



Another Inequality for Skew CR-Warped Products in Kenmotsu Manifolds

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Abstract. In this paper, we study warped products of contact skew-CR submanifolds, called contact skew CR-warped products in Kenmotsu manifolds. We obtain a lower bound relationship between the squared norm of the second fundamental form and the warping function. Furthermore, the equality case is investigated and some applications of derived inequality are given.

1. Introduction

As a generalized class of holomorphic, totally real, CR, slant and semi-slant submanifolds G. S. Ronsse [25] introduced skew CR-submanifolds of almost Hermitian manifolds. Later on, in contact geometry, M.M. Tripathi [28] extended this idea for almost semi-invariant submanifolds of contact metric manifolds.

Recently, B. Sahin [27] studied the warped product skew CR-submanifolds of Kaehler manifolds as a generalization of CR-warped products introduced by B.-Y. Chen in his seminal work [10–13] and of warped product hemi-slant submanifolds, studied by B. Sahin in [26]. The contact version of skew CR-warped products of cosymplectic manifolds appeared in [20] and skew CR-warped products of Sasakian manifolds in [39]. For up-to-date survey on warped product manifolds and warped product submanifolds we refer to B.-Y. Chen's books [14, 16] and his survey article [15].

In the series of warped product submanifolds of Kenmotsu manifolds, we studied contact skew CR-warped product in [23]. In this paper, we study the contact skew CR-warped product submanifolds by considering the base manifold of warped product as a Riemannian product of anti-invariant and proper slant submanifolds and the fiber is an invariant submanifold.

The paper is organized as follows: In Section 2, we provide some preliminaries formulas and definitions for almost contact metric manifolds and their submanifolds. In Section 3, we study warped product skew CR-submanifolds of contact metric manifolds. In this, section, first we find some useful lemmas and then we derive a relation for the squared norm of the second fundamental form in terms of components of the gradient of warping function. The equality case is also considered. In Section 4, we give some applications of Theorem 4 as special cases.

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2. Preliminaries

A $(2m + 1)$ -dimensional differentiable manifold \tilde{M} is called an *almost contact metric manifold* if there is an almost contact metric structure (φ, ξ, η, g) consisting of a $(1, 1)$ tensor field φ , a vector field ξ , a 1-form η and the compatible metric g satisfying [3]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0; \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any $X, Y \in \Gamma(T\tilde{M})$, where the $\Gamma(T\tilde{M})$ is the Lie algebra of vector fields on \tilde{M} and $I : T\tilde{M} \rightarrow T\tilde{M}$ is the identity mapping. As an immediate consequence of (2), one has $\eta(X) = g(X, \xi)$, $\eta(\xi) = 1$ and $g(\varphi X, Y) = -g(X, \varphi Y)$. An almost contact metric manifold is Kenmotsu if and only if [19]

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \tilde{\nabla}_X \xi = X - \eta(X)\xi \tag{3}$$

for any $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of g .

Let M be a Riemannian manifold isometrically immersed in an another Riemannian manifold \tilde{M} . Then formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{5}$$

for any vector field $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where ∇^\perp is the normal connection in the normal bundle, h is the second fundamental form and A is the shape operator of the submanifold. The second fundamental form and the shape operator are related by

$$g(h(X, Y), N) = g(A_N X, Y) \tag{6}$$

We g denotes the inner product of M as well as \tilde{M} .

A submanifold M is said to be totally geodesic if $h = 0$ and totally umbilical if $h(X, Y) = g(X, Y)H$, $\forall X, Y \in \Gamma(TM)$, where $H = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$ is the mean curvature vector of M . For any $x \in M$ and $\{e_1, \dots, e_n, \dots, e_{2m+1}\}$ is an orthonormal frame of the tangent space $T_x \tilde{M}$ such that e_1, \dots, e_n are tangent to M at x . Then, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, \quad r \in \{n + 1, \dots, 2m + 1\}, \tag{7}$$

$$\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)). \tag{8}$$

B.-Y. Chen [8, 9] introduced a generalized class of holomorphic (invariant) and totally real (anti-invariant) submanifolds known as slant submanifolds in complex geometry. Later, A. Lotta [22] has extended Chen’s idea for contact metric manifolds.

Let M be a submanifold of an almost contact metric manifold \tilde{M} . Let \mathfrak{D} be a differentiable distribution on M . For any non-zero vector $X \in \mathfrak{D}_x$, the angle $\theta_{\mathfrak{D}}(X)$ between φX and \mathfrak{D}_x is a slant angle of X with respect to the distribution \mathfrak{D} . If the slant angle $\theta_{\mathfrak{D}}(X)$ is constant, i.e., it is independent of the choice $x \in M$ and $X \in \mathfrak{D}_x$, then \mathfrak{D} is called a θ -slant distribution and $\theta_{\mathfrak{D}}(X) = \theta_{\mathfrak{D}}$ is called the slant angle of the distribution \mathfrak{D} . A submanifold M tangent to ξ is said to be *slant* if for any $x \in M$ and any $X \in T_x M$, linearly independent to ξ , the angle between φX and $T_x M$ is a constant $\theta \in [0, \pi/2]$, called the *slant angle* of M in \tilde{M} . Invariant and anti-invariant submanifolds are θ -slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A

slant submanifold which is neither invariant nor anti-invariant is called *proper slant*. For more details, we refer to [5, 9].

