The (Signless Laplacian) Spectral Radius (Of Subgraphs) of Uniform Hypergraphs

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Abstract. Let $\lambda_1(G)$ and $q_1(G)$ be the spectral radius and the signless Laplacian spectral radius of a $k$-uniform hypergraph $G$, respectively. In this paper, we give the lower bounds of $d - \lambda_1(H)$ and $2d - q_1(H)$, where $H$ is a proper subgraph of a $f$(-edge)-connected $d$-regular (linear) $k$-uniform hypergraph. Meanwhile, we also give the lower bounds of $2\Delta - q_1(G)$ and $\Delta - \lambda_1(G)$, where $G$ is a nonregular $f$(-edge)-connected (linear) $k$-uniform hypergraph with maximum degree $\Delta$.

1. Introduction

A hypergraph $G = (V, E)$ is a pair consisting of a vertex set $V = \{1, 2, \ldots, n\}$, and an edge set $E = \{e_1, e_2, \ldots, e_m\}$, where $e_i$ $(1 \leq i \leq m)$ is a subset of $V$. A hypergraph is called $k$-uniform if every edge contains precisely $k$ vertices. We will use the term $k$-graphs in place of $k$-uniform hypergraphs. A hypergraph $G$ is called linear provided that each pair of the edges of $G$ has at most one common vertex [1]. Given two $k$-graphs $G = (V, E)$ and $H = (V', E')$, if $V' \subseteq V$ and $E' \subseteq E$, then $H$ is said to be a subgraph (sub-hypergraph) of $G$. If $H$ is a subgraph of a $k$-graph $G$, and $H \neq G$, then $H$ is called a proper subgraph of $G$ [11]. A tensor $\mathcal{A}$ with order $k$ and dimension $n$ over the complex field $\mathbb{C}$ is a multidimensional array

$$\mathcal{A} = (a_{i_1i_2\cdots i_k}), \ 1 \leq i_1, i_2, \ldots, i_k \leq n.$$ 

The tensor $\mathcal{A}$ is called symmetric if its entries are invariant under any permutation of their indices. For a vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$, $\mathcal{A}^{k-1}$ is a vector in $\mathbb{C}^n$ whose $i$-th component is the following

$$(\mathcal{A}^{k-1})_i \ = \ \sum_{i_2, \ldots, i_k=1}^{n} a_{i_2i_3\cdots i_k}x_{i_2}\cdots x_{i_k}, \ \forall \ i \in [n].$$

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Let $x^{[k-1]} = (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1})^T \in \mathbb{R}^n$. If $Ax^{[k-1]} = \lambda x^{[k-1]}$ has a solution $x \in \mathbb{R}^n \setminus \{0\}$, then $\lambda$ is called an eigenvalue of $A$ and $x$ is an eigenvector associated with $\lambda$. And the spectral radius of $A$ is defined as $\lambda_1(A) = \max\{|\lambda|\}$ $\lambda$ is an eigenvalue of $A$. Also, a tensor $A$ of order $k$ and dimension $n$ uniquely determines a $k$-th degree homogeneous polynomial function $x^k$, which is

$$x^T (Ax^{[k-1]}) = \sum_{i=1}^{\delta} a_{i_1i_2i_k} x_{i_1}x_{i_2} \cdots x_{i_k}.$$ 

The adjacency tensor $[6]$ of a $k$-graph $G$ with $n$ vertices, denoted by $A(G)$, is an order $k$ dimension $n$ symmetric tensor with entries $a_{i_1i_2i_k}$ such that

$$a_{i_1i_2i_k} = \begin{cases} \frac{1}{(n-1)!}, & \text{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\ 0, & \text{otherwise}. \end{cases}$$

Let $\lambda$ be an eigenvalue of a $k$-graph $G$ with eigenvector $x$. Since $A(G)x^{[k-1]} = \lambda x^{[k-1]}$, we know that $cx$ is also an eigenvector of $\lambda$ for any nonzero constant $c$. So we can choose $x$ such that $\sum_{i=1}^{n} x_i = 1$. In this case, we have [6, 9]

$$\lambda = x^T (A(G)x^{[k-1]}) = k \sum_{e \in E(G)} x^e,$$

where $x^e = x_{i_1}x_{i_2} \cdots x_{i_k}$, $e = \{i_1, i_2, \ldots, i_k\}$.

