Further Results on Hybrid \((b, c)\)-Inverses in Rings

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Abstract. Let \(R\) be a ring and \(b, c \in R\). In this paper, the absorption law for the hybrid \((b, c)\)-inverse in a ring is considered. Also, by using the Green’s preorders and relations, we obtain the reverse order law of the hybrid \((b, c)\)-inverse. As applications, we obtain the related results for the \((b, c)\)-inverse.

1. Introduction

Core inverse, dual core inverse, and Mary inverse, as well as classical generalized inverses, are special types of outer inverses. In [2], Drazin introduced a new class of outer inverse and called it \((b, c)\)-inverse, which encompasses the above-mentioned generalized inverses.

Definition 1.1. Let \(R\) be an associative ring and let \(b, c \in R\). An element \(a \in R\) is \((b, c)\)-invertible if there exists \(y \in R\) such that

\[y \in (bRy) \cap (yRc), \quad yab = b, \quad cay = c.\]

If such \(y\) exists, it is unique and is denoted by \(a \parallel (b \triangleleft c)\). Drazin [2] also presented an equivalent characterization for the \((b, c)\)-inverse \(y\) of \(a\) as \(yay = y, \ yR = bR\) and \(Ry = Rc\).

As generalizations of \((b, c)\)-inverses, hybrid \((b, c)\)-inverses and annihilator \((b, c)\)-inverses were introduced in [2]. The symbols \(\text{lann}(a) = \{g \in R : ga = 0\}\) and \(\text{rann}(a) = \{h \in R : ah = 0\}\) denote the sets of all left annihilators and right annihilators of \(a\), respectively.

Definition 1.2. Let \(a, b, c, y \in R\). We say that \(y\) is a hybrid \((b, c)\)-inverse of \(a\) if

\[yay = y, \quad yR = bR, \quad \text{rann}(y) = \text{rann}(c).\]

If such \(y\) exists, it is unique. In this article, we use the symbol \(a \parallel (b \{a\} c)\) to denote the hybrid \((b, c)\)-inverse of \(a\).

Definition 1.3. Let \(a, b, c, y \in R\). We say that \(y\) is an annihilator \((b, c)\)-inverse of \(a\) if

\[yay = y, \quad \text{lann}(y) = \text{lann}(b), \quad \text{lann}(y) = \text{lann}(c).\]
2. Absorption law for the hybrid $(b, c)$-inverse

Let $a, b \in R$ be two invertible elements. It is well known that

$$a^{-1} + b^{-1} = a^{-1}(a + b)b^{-1}.$$ 

The above equality is known as the absorption law of invertible elements. In general, the absorption law does not hold for generalized inverses (see [9, 10]). In this section, the absorption laws for the hybrid $(b, c)$-inverse are obtained. For future reference we state some known results.

**Lemma 2.1.** ([14, Proposition 2.1]) Let $a, b, c, y \in R$. Then the following conditions are equivalent:

(i) $a$ is hybrid $(b, c)$-invertible and $y$ is the hybrid $(b, c)$-inverse of $a$.

(ii) $yab = b$, $cay = c$, $yR \subseteq bR$ and $rann(c) \subseteq rann(y)$.

**Lemma 2.2.** ([2, P.1992]) Let $a, b, c \in R$. Then $a$ has a hybrid $(b, c)$-inverse if and only if $c \in cabR$ and $rann(cab) \subseteq rann(b)$.

**Lemma 2.3.** Let $a, b, c, d \in R$. If $a^{(b, c)}$ is the hybrid $(b, c)$-inverse of $a$ and $d^{(b, c)}$ is the hybrid $(b, c)$-inverse of $d$, then $a^{(b, c)} + d^{(b, c)} = a^{(b, c)} + d^{(b, c)}$.