For any vector field $X \in \Gamma(TM)$, we have

$$\varphi X = TX + FX, \tag{9}$$

where TX and FX are the tangential and normal components of φX , respectively. For a slant submanifold of almost contact metric manifolds we have the following useful result.

Theorem 2.1. [5] *Let M be a submanifold of an almost contact metric manifold \tilde{M} , such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{10}$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2 \theta$.

Following relations are straightforward consequence of (10)

$$g(TX, TY) = \cos^2 \theta [g(X, Y) - \eta(X)\eta(Y)] \tag{11}$$

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X)\eta(Y)] \tag{12}$$

for any $X, Y \in \Gamma(TM)$.

In [28], M.M. Tripathi introduced the concept of contact skew CR-submanifolds under the name almost semi-invariant submanifolds by exploiting the behavior of a natural bounded symmetric linear operator T^2 on the submanifold. From (2) and (9), it is easy to see that $g(TX, Y) = -g(X, TY)$, for any $X, Y \in \Gamma(TM)$, which implies that $g(T^2X, Y) = g(X, T^2Y)$, i.e., T^2 is a symmetric operator, therefore its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are bounded by -1 and 0 .

Since $\xi \in \Gamma(TM)$, then we have $TM = \langle \xi \rangle \oplus \langle \xi \rangle^\perp$ where $\langle \xi \rangle$ is the distribution spanned by ξ and $\langle \xi \rangle^\perp$ is the orthogonal complementary distribution of $\langle \xi \rangle$ in M . For any $x \in M$, we may write

$$\mathfrak{D}_x^\lambda = \ker(T^2 + \lambda^2(x)I)_x,$$

where I is the identity transformation and $\lambda(x) \in [0, 1]$ such that $-\lambda^2(x)$ is an eigenvalue of $T^2(x)$. We note that $\mathfrak{D}_x^1 = \ker F$ and $\mathfrak{D}_x^0 = \ker T$. \mathfrak{D}_x^1 is the maximal φ -invariant subspace of T_xM and \mathfrak{D}_x^0 is the maximal φ -anti-invariant subspace of T_xM .

From now on, we denote the distributions \mathfrak{D}^1 and \mathfrak{D}^0 by $\mathfrak{D} \oplus \langle \xi \rangle$ and \mathfrak{D}^\perp , respectively. Since T^2 is symmetric and diagonalizable, for some integer k if $-\lambda_1^2(x), \dots, -\lambda_k^2(x)$ are the eigenvalues of T^2 at $x \in M$, then $\langle \xi \rangle_x^\perp$ can be decomposed as direct sum of mutually orthogonal eigenspaces, i.e.

$$\langle \xi \rangle_x^\perp = \mathfrak{D}_x^{\lambda_1} \oplus \mathfrak{D}_x^{\lambda_2} \dots \oplus \mathfrak{D}_x^{\lambda_k}.$$

Each $\mathfrak{D}_x^{\lambda_i}, 1 \leq i \leq k$, is a T -invariant subspace of T_xM . Moreover if $\lambda_i \neq 0$, then $\mathfrak{D}_x^{\lambda_i}$ is even dimensional. We say that a submanifold M of an almost contact metric manifold \tilde{M} is a generic submanifold if there exists an integer k and functions $\lambda_i, 1 \leq i \leq k$ defined on M with values in $(0, 1)$ such that

- (1) Each $-\lambda_i^2(x), 1 \leq i \leq k$ is a distinct eigenvalue of T^2 with

$$T_xM = \mathfrak{D}_x \oplus \mathfrak{D}_x^\perp \oplus \mathfrak{D}_x^{\lambda_1} \oplus \dots \oplus \mathfrak{D}_x^{\lambda_k} \oplus \langle \xi \rangle_x$$

for any $x \in M$.

- (2) The dimensions of $\mathfrak{D}_x, \mathfrak{D}_x^\perp$ and $\mathfrak{D}_x^{\lambda_i}, 1 \leq i \leq k$ are independent on $x \in M$.

Moreover, if each λ_i is constant on M , then M is called a skew CR-submanifold. Thus, we observe that CR-submanifolds are a particular class of skew CR-submanifolds with $k = 0$, $\mathfrak{D} \neq \{0\}$ and $\mathfrak{D}^\perp \neq \{0\}$. And slant submanifolds are also a particular class of skew CR-submanifolds with $k = 1$, $\mathfrak{D} = \{0\}$, $\mathfrak{D}^\perp = \{0\}$ and λ_1 is constant. Moreover, if $\mathfrak{D}^\perp = \{0\}$, $\mathfrak{D} \neq 0$ and $k = 1$, then M is a semi-slant submanifold. Furthermore, if $\mathfrak{D} = \{0\}$, $\mathfrak{D}^\perp \neq \{0\}$ and $k = 1$, then M is a pseudo-slant (or hemi-slant) submanifold.