For a $k$-graph $G$, we denote $N_G(v)$ as the set of neighbours of $v$ in $G$, and $E_G(v)$ as the set of edges containing $v$ in $G$. The degree of a vertex $v$ in $G$, denoted by $d_v = d_G(v)$, is $|E_G(v)|$. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimum degree and the maximum degree of $G$, respectively. If all vertices of $G$ have the same degree, then $G$ is called regular. Let $D = D(G)$ be a $k$-th order $n$-dimensional diagonal tensor with its diagonal element $d_{i,i}$ being $d_i$, the degree of vertex $i$ of $G$, for all $i \in [n]$. Then $Q(G) = D(G) + A(G)$ is the signless Laplacian tensor of the hypergraph $G$ [16]. The signless Laplacian eigenvalues refer to the eigenvalues of the signless Laplacian tensor. Let $q_1(G)$ be the signless Laplacian spectral radius of $G$.

In a $k$-graph $G$, a path of length $l$ is defined to be an alternating sequence of vertices and edges $u_1, e_1, u_2, \ldots, u_l, e_l, u_{l+1}$, where $u_1, \ldots, u_{l+1}$ are distinct vertices of $G$, $e_1, \ldots, e_l$ are distinct edges of $G$ and $u_i, u_{i+1} \in e_i$ for $i = 1, \ldots, l$. For any two vertices $u$ and $v$ of $G$, if there exists a path connecting $u$ and $v$, then $G$ is called connected. A hypergraph $G$ is called $f$-edge-connected if $G - U$ is connected for any edge subset $U \subseteq E(G)$ satisfying $|U| < f$. A hypergraph $G$ is called $f$-connected if there exist $f$ paths connecting $u$ and $v$ in $G$, where no pair of them have any other elements in common except $u$ and $v$, for any $u, v \in V(G)$ [27].

Spectral graph theory has a long history behind its development [2, 7]. It is natural to generalize spectral theory of graphs to hypergraphs. Recently, there are many work about the spectral theory of hypergraphs [8, 12, 14, 16, 17, 21–23]. In [20], Stevanović proposed a question: How small $\Delta - \lambda_1(G)$ can be when $G$ is an irregular graph with maximum degree $\Delta$ and spectral radius $\lambda_1(G)$? Ciobă et al. [5] gave a lower bound on $\Delta - \lambda_1(G)$ for irregular graphs, which improved previous bounds of Stevanović [20] and of Zhang [26]. Ciobă [4] obtained a lower bound on $\Delta - \lambda_1(G)$ for an irregular graph $G$ with maximum degree $\Delta$ and diameter $D$. Nikiforov [13] presented a lower bound on $\lambda_1(G) - \lambda_1(H)$ for a proper subgraph $H$ of a connected regular graph $G$. Shi [18] obtained a lower bound on $\Delta - \lambda_1(G)$ for a connected irregular graph $G$ in terms of its diameter and average degree. Ning et al. [15] gave a lower bound on $2\Delta - q_1(G)$ for a connected irregular graph $G$ in terms of the diameter. Shui et al. [19] gave a lower bound on $2\Delta - q_1(G)$ and $2\Delta - q_1(H)$ for a $k$-connected irregular graph $G$ and a proper spanning subgraph $H$ of a $k$-regular $k$-connected graph, respectively. Li et al. [10] obtained the lower bounds on $\lambda_1(G)$ for irregular connected $k$-graphs in terms of vertex degrees, the diameter, and the number of vertices and edges. Yuan et al. [25] gave some bounds on $\lambda_1(G)$ and $q_1(G)$ for a $k$-graph $G$ in terms of its degrees of vertices. Chen et al. [3] presented several upper bounds on $\lambda_1(G)$ and $q_1(G)$ for a $k$-graph $G$ in terms of degree sequences. We are inspired by two articles of Shui et al. [19] and Li et al. [10]. In this paper, we give the bounds of (signless Laplacian) spectral radius of subgraphs of $(f$-edge-connected $d$-regular (linear) $k$)-graphs. We also give the bounds of (signless Laplacian) spectral radius of connected nonregular (linear) $k$-graphs.
2. Preliminaries

In this section, we give some useful lemmas.

Let $G$ be a connected $k$-graph. By Perron-Frobenius theorem of nonnegative tensors [24], $\lambda_1(G)$ (resp., $q_1(G)$) is an eigenvalue of $A(G)$ (resp., $Q(G)$), and there exists a unique positive eigenvector $x = (x_1, \ldots, x_n)^T$ corresponding to $\lambda_1(G)$ (resp., $q_1(G)$) with $\sum_{i=1}^{n} x_i = 1$, and $x$ is called the principal eigenvector of $A(G)$ (resp., $Q(G)$).

The following Lemma 2.1 is from the proof of Theorem 4.1 in [10].