**Proof.** Let $x = a^{(b, c)}$ and $y = d^{(b, c)}$. Then by Lemma 2.1, we have $y \in bR$ and $xab = b$. This gives that $y = bs$ for some $s \in R$, and $sab = s$ and $y = y$. Moreover, by Lemma 2.1, we have $c = cax = cdy$, which means that $ax - dy \in rann(c)$. Note that since $rann(c) \subseteq rann(y)$, it follows $y(ax - dy) = 0$ and $yax = ydy = y$. Here, we prove that $d^{(b, c)} = d^{(b, c)} + d^{(b, c)}$. Similarly, we can also get $d^{(b, c)} = d^{(b, c)} + d^{(b, c)}$. □

Next, we will consider when $d$ is hybrid $(b, c)$-invertible if $a^{(b, c)}$ exists. In fact, whether we discuss about the absorption law or the reverse order law for the hybrid $(b, c)$-inverse, we always assume that $a$ and $d$ are both hybrid $(b, c)$-invertible first. Moreover, this kind of problems frequently were studied in optimization theory. It is of interest to know that, in $C^{s}$ algebras, if $a$ contains some properties, wether $d$ and $a + \epsilon$ also contains the similar properties when $\epsilon \to 0$. In the following, we will give existence criteria for the hybrid $(b, c)$-inverse of $d$, when $a$ is hybrid $(b, c)$-invertible. By Lemma 2.1, it is easy to conclude that if $a$ is hybrid $(b, c)$-invertible, then $b$ is regular. An element $a \in R$ is called (von Neumann) regular if there exists $x \in R$ such that $a = axa$. Such an $x$ is called an inner inverse of $a$ and is denoted by $a^{-}$. Before we investigate the existence criteria for the hybrid $(b, c)$-inverse, the following lemma is necessary.

**Lemma 2.4.** ([8]) Let $a, e \in R$ with $c^{2} = e$. Then the following statements are equivalent:

(i) $e \in eaeR \cap Reae$.

(ii) $eae + 1 - e$ is invertible (or $ae + 1 - e$ is invertible).
Theorem 2.5. Let $a, b, c, d \in R$. Assume that $a^{||b, c||}$ exists. Let $b^-$ be any inner inverses of $b$ and set $e = bb^-$. Then the following statements are equivalent:

(i) $d$ has a hybrid $(b, c)$-inverse.
(ii) $e \in e^{||b, c||}dR \cap R^{-1}e^{||b, c||}de$.
(iii) $a^{||b, c||}d e + 1 - e$ is invertible.

In this case, $d^{||b, c||} = (a^{||b, c||}d e + 1 - e)^{-1}a^{||b, c||}$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $d^{||b, c||}$ exists. Let $x = a^{||b, c||}$ and $y = d^{||b, c||}$. It follows from Lemma 2.3 that $x = x a y$ and $y = y a x$. As $a^{||b, c||}$ exists, we have $x \in bR$, $y \in bR$ and $b = x a b$ by Lemma 2.1. Therefore $b = x a b = e(x a y) a b = e x a y$ since $e y = y$. Multiplying on the right by $b^-$ gives $e = e x a y e d e$ and $e \in e x d e R$. Moreover, as $d^{||b, c||}$ exists we have $y \in bR$ and $b = y d b$. Therefore $b = y d b = e(y a x) d b = e y a x d b$ since $e x = x$. Multiplying on the right by $b^-$ we obtain $e = e y a x d e$ and $e \in e R x e d$.

(ii) $\Rightarrow$ (i) See Lemma 2.4.

(iii) $\Rightarrow$ (i) Set $x = a^{||b, c||}$. Firstly we note that $e x = x b R$. Set $u = e x d e + 1 - e$. It is clear that $e u = u e$ and $e u r^{-1} = u^{-1} e$. Write $y = u^{-1} x$. Next, we verify that $y$ is the hybrid $(b, c)$-inverse of $d$.