A submanifold M of an almost contact metric manifold \tilde{M} is said to be a Contact skew CR-submanifold of order 1 if M is a skew CR-submanifold with $k = 1$ and λ_1 is constant. In this case, the tangent bundle of M is decomposed as

$$TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle. \tag{13}$$

The normal bundle $T^\perp M$ of a contact skew CR-submanifold M is decomposed as

$$T^\perp M = \varphi \mathfrak{D}^\perp \oplus F\mathfrak{D}^\theta \oplus \nu, \tag{14}$$

where ν is a φ -invariant normal subbundle of $T^\perp M$.

Now, we provide a non-trivial example of skew CR-submanifolds of an almost contact metric manifold.

Example 2.2. Consider the Euclidean space \mathbf{R}^9 with coordinates $(x_1, \dots, x_4, y_1, \dots, y_4, z)$. Let \mathbf{R}^9 has the almost contact structure given by

$$\varphi \left(\frac{\partial}{\partial x_i} \right) = -\frac{\partial}{\partial y_i}, \quad \varphi \left(\frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial x_j}, \quad \varphi \left(\frac{\partial}{\partial z} \right) = 0, \quad 1 \leq i, j \leq 4.$$

It is easy to see that \mathbf{R}^9 is an almost contact metric manifold with respect to the Euclidean metric tensor of \mathbf{R}^9 and the assumed almost contact structure. Let M be a submanifold defined by the immersion ψ as follows

$$\psi(u, v, w, \theta, \phi, z) = \left(\frac{1}{2}(u^2 + v^2), 0, \theta \cos \phi, \phi \cos \theta, \frac{1}{2}(u^2 - v^2), w, \theta \sin \phi, -\phi \sin \theta, z \right)$$

with $u \neq \pm v$ and $z \neq 0$. Then the tangent space TM of M at any point is spanned by the following vectors

$$U_1 = u \frac{\partial}{\partial x_1} + u \frac{\partial}{\partial y_1}, \quad U_2 = v \frac{\partial}{\partial x_1} - v \frac{\partial}{\partial y_1}, \quad U_3 = \frac{\partial}{\partial y_2}, \quad U_4 = \cos \phi \frac{\partial}{\partial x_3} - \phi \sin \theta \frac{\partial}{\partial x_4} + \sin \phi \frac{\partial}{\partial y_3} - \phi \cos \theta \frac{\partial}{\partial y_4},$$

$$U_5 = -\theta \sin \phi \frac{\partial}{\partial x_3} + \cos \theta \frac{\partial}{\partial x_4} + \theta \cos \phi \frac{\partial}{\partial y_3} - \sin \theta \frac{\partial}{\partial y_4}, \quad U_6 = \frac{\partial}{\partial z}.$$

Clearly, we have

$$\varphi U_1 = -u \frac{\partial}{\partial y_1} + u \frac{\partial}{\partial x_1}, \quad \varphi U_2 = -v \frac{\partial}{\partial y_1} - v \frac{\partial}{\partial x_1}, \quad \varphi U_3 = \frac{\partial}{\partial x_2},$$

$$\varphi U_4 = -\cos \phi \frac{\partial}{\partial y_3} + \phi \sin \theta \frac{\partial}{\partial y_4} + \sin \phi \frac{\partial}{\partial x_3} - \phi \cos \theta \frac{\partial}{\partial x_4},$$

$$\varphi U_5 = \theta \sin \phi \frac{\partial}{\partial y_3} - \cos \theta \frac{\partial}{\partial y_4} + \theta \cos \phi \frac{\partial}{\partial x_3} - \sin \theta \frac{\partial}{\partial x_4}, \quad \varphi U_6 = 0.$$

Then, we find that φU_3 is orthogonal to TM and M is a submanifold tangent to the structure vector field $\xi = \frac{\partial}{\partial z}$ with invariant, anti-invariant and proper slant distributions $\mathfrak{D} = \text{Span}\{U_1, U_2\}$, $\mathfrak{D}^\perp = \text{Span}\{U_3\}$, and $\mathfrak{D}^\theta = \text{Span}\{U_4, U_5\}$, respectively with slant angle $\Theta = \cos^{-1} \left(\frac{\theta + \phi}{\sqrt{(1 + \theta^2)(1 + \phi^2)}} \right)$. Hence, M is a skew CR-submanifold of \mathbf{R}^9 .

3. Skew CR-warped product submanifolds in Kenmotsu manifolds

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and f be a positive smooth function on B . The warped product of B and F is the Riemannian manifold

$$B \times_f F = (M = B \times F, g)$$

equipped with the warped metric $g = g_B + f^2 g_F$. The function f is called the warping function and a warped product manifold M is said to be trivial or simply a Riemannian product of B and F if f is constant (see, for instance, [2]).

Let X be a vector field on B and Z be an another vector field on F . Then, from Lemma 7.3 of [2], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{15}$$

where ∇ denotes the Levi-Civita connection on M . Now for a smooth function f on an n -dimensional manifold M , we have

$$\|\vec{\nabla} f\|^2 = \sum_{i=1}^m (e_i(f))^2 \tag{16}$$

for the given orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ on M , where $\vec{\nabla} f$ is the gradient of f defined by $g(\vec{\nabla} f, X) = X(f)$.

