1. Introduction and Preliminary

Let $X$ be a Banach lattice. If $\mathcal{F}$ is a collection of continuous linear operators with modulus on $X$, we define $|\mathcal{F}| = \{|T|; T \in \mathcal{F}\}$, where $|T|$ denotes the modulus of the operator $T$. A collection $C$ of continuous linear operators with modulus on $X$ is said to be finitely modulus-quasinilpotent at a vector $x \in X$ if
\[
\lim_{n \to \infty} \||\mathcal{F}|^n x_n\|^{1/n} = 0
\]
for every finite subset $\mathcal{F}$ of $C$.

If $A$ and $B$ are continuous linear operators on $X$ with $B$ positive, then $A$ is said to be dominated by $B$ whenever $|Ax| \leq B(|x|)$ holds for all $x \in X$.

Let $C$ be a collection of continuous positive operators on $X$, then $C'_+ \subseteq C'$ denotes the set of all continuous positive operators $S$ on $X$ such that $TS = ST$ for all $T \in C$. We say that $C'_+$ is the positive commutant of $C$.

Let $C$ be a collection of continuous linear operators with modulus on $X$, then $C'_m \subseteq C'$ denotes the set of all continuous linear operators $S$ with modulus on $X$ such that $|T||S| \leq |S||T|$ for all $T \in C$. We say that $C'_m$ is the modulus sub-commutant of $C$.

It is easy to see that if $C$ is a collection of positive operators on a Banach lattice, then $C'_+ \subseteq C'_m$, and "finitely quasi-nilpotent" and "finitely modulus-quasinilpotent" are equivalent.

A vector subspace $I$ of $X$ is said to be an (order) ideal whenever $|x| \leq |y|$ and $y \in I$ imply that $x \in I$. The ideal generated by a non-empty subset $F$ of $X$ is defined by $I_F = \{x \in X; \text{there are } x_1, \ldots, x_n \in F \text{ and } \lambda_1, \ldots, \lambda_n > 0 \text{ such that } |x| \leq \sum_{i=1}^n \lambda_i |x_i|\}$. In particular, the ideal generated by a singleton $\{x\}$ is given by $I_x = \{y \in X; \text{there is } \lambda > 0 \text{ such that } |y| \leq \lambda |x|\}$.

It is well known that if $X$ is a Banach space $X$ with an unconditional basis, then $X$ may be regarded as a Banach lattice whenever one is looking for invariant subspaces and invariant ideals. For each given positive integer $n$, define the functional $f_n$ by $f_n(x) = \alpha_n$ for every $x = \sum_{k=1}^n \alpha_k e_k$. Then $f_n$ is a continuous linear functional on $X$.
In 1954, N. Aronszajn and K. T. Smith [1] showed that every compact operator on a Banach space has a non-trivial invariant closed subspace.

But it was not until 1986 that people solved the invariant closed ideal problem for a special class of compact operators. To be more specific, B. de Pagter [7] proved that every positive quasinilpotent compact operator on a Banach lattice has a non-trivial invariant closed ideal. It is well known that Pagter’s result is an affirmative answer of a long standing open question (cf. [1], [2], [5] and [7]).

In 2007, M. Liu [3] showed that if \( C \neq \{0\} \) is a collection of continuous positive operators on a Banach space with a Schauder basis that is finitely quasinilpotent at a non-zero positive vector, then \( C \) and its positive commutant \( C' \) have a common non-trivial invariant closed subspace.

In this paper, we shall extend the result in [3] from the invariant closed subspace for collections of positive operators to the invariant closed ideal for collections of operators with modulus (may be non-positive operators). This paper can be seen as a sequel to [3].

It is well known that the non-trivial invariant closed ideal for any operator is necessarily its non-trivial invariant closed subspace and each positive has the modulus, but their converses are not necessarily true. Moreover, in section 3, we will give a collection of nonpositive continuous linear operators that satisfies the condition of our main result.

2. The Main Result

Now we are in a position to give the main result.

**Theorem 1.** Let \( X \) be a Banach space with an unconditional basis \( \{e_n\} \), and let \( C \neq \{0\} \) be a collection of continuous linear operators with modulus on \( X \) that is finitely modulus-quasinilpotent at a non-zero positive operator \( x \). Then \( C \) and its right modulus sub-commutant \( C'_d \) have a common non-trivial invariant closed subspace.

**Proof.** As in [3], it follows from \( x > 0 \) that there are an appropriate scalar \( \lambda > 0 \) and a positive integer \( n \) such that \( \lambda x_n \geq e_n > 0 \). It is clear that \( C \) is finitely modulus-quasinilpotent at \( \lambda x_n \). Let \( G \) be the multiplicative semigroup generated by \( \mathbb{C} \) (i.e. \( G = \bigcup_{n=1}^{\infty} |C|^n \)), and let \( \mathcal{A} \) be the algebra of all continuous linear operators on \( X \) such that each \( A \in \mathcal{A} \) is dominated by some operator of the form \( \sum_{i=1}^{n} |S_i|G_i \) with \( S_i \in C'_d \) and \( G_i \in G \).