Step 1. $y d y = y$. Indeed, using $e x = x$ and $e u r^{-1} = u^{-1} e$, we can check that

$$y d y = u^{-1} x = u^{-1} e x d e u^{-1} x$$
$$= u^{-1} (e x d e + 1 - e) e u^{-1} x$$
$$= e u^{-1} x = u^{-1} x = y.$$

Step 2. $b R = y R$. Indeed, from $(1 - e) b = 0$ and $x = e x$, we have

$$b = u^{-1} u b = u^{-1} (e x d e + 1 - e) b = u^{-1} e x d e b = u^{-1} x d e b = y d b \in y R$$

Meanwhile, $y = u^{-1} x = u^{-1} e x \in e u r^{-1} e x = b R$. This shows that $b R = y R$.

Step 3. $\text{rann}(c) = \text{rann}(y)$. Since $u$ is invertible element in $R$, we have $\text{rann}(y) = \text{rann}(x)$. Moreover, from Lemma 2.1, we have $\text{rann}(x) = \text{rann}(c)$. This leads to $\text{rann}(c) = \text{rann}(y)$.

Next, the absorption law for the hybrid $(b, c)$-inverse is given when $a$ and $d$ are both hybrid $(b, c)$-invertible.

Theorem 2.6. Let $a, b, c, d \in R$. If $a$ is hybrid $(b, c)$-invertible and $d$ is hybrid $(b, c)$-invertible, then $a^{||b, c||} + d^{||b, c||} = a^{||b, c||}(a + d)^{||b, c||}$.

Proof. Let $x = a^{||b, c||}$ and $y = d^{||b, c||}$. It follows from Lemma 2.3 that $x a y = y$ and $x d y = x$, and consequently $x(a + d)y = x a y + x d y = y + x$.

By Theorem 2.6, we have the following corollary.

Corollary 2.7. Let $a, b, c, d \in R$. If $a$ is $(b, c)$-invertible and $d$ is $(b, c)$-invertible, then $a^{||b, c||} + d^{||b, c||} = a^{||b, c||}(a + d)^{||b, c||}$.

Proof. If $a$ is hybrid $(b, c)$-invertible and $d$ is hybrid $(b, c)$-invertible, then $a$ is hybrid $(b, c)$-invertible and $d$ is hybrid $(b, c)$-invertible. Let $x = a^{||b, c||}$ and $y = d^{||b, c||}$. Then we have $x = a^{||b, c||}$ and $y = d^{||b, c||}$. It follows from Theorem 2.6 that $x + y = x(a + d)y$, and consequently $a^{||b, c||} + d^{||b, c||} = a^{||b, c||}(a + d)^{||b, c||}$.

Let $a, b, c, d \in R$. If $a$ and $d$ are both hybrid $(b, c)$-invertible, then the absorption law for the hybrid $(b, c)$-inverse holds by Theorem 2.6. If $a$ is hybrid $(b, c)$-invertible and $d$ is hybrid $(u, v)$-invertible for some $u, v \in R$, does the absorption law for $a^{||b, c||}$ and $d^{||b, c||}$ hold?

Example 2.8. Let $\mathbb{C}^{2 \times 2}$ denote the set of all $2 \times 2$ complex matrices over the complex field $\mathbb{C}$. Consider $a = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$,

$$d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad u = v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Then it is easy to check that $a^{||b, c||} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $d^{||b, c||} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. It is clear that $a^{||b, c||} + d^{||b, c||} = a^{||b, c||}(a + d)^{||b, c||}$. 