Remark 3.1. It is also important to note that for a warped product $M = B \times_f F$; B is totally geodesic and F is totally umbilical in M [2, 10].

The purpose of this section is to study contact skew CR-warped products of Kenmotsu manifolds which we define as: A warped product submanifold $M = B \times_f M_T$ is called a *contact skew CR-warped product submanifold* if $B = M_\perp \times M_\theta$ is the product of an anti-invariant submanifold M_\perp and a proper slant submanifold M_θ of a Kenmotsu manifold \tilde{M} , where M_T is invariant submanifold of \tilde{M} . Throughout this paper, we assume the structure vector field ξ tangent to the submanifold. In case of $\xi \in \Gamma(TM)$, we have two cases, either ξ is tangent to M_T or ξ is tangent to B . When $\xi \in \Gamma(TM_T)$, then from (3) we have $\tilde{\nabla}_U \xi = U$, for any $U \in \Gamma(TB)$. Using (4) and (15), we find $U(\ln f)\xi = U$ and taking the inner product with ξ , we get $U(\ln f) = 0$, which means that f is constant.

From now, for the simplicity we denote the tangent spaces of M_T, M_\perp and M_θ by the same symbols $\mathfrak{D}, \mathfrak{D}^\perp$ and \mathfrak{D}^θ , respectively.

Now, if we consider $\xi \in \Gamma(TB)$, then there are two possibilities that either ξ is tangent to M_T or tangent to M_θ . For this, we have the following useful results.

Lemma 3.2. Let $M = B \times_f M_T$ be a contact skew CR-warped product submanifold of order 1 of a Kenmotsu manifold \tilde{M} such that ξ is tangent to B and $B = M_\perp \times M_\theta$, where M_T, M_\perp and M_θ are invariant, anti-invariant and proper slant submanifolds of \tilde{M} , respectively. Then, we have

- (i) $\xi(\ln f) = 1$,
- (ii) $g(h(X, Y), \varphi Z) = (Z(\ln f) - \eta(Z))g(X, \varphi Y)$,
- (iii) $g(h(X, Z), \varphi W) = 0$,

for any $X, Y \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp \oplus \langle \xi \rangle)$.

Proof. For any $X \in \Gamma(\mathfrak{D})$, by using (3) we have $\tilde{\nabla}_X \xi = X$. Then using (4) and (15), we find that $\xi(\ln f) = 1$, which is first part of the lemma. For the second part, we have

$$g(h(X, Y), \varphi Z) = g(\tilde{\nabla}_X Y, \varphi Z) = -g(\tilde{\nabla}_X \varphi Y, Z) + g((\tilde{\nabla}_X \varphi)Y, Z).$$

for any $X, Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle \xi \rangle)$. Using (3) and the orthogonality of vector fields, we derive

$$g(h(X, Y), \varphi Z) = g(\tilde{\nabla}_X Z, \varphi Y) + \eta(Z)g(\varphi X, Y) = g(\nabla_X Z, \varphi Y) - \eta(Z)g(X, \varphi Y).$$

Then, second part follows from above relation by using (3). On the other hand, for any $X \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp \oplus \langle \xi \rangle)$, we have

$$g(h(X, Z), \varphi W) = g(\tilde{\nabla}_Z X, \varphi W) = -g(\tilde{\nabla}_Z \varphi X, W) + g((\tilde{\nabla}_Z \varphi)X, W).$$

Again, from (3), (9), (15) and the orthogonality of vector fields, we obtain

$$g(h(X, Z), \varphi W) = -Z(\ln f) g(\varphi X, W) = 0,$$

which is (iii). Hence, the proof is complete. \square

Interchanging X by φX , for any $X \in \Gamma(\mathfrak{D})$ in Lemma 3.2 (ii), we derive

$$g(h(\varphi X, Y), \varphi Z) = (Z(\ln f) - \eta(Z)) g(X, Y), \tag{17}$$

Lemma 3.3. *Let $M = B \times_f M_T$ be a contact skew CR-warped product submanifold of order 1 of a Kenmotsu manifold \tilde{M} such that ξ is tangent to B . Then*

- (i) $g(h(X, Y), FV) = (V(\ln f) - \eta(V)) g(X, \varphi Y) + TV(\ln f)g(X, Y),$
- (ii) $g(h(X, Y), FTV) = TV(\ln f)g(X, \varphi Y) - \cos^2 \theta (V(\ln f) - \eta(V)) g(X, Y),$
- (iii) $g(h(X, U), FV) = 0,$

for any $X, Y \in \Gamma(\mathfrak{D})$ and $U, V \in \Gamma(\mathfrak{D}^\theta \oplus \langle \xi \rangle)$.

