in \mathcal{B}(\mathcal{H}) \) be nth hyponormal satisfying \((**)\).
1. If \( z \) and \( w \) are distinct eigen-values of \( T \) and \( x, y \in \mathcal{H} \) are corresponding eigen-vectors, respectively, then \( \langle x, y \rangle = 0 \).
2. If \( z, w \) are distinct values of \( \sigma_a(T) \) and \( \{x_m\}, \{y_m\} \) are sequences of unit vectors in \( \mathcal{H} \) such that \( (T - z)x_m \to 0 \) and \( (T - w)y_m \to 0 \) as \( m \to \infty \), then \( \lim_{m \to \infty} \langle x_m, y_m \rangle = 0 \).

**Proof.** Since (1) follows from (2), we will only prove (2). From \( (T - z)x_m \to 0 \) and \( (T - w)y_m \to 0 \) as \( m \to \infty \), we get \( (T^n - z^n)x_m \to 0 \) and \( (T^n - w^n)y_m \to 0 \). Further, because \( T^n \) is hyponormal, \( (T^n - w^n)y_m \to 0 \). Therefore,

\[
\lim_{m \to \infty} z^n(x_m, y_m) = \lim_{m \to \infty} \langle z^n x_m, y_m \rangle = \lim_{m \to \infty} \langle T^n x_m, y_m \rangle = \lim_{m \to \infty} \langle x_m, T^n y_m \rangle = \lim_{m \to \infty} w^n(x_m, y_m).
\]

In the case that \( z^n = w^n \), by \( 0 = z^n - w^n = (z - w)(z - e^{\frac{2\pi i}{n}} w)(z - e^{\frac{2\pi i 2}{n}} w) \cdots (z - e^{\frac{2\pi i (n-1)}{n}} w) \), \( z \neq w \) and \((**)\), we deduce that \( z = w = 0 \). So, \( z^n \neq w^n \) and \( \lim_{m \to \infty} \langle x_m, y_m \rangle = 0 \).
Corollary 4.6. Let \( n \in \mathbb{N} \) and \( T \in B(H) \) be \( n \)th hyponormal satisfying \((**)\). If \( z \) and \( w \) are distinct eigen-values of \( T \), then \( \ker(T - z) \perp \ker(T - w) \).

Corollary 4.7. Let \( n \in \mathbb{N} \) and \( T \in B(H) \) be \( n \)th hyponormal satisfying \((**)\). Then \( T \) has SVEP.

In a similar manner as Theorem 3.10, we prove the next result.

Theorem 4.8. Let \( n \in \mathbb{N} \) and \( T \in B(H) \) be \( n \)th hyponormal satisfying \((**)\). If \( z \) is a non-zero eigen-value of \( T \), then \( \ker(T - z) = \ker(T^n - z^n) \subset \ker(T^n - z^n) = \ker(T^n - z^n) \) and hence \( \ker(T - z) \) is a reducing subspace for \( T \).

As Theorem 3.13 and Theorem 3.15, we can verify the following theorems.

Theorem 4.9. Let \( n \in \mathbb{N} \) and \( T \in B(H) \) be \( n \)th hyponormal satisfying \((**)\). Let \( \lambda \) be an isolated point of spectrum of \( T \). Then the following statements hold.

(i) If \( \lambda = 0 \), then \( \mathcal{H}_0(T) = \ker(T^n) = \ker(T^n), \ E_T(\{0\}) \) is self-adjoint and the order of pole \( \lambda \) is not greater than \( n \).

(ii) If \( \lambda \neq 0 \), then \( \mathcal{H}_0(T - \lambda) = \ker(T - \lambda) = \ker((T - \lambda)^n), \ E_T(\{\lambda\}) \) is self-adjoint and the order of pole \( \lambda \) is 1.

Theorem 4.10. Let \( D \) be an arbitrary bounded disk in \( \mathbb{C} \), \( n \in \mathbb{N} \) and \( T \in B(H) \) be \( n \)th hyponormal satisfying \((**)\). Then the operator

\[
z - T : W^2(D; \mathcal{H}) \longrightarrow L^2(D; \mathcal{H})
\]

is one to one for every \( z \in \mathbb{C} \).

Acknowledgment. Authors would like to express their thanks to Prof. K. Tanahashi for his important suggestion of Theorem 3.15.

References