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Following Green [7], Green’s preorders and relations in a semigroup are defined. Similarly, we say the Green’s preorder and relations in rings as

\[ a \leq_L b \iff Ra \subseteq Rb \iff \text{there exists } x \in R \text{ such that } a = xb. \]
\[ a \leq_R b \iff aR \subseteq bR \iff \text{there exists } x \in R \text{ such that } a = bx. \]
\[ a \leq_H b \iff a \leq_L b \text{ and } a \leq_R b. \]
\[ a \mathcal{L} b \iff Ra = Rb \iff \text{there exist } x, y \in R \text{ such that } a = xb \text{ and } b = ya. \]
\[ a \mathcal{R} b \iff aR = bR \iff \text{there exist } x, y \in R \text{ such that } a = bx \text{ and } b = ay. \]
\[ a \mathcal{H} b \iff a \mathcal{L} b \text{ and } a \mathcal{R} b. \]

Before investigate the absorption law for \( d^{(l)} \) and \( d^{(u)} \) by using Green’s preorders and relations, the following lemma is given.

**Lemma 2.9.** Let \( a, b, c, u, v \in R \). If \( bRu \) and \( cLu \), then \( a \) is hybrid \((b,c)\)-invertible if and only if \( a \) is hybrid \((u,v)\)-invertible. In this case, we have \( d^{(l)} = d^{(u)} \).

**Proof.** We present a proof of the necessity. As \( bRu \), then we have \( u = by \) and \( b = u\delta \) for some \( \gamma, \delta \in R \). Moreover, by \( cLu \), it gives that \( v = ac \) and \( c = \beta v \) for some \( a, \beta \in R \). Since \( a \) is hybrid \((b,c)\)-invertible, by Lemma 2.2, there is \( w \in R \) such that \( c = cabw \). It follows that \( v = ac = a(cabw) = (ac)abw = cabw = c\beta uv \), and consequently \( vR = vauR \). For any \( x \in rann(vau) \), by \( c = \beta v \), then \( vux = 0 \) and \( caux = (\beta v)aux = \beta vaux = 0 \). Note that \( u = by \), then \( caux = cabyx = 0 \). Again, from Lemma 2.2, it follows \( \gamma xv \in rann(cab) = rann(b) \). This implies that \( byx = 0 \) and \( uax = 0 \), which gives \( rann(vau) \subseteq rann(u) \). So, by Lemma 2.1, one can see that \( a \) is hybrid \((u,v)\)-invertible. Moreover, from Lemma 2.1, it is not difficult to directly check that \( a^{(l)} = a^{(u)} \). \( \square \)

**Theorem 2.10.** Let \( a, b, c, u, v \in R \) with \( bRu \) and \( cLu \). If \( d^{(l)} \) and \( d^{(u)} \) exist, then \( d^{(l)} + d^{(u)} = d^{(l)}(a + d)\).

**Proof.** Since \( bRu \) and \( cLu \), by Lemma 2.9 we have \( d^{(u)} = d^{(l)} \). Therefore, by Theorem 2.6, one can see that

\[
\begin{align*}
\quad d^{(l)} + d^{(u)} & = d^{(l)} + d^{(u)} \\
& = d^{(l)}(a + d) + d^{(u)} \\
& = d^{(l)}(a + d) + d^{(u)} (a + d).
\end{align*}
\]

\( \square \)

An involutory ring \( R \) means that \( R \) is a unital ring with involution, i.e., a ring with unity 1, and a mapping \( a \mapsto a^\dagger \) from \( R \) to \( R \) such that \((a^\dagger)^\dagger = a \), \((ab)^\dagger = b^\dagger a^\dagger\) and \((a + b)^\dagger = a^\dagger + b^\dagger\), for all \( a, b \in R \). Let \( R \) be an involutory ring and \( a \in R \). By [2, P.1910] and [12, Theorem 3.10], we have that \( a \) is Moore-Penrose invertible if and only if \( a \) is \((a, a^\dagger)\)-invertible if and only if \( a \) is hybrid \((a, a^\dagger)\)-invertible. Let \( R \) be an associative ring and \( a \in R \). \( a \) is Drazin invertible if and only if there exists \( k \in \mathbb{N} \) such that \( a^{(k)} \) is \((a, a^\dagger)\)-invertible if and only if there exists \( k \in \mathbb{N} \) such that \( a \) is \((a^k, a^k)\)-invertible, where the positive integer \( k \) is the Drazin index of \( a \), denoted by \( \text{ind}(a) \). \( a \) is group invertible if and only if there exists \( k \in \mathbb{N} \) such that \( a \) is \((a, a^k)\)-invertible, where the positive integer \( k \) is the Drazin index of \( a \), denoted by \( \text{ind}(a) \). \( a \) is Moore-Penrose inverse, the group inverse and the Drazin inverse of \( a \).