We consider two cases separately.

Case 1. If there is an operator \( S \in \mathcal{A} \) such that \( A_i e_n \neq 0 \), then the ideal \( \mathcal{I}_{Ae_n} \) generated by \( A e_n \) is a non-zero ideal in \( X \), where \( \mathcal{A}_{e_n} := \{ A e_n : A \in \mathcal{A} \} \).

First we show that \( \overline{\mathcal{I}_{Ae_n}} \neq X \). As in [3], let \( P \) denote the natural projection from \( X \) onto the vector subspace generated by \( e_n \). By a modification of the corresponding part of [3], we can prove that

\[
P(S|G)e_n = 0
\]

for all \( S \in C'_d \) and all \( G \in G \). (Indeed, it suffices to replace \( S, T, \mathcal{F} \) and \( C \) by \( |S|, |T|, |\mathcal{F}| \) and \( |C| \) respectively.) For every \( x \in \mathcal{I}_{Ae_n} \), the definition of \( \mathcal{I}_{Ae_n} \) implies that there are operators \( A_1, A_2, \ldots, A_m \in \mathcal{A} \) such that \( x = \sum_{i=1}^{m} |A_i e_n| \) and so \( x^+ \leq \sum_{i=1}^{m} |A_i e_n| \) and \( x^- \leq \sum_{i=1}^{m} |A_i e_n| \). For each \( i = 1, 2, \ldots, m \), by the definition of \( \mathcal{A} \) there are operators \( S_{ij} \in C'_d \), \( G_{ij} \in G \) (\( j = 1, 2, \ldots, n(i) \)) such that \( |A_i e_n| \leq \sum_{j=1}^{n(i)} |S_{ij}|G_{ij}e_n \). Thus we have

\[
x^+ \leq \sum_{i=1}^{m} |A_i e_n| \leq \sum_{i=1}^{m} \sum_{j=1}^{n(i)} |S_{ij}|G_{ij}e_n.
\]

Thus by (1) we obtain \( P(x^+) \leq \sum_{i=1}^{m} \sum_{j=1}^{n(i)} |S_{ij}|G_{ij}e_n = 0 \), and so \( P(x^+) = 0 \). Hence it is easy to obtain that \( f_n(x^+) = f_n(Px^+) = 0 \). Similarly, \( f_n(x^-) = 0 \). Thus we have \( f_n(x) = 0 \) for every \( x = x^+ - x^- \in \mathcal{I}_{Ae_n} \). For the complex space \( X \), we can obtain \( f_n((\text{Re} x)^+) = f_n((\text{Im} x)^+) = f_n((\text{Im} x)^-) = 0 \), thus we have \( f_n(x) = 0 \) for every \( x = (\text{Re} x)^+ - (\text{Re} x)^- + i((\text{Im} x)^+ - (\text{Im} x)^-) \in \mathcal{I}_{Ae_n} \). Consequently \( f_n(x) = 0 \) for every \( x \in \mathcal{I}_{Ae_n} \). Thus by \( f_n(e_n) = 1 \), we obtain \( \overline{\mathcal{I}_{Ae_n}} \neq X \).
We now prove that \( \mathcal{J} \) is invariant under \( \mathcal{C} \) and \( \mathcal{C}' \). To this end, take \( x \in \mathcal{J} \), \( T \in \mathcal{C} \) and \( S \in \mathcal{C}' \). Then by (2) we obtain

\[
|Tx^*| \leq |T(x^*)| \leq \sum_{j=1}^{m} \left( \sum_{i=1}^{n} |T||S_{ij}||G_{ij}|e_n \right) e_n.
\]

(3)

It is easy to see that \( |T||G_{ij}| \in \mathcal{G} \), and so \( \sum_{j=1}^{m} |T||S_{ij}||G_{ij}| \in \mathcal{A} \). Thus by (3) we obtain \( Tx^* \in \mathcal{J} \). Similarly, \( Tx^- \in \mathcal{J} \), and so \( Tx \in \mathcal{J} \)

Again, by (2) we obtain

\[
|Sx^*| \leq |Sx^*| \leq \sum_{j=1}^{m} \left( \sum_{i=1}^{n} |S||S_{ij}||G_{ij}|e_n \right) e_n.
\]

(4)

Since \( |S||S_{ij}| \in \mathcal{C}' \), we have \( \sum_{j=1}^{m} |S||S_{ij}||G_{ij}| \in \mathcal{A} \). Thus by (4) we obtain \( Sx^* \in \mathcal{J} \). Similarly, \( Sx^- \in \mathcal{J} \), and so \( Sx \in \mathcal{J} \).

From the above we see that \( \mathcal{J} \) is a common non-trivial invariant closed ideal for \( \mathcal{C} \) and \( \mathcal{C}' \).