Proof. For any $X, Y \in \Gamma(\mathfrak{D})$ and $V \in \Gamma(\mathfrak{D}^\theta \oplus \langle \xi \rangle)$, we have

$$\begin{aligned} g(h(X, Y), FV) &= g(\tilde{\nabla}_X Y, \varphi V - TV) \\ &= -g(\tilde{\nabla}_X \varphi Y, V) + g((\tilde{\nabla}_X \varphi)Y, V) + g(\tilde{\nabla}_X TV, Y) \\ &= g(\tilde{\nabla}_X V, \varphi Y) + \eta(V)g(\varphi X, Y) + TV(\ln f)g(X, Y). \end{aligned}$$

First part follows from above relation by using (15). Second part immediately follows from (i) by interchanging V by TV . For the third part of the lemma, we have

$$g(h(X, U), FV) = g(\tilde{\nabla}_U X, \varphi V) - g(\tilde{\nabla}_U X, TV) = -g(\tilde{\nabla}_U \varphi X, V) + g((\tilde{\nabla}_U \varphi)X, V) - U(\ln f)g(X, TV)$$

for any $X \in \Gamma(\mathfrak{D})$ and $U, V \in \Gamma(\mathfrak{D}^\theta \oplus \langle \xi \rangle)$. Using (3), (4), (15) and orthogonality of vector fields, we easily get (iii) from above relation, which proves the lemma completely. \square

Lemma 3.4. *Let $M = B \times_f M_T$ be a contact skew CR-warped product submanifold of order 1 of a Kenmotsu manifold \tilde{M} such that ξ is tangent to B . Then, we have*

- (i) $g(h(X, Z), FV) = 0,$
- (ii) $g(h(X, V), \varphi Z) = 0,$

for any $X, Y \in \Gamma(\mathfrak{D}), Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle \xi \rangle), V \in \Gamma(\mathfrak{D}^\theta \oplus \langle \xi \rangle)$.

Proof. For any $X \in \Gamma(\mathfrak{D}), Z \in \Gamma(\mathfrak{D}^\perp)$ and $V \in \Gamma(\mathfrak{D}^\theta \oplus \langle \xi \rangle)$, we have

$$g(h(X, Z), FV) = g(\tilde{\nabla}_Z X, \varphi V) - g(\tilde{\nabla}_Z X, TV) = -g(\tilde{\nabla}_Z \varphi X, V) + g((\tilde{\nabla}_Z \varphi)X, V) - Z(\ln f)g(X, TV).$$

Using (3), (4), (15) and the orthogonality of vector fields, we find (i). In a similar way, we can prove the second part of the lemma. \square

A warped product $M = B \times_f F$ is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any $X \in \Gamma(TB)$ and $Z \in \Gamma(TF)$.

Now, we construct the following frame fields for the contact skew CR-warped product submanifold M of Kenmotsu manifold \tilde{M} . Let $M = B \times_f M_T$ be a n -dimensional contact skew CR-warped product submanifold of a $(2m + 1)$ -dimensional Kenmotsu manifold \tilde{M} with $B = M_\perp \times M_\theta$ and ξ is tangent to B where M_\perp, M_θ and M_T are anti-invariant, proper slant and invariant submanifolds of \tilde{M} with their real dimensions as $\dim(M_\perp) = m_1, \dim(M_\theta) = m_2$ and $\dim(M_T) = m_3$, respectively. Then, clearly we have $n = m_1 + m_2 + m_3$. We denote the tangent bundle of M_T, M_\perp and M_θ by $\mathfrak{D}, \mathfrak{D}^\perp$ and \mathfrak{D}^θ , respectively. Since, $\xi \in \Gamma(TB)$, then we have two cases: either $\xi \in \Gamma(\mathfrak{D}^\perp)$ or $\xi \in \Gamma(\mathfrak{D}^\theta)$. If we consider $\xi \in \Gamma(\mathfrak{D}^\theta)$ then we set the orthonormal frame fields of M as follows: $\mathfrak{D}^\perp = \text{Span}\{e_1, \dots, e_{m_1}\}, \mathfrak{D}^\theta = \text{Span}\{e_{m_1+1} = e_1^*, \dots, e_{m_1+p} = e_p^*, e_{m_1+p+1} = e_{p+1}^* = \sec \theta \text{Te}_1^* \dots, e_{m_1+2p} = e_{2p}^* = \sec \theta \text{Te}_p^*, e_{m_1+m_2} = e_{2p+1}^* = \xi\}$ and $\mathfrak{D} = \text{Span}\{e_{m_1+m_2+1} = \bar{e}_1, \dots, e_{m_1+m_2+q} = \bar{e}_q, e_{m_1+m_2+q+1} = \varphi e_1, \dots, e_n = \varphi e_q\}$. Then, the normal subbundles of $T^\perp M$ are spanned by $\varphi \mathfrak{D}^\perp = \text{Span}\{e_{n+1} = \bar{e}_1 = \varphi e_1, \dots, e_{n+m_1} = \bar{e}_{m_1} = \varphi e_{m_1}\}, F\mathfrak{D}^\theta = \text{Span}\{e_{n+m_1+1} = \bar{e}_{m_1+1} = \csc \theta \text{Fe}_1^*, \dots, e_{n+m_1+p} = \bar{e}_{m_1+p} = \csc \theta \text{Fe}_p^*, e_{n+m_1+p+1} = \bar{e}_{m_1+p+1} = \csc \theta \sec \theta \text{FTe}_1^*, \dots, e_{n+m_1+m_2-1} = \bar{e}_{m_1+m_2-1} = \csc \theta \sec \theta \text{FTe}_p^*\}$ and $\mu = \text{Span}\{e_{n+m_1+m_2} = \bar{e}_{m_1+m_2}, \dots, e_{2m+1} = \bar{e}_{2(m-m_1-m_2+1)-m_3}\}$.