**Corollary 2.11.** Let \( a, b \in R \). Then

(i) If \( a^\dagger \) and \( b^\dagger \) exist with \( a\mathcal{H}b \), then \( a^\dagger + b^\dagger = a^\dagger(a + b)b^\dagger \).

(ii) If \( a^\dagger \) and \( b^\dagger \) exist with \( a\mathcal{H}b \), then \( a^\dagger + b^\dagger = a^\dagger(a + b)b^\dagger \).

(iii) If \( a^D \) and \( b^D \) exist with \( a\mathcal{H}b^m \), where \( \text{ind}(a) = n \) and \( \text{ind}(b) = m \), then \( a^D + b^D = a^D(a + b)b^D \).
3. Reverse order law for the hybrid \((b,c)\)-inverse

Let \(a, b \in R\) be two invertible elements. It is well known that
\[
(ab)^{-1} = b^{-1}a^{-1}.
\]
The above equality is known as the reverse order law of invertible elements. In general, the reverse order law does not hold for generalized inverses (see [1, 11]). In this section, the reverse order laws for the hybrid \((b,c)\)-inverse are obtained.

**Theorem 3.1.** Let \(a, b, c, d \in R\) such that \(a^{{(b,c)}}\) and \(d^{{(b,c)}}\) exist. If \(ad^{{(b,c)}} = a^{{(b,c)}}d\), then \(ad\) is hybrid \((b,c)\)-invertible and \((ad)^{{(b,c)}} = d^{{(b,c)}}a^{{(b,c)}}\).

*Proof.* Let \(x = a^{{(b,c)}}, y = d^{{(b,c)}}\) and \(z = yx\). We verify that \(z\) is the hybrid \((b,c)\)-inverse of \(ad\).

Step 1. \(zad = z\). Indeed, by Lemma 2.3, we know that \(xay = x\), \(y = y\), which give that \(z(ad)z = yx\) \(\in x\). Hence, as \(xt = b\), then we have \(zR = yxR \subseteq bR = yR = yxR = yxR \subseteq yxR = zR\), which gives \(zR = bR\).

Step 2. \(z = bR\). Indeed, as \(yR = bR\), then \(zR = xR \subseteq bR = yR = yxR = xzR \subseteq yxR = zR\), which gives \(zR = bR\).

Step 3. \(rann(z) = rann(c)\). It is easy to get \(rann(c) = rann(x) \subseteq rann(y) = rann(z)\).

Next, we claim that \(rann(z) \subseteq rann(c)\). Given any \(t \in rann(z)\), then \(xt = 0\), i.e., \(xt \in rann(y) = rann(c)\).

Remark 3.2. By [12, Proposition 3.3], we know that if \(a\) is \((b, c)\)-invertible, then \(b\) and \(c\) are both regular. Moreover, from Theorem 3.1, if \(a^{{(b,c)}}\) and \(d^{{(b,c)}}\) exist with \(ad^{{(b,c)}} = a^{{(b,c)}}d\), then \(z = d^{{(b,c)}}a^{{(b,c)}}\) is regular and \(rann(z) = rann(c)\).

**Lemma 3.3.** [12, Lemma 3.2] Let \(a \in R\) be regular. Then \(lann(rann(a)) = Ra\).

In view of Remark 3.2 and Lemma 3.3, we obtain the following result.