Case 2. If \( \mathcal{J} \) is non-trivial closed ideal in \( \mathcal{X} \). The identity operator \( I \) is a common non-trivial invariant closed ideal for \( \mathcal{C} \) and \( \mathcal{C}' \). To this end, take \( x \in \mathcal{J} \), \( T \in \mathcal{C} \) and \( S \in \mathcal{C}' \). For any \( A \in \mathcal{X} \), it follows from the definition of \( \mathcal{A} \) that there are operators \( S_1, S_2, \cdots, S_n \in \mathcal{C}' \) and \( G_1, G_2, \cdots, G_n \in \mathcal{G} \) such that \( |Ay| \leq \sum_{j=1}^{n} |S_j||G_j|(|y|) \) for all \( y \in \mathcal{X} \). Thus we have

\[
|A(|Tx|)| \leq \sum_{j=1}^{n} |S_j||G_j|(|Tx|) \leq \sum_{j=1}^{n} |S_j||G_j||(|x|).
\]

(5)

Observing \( G_j ||T_j| \in \mathcal{G} \), we see that \( \sum_{j=1}^{n} |S_j||G_j|T_j| \in \mathcal{A} \). Since \( x \in \mathcal{J} \), it follows that \( \sum_{j=1}^{n} |S_j||G_j|(|x|) = 0 \). Thus by (5) we have \( |A(|Tx|)| = 0 \), and so \( |A|T| = 0 \) for all \( A \in \mathcal{A} \). Consequently \( Tx \in \mathcal{J} \). Similarly, we have

\[
|A(|Sx|)| \leq \sum_{j=1}^{n} |S_j||G_j|(|Sx|) \leq \sum_{j=1}^{n} |S_j||G_j||(|x|).
\]

Since \( G_j \in \mathcal{G} \), \( G_j \) is an operator of the form \( |T_j||T_j| \cdots |T_j| \) where \( T_j, T_j, \cdots, T_j \in \mathcal{C} \). Thus we obtained

\[
|A(|Sx|)| \leq \sum_{j=1}^{n} |S_j||T_j||T_j| \cdots |T_j||(|x|)
\]

\[
\leq \sum_{j=1}^{n} |S_j||S||T_j||T_j| \cdots |T_j||(|x|) = \sum_{j=1}^{n} |S_j||S||G_j|(|x|).
\]

(6)

Since \( |S_j||S| \in \mathcal{C}' \), it follows that \( \sum_{j=1}^{n} |S_j||S||G_j| \in \mathcal{A} \). Thus by \( x \in \mathcal{J} \) we obtain \( \sum_{j=1}^{n} |S_j||S||G_j|(|x|) = 0 \). Thus by (6) we have \( |A(|Sx|)| = 0 \), and so \( |A|S| = 0 \) for all \( A \in \mathcal{A} \). Consequently \( Sx \in \mathcal{J} \).

From the above we conclude that \( \mathcal{J} \) is a common non-trivial invariant closed ideal for \( \mathcal{C} \) and \( \mathcal{C}' \), and this completes the proof.

3. An example

We conclude this paper with the following example for a non-commutative finitely modulus-quasinilpotent collection \( \mathcal{C} \) of continuous non-positive operators that satisfies the conditions of Theorem 1.
Let $T_a$, $S_a$ and $B_a$ be operators on the sequence space $\ell^p$ ($1 \leq p < \infty$) with matrix respectively

\[
\begin{pmatrix}
  a_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  a_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & \frac{1}{2}a_2 & 0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & \frac{1}{3}a_3 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 0 & \frac{1}{4}a_4 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & \frac{1}{5}a_5 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
  0 & a_1 & 0 & 0 & 0 & 0 & \cdots \\
  0 & 0 & a_2 & 0 & 0 & 0 & \cdots \\
  0 & 0 & 0 & a_3 & 0 & 0 & \cdots \\
  0 & 0 & 0 & 0 & a_4 & 0 & \cdots \\
  0 & 0 & 0 & 0 & 0 & a_5 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where $a = (a_0, a_1, a_2, \cdots)$, $a_k = 1$ or $a_k = -1$. Set $C = [T_a, S_a; a = (a_0, a_1, a_2, \cdots), a_k = 1$ or $a_k = -1]$. Then the collection $C$ of operators satisfies our demands. Indeed, as in [3], we can show that $T_a S_a e_1 \neq S_a T_a e_1$ and $\lim_{n \to \infty} \|F^n e_1\|^{1/n} = 0$ for every subset $F$ of $C$, where $e_n$ denotes the vector in $\ell^p$ whose $n$-th component is one and every other zero.

Moreover, it is clear that $[B_a; a = (a_0, a_1, a_2, \cdots), a_k = 1$ or $a_k = -1] \subset C'_{BR}$. Thus by Theorem 1 all operators in the collection $[T_a, S_a, B_a; a = (a_0, a_1, a_2, \cdots), a_k = 1$ or $a_k = -1]$ of non-positive operators have a common non-trivial invariant closed ideal.

It should be noticed that operators in the main result of [3] are positive, while operators in Example 1 and the main result of this paper may be non-positive.

References