Now, using the above orthonormal frame field and some results of previous sections, we derive the following main result of this paper.

Theorem 3.5. *Let $M = B \times_f M_T$ be a contact skew CR-warped product submanifold of order 1 of a Kenmotsu manifold \tilde{M} such that ξ is tangent to B and $B = M_\perp \times M_\theta$, where M_\perp, M_θ and M_T are anti-invariant, proper slant and invariant submanifolds of \tilde{M} with their real dimensions m_1, m_2 and m_3 , respectively. Then we have:*

(i) *If ξ is tangent to M_θ , then*

$$\|h\|^2 \geq 2m_3 \left(\|\vec{\nabla}^\perp(\ln f)\|^2 \right) + m_3 \left(1 + 2 \cot^2 \theta \right) \left(\|\nabla^\theta(\ln f)\|^2 - 1 \right).$$

(ii) *If ξ is tangent to M_\perp , then*

$$\|h\|^2 \geq 2m_3 \left(\|\vec{\nabla}^\perp(\ln f)\|^2 - 1 \right) + m_3 \left(1 + 2 \cot^2 \theta \right) \|\nabla^\theta(\ln f)\|^2.$$

where $\vec{\nabla}^\perp(\ln f)$ and $\nabla^\theta(\ln f)$ are the gradient components along M_\perp and M_θ , respectively.

(iii) *If the equality sign holds in above inequalities, then B is a totally geodesic submanifold of \tilde{M} and M_T is totally umbilical in \tilde{M} . Moreover, M is a \mathfrak{D} -mixed totally geodesic submanifold of \tilde{M} .*

Proof. From the definition, we have

$$\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n g(h(e_i, e_j), e_r).$$

According to the constructed frame field, the above relation takes the form

$$\begin{aligned} \|h\|^2 &= \sum_{r=n+1}^{n+m_1} \sum_{i,j=1}^{m_1+m_2+m_3} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+m_1+1}^{n+m_1+m_2-1} \sum_{i,j=1}^{m_1+m_2+m_3} g(h(e_i, e_j), e_r)^2 \\ &+ \sum_{r=n+m_1+m_2}^{2m+1} \sum_{i,j=1}^{m_1+m_2+m_3} g(h(e_i, e_j), e_r)^2. \end{aligned} \tag{18}$$

Leaving the last positive v -components term in the right hand side of (18). Then, we derive

$$\|h\|^2 \geq \sum_{r=1}^{m_1} \sum_{i,j=1}^{m_1+m_2+m_3} g(h(e_i, e_j), \bar{e}_r)^2 + \sum_{r=m_1+1}^{m_1+m_2-1} \sum_{i,j=1}^{m_1+m_2+m_3} g(h(e_i, e_j), \bar{e}_r)^2.$$

Above relation decomposes for the assumed frame fields as follows.

$$\begin{aligned}
 \|h\|^2 \geq & \sum_{r=1}^{m_1} \sum_{i,j=1}^{m_1} g(h(e_i, e_j), \varphi e_r)^2 + \sum_{r=1}^{m_1} \sum_{i,j=1}^{m_2} g(h(e_i^*, e_j^*), \varphi e_r)^2 + \sum_{r=1}^{m_1} \sum_{i,j=1}^{m_3} g(h(\bar{e}_i, \bar{e}_j), \varphi e_r)^2 \\
 & + \csc^2 \theta \sum_{r=1}^p \sum_{i,j=1}^{m_1} g(h(e_i, e_j), F e_r^*)^2 + \sec^2 \theta \csc^2 \theta \sum_{r=1}^p \sum_{i,j=1}^{m_1} g(h(e_i, e_j), F T e_r^*)^2 \\
 & + \csc^2 \theta \sum_{r=1}^p \sum_{i,j=1}^{m_2} g(h(e_i^*, e_j^*), F e_r^*)^2 + \sec^2 \theta \csc^2 \theta \sum_{r=1}^p \sum_{i,j=1}^{m_2} g(h(e_i^*, e_j^*), F T e_r^*)^2 \\
 & + \csc^2 \theta \sum_{r=1}^p \sum_{i,j=1}^{m_3} g(h(\bar{e}_i, \bar{e}_j), F e_r^*)^2 + \sec^2 \theta \csc^2 \theta \sum_{r=1}^p \sum_{i,j=1}^{m_3} g(h(\bar{e}_i, \bar{e}_j), F T e_r^*)^2 \\
 & + 2 \sum_{r=1}^{m_1} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g(h(e_i, e_j^*), \varphi e_r)^2 + 2 \sum_{r=1}^{m_1} \sum_{i=1}^{m_1} \sum_{j=1}^{m_3} g(h(e_i, \bar{e}_j), \varphi e_r)^2 + 2 \sum_{r=1}^{m_1} \sum_{i=1}^{m_2} \sum_{j=1}^{m_3} g(h(e_i^*, \bar{e}_j), \varphi e_r)^2 \\
 & + 2 \csc^2 \theta \sum_{r=1}^p \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g(h(e_i, e_j^*), F e_r^*)^2 + 2 \sec^2 \theta \csc^2 \theta \sum_{r=1}^p \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g(h(e_i, e_j^*), F T e_r^*)^2 \\
 & + 2 \csc^2 \theta \sum_{r=1}^p \sum_{i=1}^{m_2} \sum_{j=1}^{m_3} g(h(e_i^*, \bar{e}_j), F e_r^*)^2 + 2 \sec^2 \theta \csc^2 \theta \sum_{r=1}^p \sum_{i=1}^{m_2} \sum_{j=1}^{m_3} g(h(e_i^*, \bar{e}_j), F T e_r^*)^2 \\
 & + 2 \csc^2 \theta \sum_{r=1}^p \sum_{i=1}^{m_1} \sum_{j=1}^{m_3} g(h(e_i, \bar{e}_j), F e_r^*)^2 + 2 \sec^2 \theta \csc^2 \theta \sum_{r=1}^p \sum_{i=1}^{m_1} \sum_{j=1}^{m_3} g(h(e_i, \bar{e}_j), F T e_r^*)^2. \tag{19}
 \end{aligned}$$