**Corollary 3.4.** Let \(a, b, c, d \in R\) such that \(a^{{(b,c)}}\) and \(d^{{(b,c)}}\) exist. If \(a^{{(b,c)}}d = a^{{(b,c)}}d\) then \(ad\) is \((b,c)\)-invertible and \((ad)^{{(b,c)}} = d^{{(b,c)}}a^{{(b,c)}}\).

*Proof.* From Theorem 3.1 and Remark 3.2, one can see \(z = d^{{(b,c)}}a^{{(b,c)}}\) is regular and \(rann(z) = rann(c)\). As \(a^{{(b,c)}}\) exists, it follows from [12, Proposition 3.3] that \(c\) is regular. Then, we obtain \(Rz = lann(rann(z)) = lann(rann(c)) = Rc\). On account of [2, Proposition 6.1], we conclude that \(ad\) is \((b,c)\)-invertible and \((ad)^{{(b,c)}} = d^{{(b,c)}}a^{{(b,c)}}\).

**Lemma 3.5.** Let \(a, b, c \in R\) with \(ab \leq R\) and \(ca \leq c\). If \(a^{{(b,c)}}\) exists, then \(a^{{(b,c)}} = a^{{(b,c)}}a\).

*Proof.* Let \(x = a^{{(b,c)}}\). Since \(ab \leq R\) and \(ca \leq c\), there is \(ab = ba\mu\) and \(ca = vac\) for some \(\mu, v \in R\).

Hence, it follows from \(\mu = v\) that \(ca = vac = vac\mu = vac\mu = vac\mu = vca\mu = vac\mu = vac\mu = vca\mu = vac\mu = vca\mu = vac\mu = vca\mu = vac\mu = vca\mu = vac\mu = vca\mu = vac\mu = vca\mu = vac\mu = vca\mu\) and consequently \(a \leq R\) \(\in rann(c) = rann(x),\) which implies \(xa = xa\mu\).

Moreover, by \(R = bR\), we have \(x = bs\) for some \(s \in R\). On account of \(b = xab\), we conclude that \(ax = a(bs) = (ab)s = (b\mu)s = (xab\mu)s = xab\mu s = xa(bs) = xa\mu\). Thus, \(ax = xa\mu\), as required.

In view of Theorem 3.1 and Lemma 3.5, we obtain the following result.

**Theorem 3.6.** Let \(a, b, c, d \in R\) with \(ab \leq R\) and \(ca \leq c\). If \(a^{{(b,c)}}\) and \(d^{{(b,c)}}\) exist, then \(ad\) is hybrid \((b,c)\)-invertible and \((ad)^{{(b,c)}} = d^{{(b,c)}}a^{{(b,c)}}\).

**Corollary 3.7.** Let \(a, b, c, d \in R\) such that \(ab = ba\) and \(ca = ca\). If \(a^{{(b,c)}}\) and \(d^{{(b,c)}}\) exist, then \(ad\) is hybrid \((b,c)\)-invertible and \((ad)^{{(b,c)}} = d^{{(b,c)}}a^{{(b,c)}}\).
In view of Lemma 3.3 and Corollary 3.7, we obtain the following result.

**Corollary 3.8.** Let \( a, b, c, d \in R \) such that \( ab = ba \) and \( ac = ca \). If \( ad \) and \( d \) exist, then \( ad \) is \((b, c)\)-invertible and \( (ad)\) is \((b, c)\)-invertible.

**Theorem 3.9.** Let \( a, b, c, d \in R \) with \( bd \leq b \) and \( ca \leq c \). If \( a \) and \( d \) exist, then \( ad \) is hybrid \((b, c)\)-invertible and \( (ad)\) is hybrid \((b, c)\)-invertible.

**Proof.** Let \( x = a \), \( y = d \), and \( z = yx \). Since \( db \leq b \) and \( ca \leq c \), there is \( db = bd \) and \( ca = vac \). Therefore, \( z = yx \) and \( ca = vac \). Since \( yxR \leq R \) and \( bR = z \), we have \( z = a \) and \( bR \leq R \). Note that \( zR = a \) and \( c = c \). Then \( bR \leq R \). On account of [14, Proposition 2.1] we conclude that \( ad \) is hybrid \((b, c)\)-invertible and \( (ad)\) is hybrid \((b, c)\)-invertible.

**Corollary 3.10.** Let \( a, b, c, d \in R \) such that \( bd = db \) and \( ac = ca \). If \( a \) and \( d \) exist, then \( ad \) is hybrid \((b, c)\)-invertible and \( (ad)\) is hybrid \((b, c)\)-invertible.

In view of Lemma 3.3 and Corollary 3.10, we obtain the following result.

**Corollary 3.11.** Let \( a, b, c, d \in R \) such that \( bd = db \) and \( ac = ca \). If \( a \) and \( d \) exist, then \( ad \) is \((b, c)\)-invertible and \( (ad)\) is \((b, c)\)-invertible.

Since \( a \) is an outer inverse of \( a \) when it exists, both \( a \) and \( a \) are idempotents. These will be referred to as the hybrid \((b, c)\)-idempotents associated with \( a \). We are interested in finding characterizations of those elements in the ring with equal hybrid \((b, c)\)-idempotents. In fact, it is also closely related to the reverse order law. We use the symbol \( R^2 \) to denote the set of all group invertible elements.

**Theorem 3.12.** Let \( a, b, c, d \in R \) such that \( a \) and \( d \) exist. Then the following statements are equivalent:

(i) \( a \) and \( d \) exist.

(ii) \( ad = da \).

(iii) \( a \) and \( d \) exist.

(iv) \( ad = da \).

(v) \( ad \) and \( da \) exist.

Proof. (i) \( \iff \) (ii). Let \( x = a \) and \( y = d \). From Lemma 2.3 we obtain

\[
\begin{align*}
x &= yx \quad \Rightarrow \quad xdy = dyx, \\
y &= xay \quad \Rightarrow \quad aydx = dxy.
\end{align*}
\]

Hence,

\[
\begin{align*}
ax &= dy \quad \Leftrightarrow \quad axdy = dyax \\
&\quad \Leftrightarrow \quad aydx = dxy.
\end{align*}
\]

(iii) \( \iff \) (iv). Set \( g = da \). We will prove that \( x \) is the group inverse of \( ad \). Using (iii) and Lemma 2.3, we get

\[
\begin{align*}
gad &= dxay = aydx = ad \\
adg &= a(ydx)y = a(xay) = ay = ad \\
g &= g(yax) = ga(ydx) = g(ax) = dxax = dx = g.
\end{align*}
\]

This implies that \( ad \) and \( ad \) exist. Conversely, if the latter holds, then \( gad = adg \) i.e., \( ad = ad \) and \( ad = ad \).

Proof. (i) \( \iff \) (ii). The proof is similar to the previous equivalence.
Next, we consider conditions under which the reverse order law for the hybrid \((b, c)\)-inverse of the product \(ad\), \((ad)\^H\) holds.

**Theorem 3.13.** Let \(a, b, c, d \in R\) such that \(a\^H\) and \(d\^H\) exist. Then the following statements are equivalent:

(i) \(ad\) has a hybrid \((b, c)\)-inverse of the form \((ad)\^H = d\^H a\^H\).

(ii) \(d\^H a\) has a hybrid \((b, c)\)-inverse of the form \((d\^H a)\^H = d\^H a\^H\).

(iii) \(a\^H d\) has a hybrid \((b, c)\)-inverse of the form \((a\^H d)\^H = d\^H a\^H\).

Proof. (i) \(\iff\) (ii). Suppose that \(ad\) has a hybrid \((b, c)\)-inverse, and \((ad)\^H = d\^H a\^H\). Then Lemma 2.3 is true for \((ad)\^H\) in place of \(a\^H\). It follows that

\[d\^H a\] \(\Rightarrow\) \(d\^H a\^H = d\^H a\^H\] yields

\[d\^H a\] \(\Leftarrow\) \(d\^H a\) such that \((ad)\^H = d\^H a\^H\).