Leaving the first, second, fourth, fifth, sixth, seventh, tenth, thirteenth and fourteenth positive terms of (20) and using Lemma 3.2, Lemma 3.3 and Lemma 3.4, we derive

$$\begin{aligned}
 \|h\|^2 \geq & 2m_3 \sum_{r=1}^{m_1} (e_r(\ln f))^2 + 2 \csc^2 \theta (1 + \sec^2 \theta) \sum_{r=1}^p \sum_{i,j=1}^q (T e_r^*(\ln f))^2 (g(\bar{e}_i, \bar{e}_j))^2 \\
 & + 2 \csc^2 \theta (1 + \cos^2 \theta) \sum_{r=1}^p \sum_{i,j=1}^q (e_r^*(\ln f) - \eta(e_r^*))^2 (g(\bar{e}_i, \bar{e}_j))^2. \tag{20}
 \end{aligned}$$

(i) When ξ is tangent to M_θ , then with the help of (16), the above inequality takes the form

$$\begin{aligned}
 \|h\|^2 \geq & 2m_3 \|\vec{\nabla}^\perp(\ln f)\|^2 + 2q \csc^2 \theta (1 + \sec^2 \theta) \sum_{r=1}^p (T e_r^*(\ln f))^2 + 2q \csc^2 \theta (1 + \cos^2 \theta) \sum_{r=1}^p (e_r^*(\ln f))^2 \\
 = & 2m_3 \|\vec{\nabla}^\perp(\ln f)\|^2 + m_3 \csc^2 \theta (1 + \sec^2 \theta) \sum_{r=1}^{2p+1} (T e_r^*(\ln f))^2 - m_3 \csc^2 \theta (1 + \sec^2 \theta) \sum_{r=p+1}^{2p} (T e_r^*(\ln f))^2 \\
 & - m_3 \csc^2 \theta (1 + \sec^2 \theta) (T e_{2p+1}^*(\ln f))^2 + m_3 \csc^2 \theta (1 + \cos^2 \theta) \sum_{r=1}^p (e_r^*(\ln f))^2. \tag{21}
 \end{aligned}$$

Since $e_{2p+1}^* = \xi$ and $T\xi = 0$, then the second last term in the right hand side of (21) is identically zero. Hence,

we derive

$$\begin{aligned} \|h\|^2 \geq & 2m_3\|\vec{\nabla}^\perp(\ln f)\|^2 + m_3 \csc^2 \theta (1 + \cos^2 \theta) (\|\nabla^\theta(\ln f)\|^2 - 1)^2 + m_3 \csc^2 \theta (1 + \cos^2 \theta) \sum_{r=1}^p (e_r^*(\ln f))^2 \\ & - m_3 \csc^2 \theta \sec^2 \theta (1 + \sec^2 \theta) \sum_{r=1}^p (g(Te_r^*, T \ln f))^2. \end{aligned} \tag{22}$$

Using (11) and (16), we get the inequality (i). If ξ is tangent to M_\perp , then the inequality follows from (20) and the orthonormal frame fields such as $\xi \in \Gamma(\mathfrak{D}^\perp)$. For the equality case, from the leaving term of (18), we find

$$h(TM, TM) \perp \nu. \tag{23}$$

From the leaving first term of (19), we obtain

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\perp) \perp \varphi \mathfrak{D}^\perp. \tag{24}$$

Also, from the leaving second term in the right hand side of (19), we derive

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) \perp \varphi \mathfrak{D}^\perp. \tag{25}$$

Similarly, from the leaving fourth and fifth terms in right hand side of (19), we find