Conversely, if the latter identities hold, we claim that \(z = d\^H a\) is the hybrid \((b, c)\)-inverse of \(ad\). Write \(x = a\) and \(y = d\). Indeed, it is clear that \(z = yx \in yR = bR\). Moreover, it is also easy to find \(rann(c) = rann(xy) \subseteq rann(z)\). On account of \(yd\) in the condition (ii), we conclude that

\[zad = yxady = yxdy = x = b\.

Similarly, in view of \(y = yd\) in the condition (ii) and \(cdy = c\), one can see that

\[cad = cadx = cdx\] \(\Leftarrow\) \(cdy = c\).

Then \(ad\) has a hybrid \((b, c)\)-inverse of the form \((ad)\^H = d\^H a\^H\) by [14, Proposition 2.1].

(ii) \(\Rightarrow\) (iii). By Lemma 2.3 we have \(x = xdy = ydx\). From the condition (ii), one can see that

\[x = xdy = xdyad\] \(\Leftarrow\) \(xady = ydx\).

That is, \(a\^H d\) has a hybrid \((b, c)\)-inverse of the form \((a\^H d)\^H = d\^H a\^H\). Moreover, again from the condition (ii), it follows

\[x = ydx = ydxady = ydxady\] \(\Leftarrow\) \(xady = ydx\).

That is, \(a\^H d\) has a hybrid \((b, c)\)-inverse of the form \((a\^H d)\^H = d\^H a\^H\).

(iii) \(\Rightarrow\) (ii). The proof is similar to (ii) \(\Rightarrow\) (iii).

We close this section with the characterization of \(a\^H d\) in rings.

**Theorem 3.14.** Let \(a, b, c, d \in R\) such that \(a\^H\) and \(d\^H\) exist. Then the following statements are equivalent:

(i) \(a\^H \cdot a = d\^H\).

(ii) \(a\^H d\) has a hybrid \((b, c)\)-inverse of the form \((a\^H d)\^H = d\^H a\^H\).

(iii) \(d\^H a\) has a hybrid \((b, c)\)-inverse of the form \((d\^H a)\^H = a\^H d\^H\).

(iv) \(a\^H d\) has a hybrid \((b, c)\)-inverse of the form \((a\^H d)\^H = d\^H a\^H\).

If any of the previous statements is valid, then \((ad)\^H = d\^H a\^H\).

Proof. Let \(x = a\) and \(y = d\). From Lemma 2.3 we obtain (3.1), that is,

\[x = xdy = ydx;\]

\[y = ydx = xady.\]

(i) \(\iff\) (ii) \(\iff\) (iii). By (1), it is clear that

\[xa = xady = ydx;\]

\[dy = dyax = dxy.\]
Hence, it follows that
\[ xa = dy \iff xdya = dyax \iff ydxa = dxay. \]

(i) \iff (iv). The necessary condition is immediate. Next, we assume that \( x = dyx \) and \( y = yxa \). Then we have \( xa = dxyax \) and \( dy = dyxa \), consequently \( xa = dy \), as desired.

(v) \iff (i). The proof is similar to the above.

Finally, we will prove that \( dy = xa \) implies that \( ad \) has a hybrid \((b,c)\)-inverse given by \((ad)_{[b\parallel c]} = \left( d^{[b\parallel c]}a_{[b\parallel c]} \right) \). From \( y = ydy \) and \( dy = xa \), it gives \( y = yxa \), and consequently \( y = ydy = (yxa)dy \). Moreover, note that \( y = yax \) and \( dy = xa \), it follows that \( y = yax = (yax)ax = ya(dy)x \). By Theorem 3.13 (ii) our assertion is proved. \( \square \)

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**References**