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\perp) \perp F\mathfrak{D}^\theta. \tag{26}$$

And from the leaving sixth and seventh terms of (19), we obtain

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) \perp F\mathfrak{D}^\theta. \tag{27}$$

From the leaving tenth term of (19), we get

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\theta) \perp \varphi \mathfrak{D}^\perp. \tag{28}$$

Also, from the leaving thirteenth and fourteenth terms of (19), we obtain

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\theta) \perp F\mathfrak{D}^\theta. \tag{29}$$

Then from (23), (24) and (26), we conclude that

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\perp) = 0. \tag{30}$$

Similarly, from (23), (25) and (27), we deduce that

$$h(\mathfrak{D}^\theta, \mathfrak{D}^\theta) = 0. \tag{31}$$

From the leaving eleventh term of (19) with Lemma 3.2 (iii), we get

$$h(\mathfrak{D}, \mathfrak{D}^\perp) \perp \varphi \mathfrak{D}^\perp. \tag{32}$$

Similarly, from the leaving twelfth term of (19) with Lemma 3.4 (ii), we find

$$h(\mathfrak{D}, \mathfrak{D}^\theta) \perp \varphi \mathfrak{D}^\perp. \tag{33}$$

Also, from the leaving fifteenth and sixteenth terms of (19) with Lemma 3.3 (iii), we obtain

$$h(\mathfrak{D}, \mathfrak{D}^\theta) \perp F\mathfrak{D}^\theta. \tag{34}$$

And leaving seventeenth and eighteenth terms of (19) with Lemma 3.4 (i), we deduce that

$$h(\mathfrak{D}, \mathfrak{D}^\perp) \perp F\mathfrak{D}^\theta. \quad (35)$$

Then, from (23), (32) and (35), we conclude that

$$h(\mathfrak{D}, \mathfrak{D}^\perp) = 0. \quad (36)$$

And from (23), (33) and (34), we find

$$h(\mathfrak{D}, \mathfrak{D}^\theta) = 0. \quad (37)$$

Also, from (23), (28) and (29), we obtain

$$h(\mathfrak{D}^\perp, \mathfrak{D}^\theta) = 0. \quad (38)$$

Then, from (30), (31) and (38) with the Remark 3.1, we conclude that B is totally geodesic in \tilde{M} . Since B is totally umbilical in M (Remark 3.1), then using this fact with (23)-(35) and (38), we get B is totally umbilical in \tilde{M} . All conditions from (23)-(38) imply that M is \mathfrak{D} -mixed totally geodesic in \tilde{M} , which proves the theorem completely. \square

4. Applications of Theorem 3.5

We have the following well known applications of Theorem 3.5.

1. If $\dim(M_\theta) = 0$ in a contact skew CR-warped product, then it reduces to contact CR-warped products of the form $M = M_\perp \times_f M_T$ studied in [32]. In this case, the statement of Theorem 3.5 will be: *Let $M = M_\perp \times_f M_T$ be a contact CR-warped product submanifold of a Kenmotsu manifold \tilde{M} such that ξ is tangent to M_\perp , where M_T and M_\perp are invariant and anti-invariant submanifolds of \tilde{M} with their real dimensions m_1, m_2 , respectively. Then we have:*

(i) *The squared norm of the second fundamental form h satisfies*

$$\|h\|^2 \geq 2m_1 \left(\|\vec{\nabla}^\perp(\ln f)\|^2 - 1 \right).$$

where $\vec{\nabla}^\perp(\ln f)$ is the gradient of $\ln f$ along M_\perp .

(ii) *If the equality sign holds in above inequality, then M_\perp is totally geodesic and M_T is a totally umbilical in \tilde{M} . Moreover, M is $\mathfrak{D} - \mathfrak{D}^\perp$ mixed totally geodesic submanifold of \tilde{M} .*

Which is the main result of [32].

2. Similarly, if $\dim(M_\perp) = 0$ in a contact skew CR-warped product, then it will change into a warped product semi-slant submanifold of the form $M = M_\theta \times_f M_T$ studied in [36]. In this case, Theorem 4.2 of [36] is a particular case of Theorem 3.5 as follows:

Corollary 4.1. *(Theorem 4.2 of [36]) Let $M = M_\theta \times_f M_T$ be a warped product semi-slant submanifold of a Kenmotsu manifold \tilde{M} such that ξ is tangent to M_θ , where M_θ is a proper slant submanifold and M_T is an m_2 -dimensional invariant submanifold of \tilde{M} . Then we have:*

(i) *The squared norm of the second fundamental form of M satisfies*

$$\|h\|^2 \geq m_2 \left(1 + 2 \cot^2 \theta \right) \left(\|\nabla^\theta(\ln f)\|^2 - 1 \right)$$

where $\nabla^\theta \ln f$ is the gradient of $\ln f$ along M_θ .

(ii) *If the equality sign in (i) holds identically, then M_θ is totally geodesic in \tilde{M} and M_T is a totally umbilical submanifold of \tilde{M} . Moreover, M is $\mathfrak{D} - \mathfrak{D}^\theta$ mixed totally geodesic submanifold of \tilde{M} .*

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