**Lemma 2.1.** ([10]) Let $G$ be a connected $k$-graph with $n$ vertices and minimum degree $\delta$. Then $x_u = \max_{i \in V(G)} |x_i|$ and $x_v = \min_{i \in V(G)} |x_i|$. Let $P : u = u_0, e_1, u_1, \ldots, u_r = v$ be a path from $u$ to $v$, where $e_i$ is an edge containing vertices $u_{i-1}$ and $u_i$. Then

$$\sum_{w_{r_i} \in E(P)} (x_w^2 - x^2) \geq \frac{k}{2} (x_u^2 - x_v^2).$$

**Lemma 2.2.** ([18]) Let $a_1, \ldots, a_n$ be nonnegative real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} - (a_1 \cdots a_n)^{\frac{1}{n}} \geq \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} (\sqrt[n]{a_i} - \sqrt[n]{a_j})^2,$$

equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

**Lemma 2.3.** ([18]) Let $a, b, y_1, y_2$ be positive numbers. Then

$$a(y_1 - y_2)^2 + b y_2^2 \geq \frac{ab}{a+b} y_1^2,$$

equality holds if and only if $y_2 = \frac{ay_1}{a+b}$.

Two paths $P_1, P_2$ are called edge-disjoint if the edges of $P_1$ have no common with the edges of $P_2$.

**Lemma 2.4.** ([27]) A hypergraph $G$ is $f$-edge-connected if and only if there are $f$ mutual edge-disjoint paths between each pair of vertices.

**Lemma 2.5.** ([27]) If a hypergraph $G$ is $f$-connected, then there are $f$ mutual vertex-disjoint paths between each pair of vertices.

**Lemma 2.6.** ([10]) Let $G$ be a connected $k$-graph with $n$ vertices, minimum degree $\delta$ and maximum degree $\Delta$, and let $x = (x_1, \ldots, x_n)^T$ be the principal eigenvector of $A(G)$. Then $x_{\max} \geq ((\frac{k}{\Delta} + 1) n - 1)^{\frac{1}{k}}$, where $x_{\max} = \max_{1 \leq i \leq n} |x_i|$.

In fact, we can prove similarly that Lemma 2.6 also holds for the principal eigenvector of $Q(G)$, where $G$ is a connected $k$-graph with $n$ vertices.

3. The (signless Laplacian) spectral radius of subgraphs of $f$-edge-connected $d$-regular $k$-graphs

In this section, we will give a bound of the spectral radius and the signless Laplacian spectral radius of a subgraph of a $f$-edge-connected $d$-regular $k$-graph $G$, respectively. And we will give a bound on the the spectral radius and the signless Laplacian spectral radius of a subgraph of a $f$-connected $d$-regular linear $k$-graph $G$, respectively.
Lemma 3.1. Let $H$ be a maximal proper subgraph of a $f$-edge-connected $d$-regular $k$-graph $G$ such that $f \geq 2$, and $\lambda_1(H)$ be the spectral radius of $H$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Then

$$d - \lambda_1(H) = \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e=\{v_i,v_{i+1},\ldots,v_n\}} (x_{v_i}^k + \cdots + x_{v_n}^k - kx^e),$$

where $d_i$ is the degree of the vertex $i$ of $H$.

Proof. Let $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that $H$ is connected since $f \geq 2$. Let $x_u = \max_{v \in V(H)}[x_v]$ and $x_v = \min_{v \in V(H)}[x_v]$. We claim $u \not= u_i$ for any $1 \leq i \leq k$. Indeed, if $u = u_i$ for some $1 \leq i \leq k$, then

$$\lambda_1(H)x_{u_i}^{k-1} = \sum_{e = \{u_i,u_{i+1},\ldots,u_k\} \in E(H)} a_{u_i,u_{i+1},\ldots,u_k}x_{u_{i+1}} \cdots x_{u_k} \leq (d - 1)x_{u_i}^{k-1},$$

and thus $\lambda_1(H) \leq d - 1$, contradicting the fact that $\lambda_1(H) > \frac{|E(H)|}{n} = d - \frac{k}{n} > d - 1$. We also find that

$$d - \lambda_1(H) = d \sum_{i=1}^{n} x_i^k - k \sum_{e \in E(H)} x^e$$

$$= d \sum_{i=1}^{n} x_i^k - d \sum_{i=1}^{n} d_i x_i^k + \sum_{e \in E(H)} d_i x_i^k - k \sum_{e \in E(H)} x^e$$

$$= \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e = \{v_i,v_{i+1},\ldots,v_n\} \in E(H)} (x_{v_i}^k + \cdots + x_{v_n}^k - kx^e).$$

□

Lemma 3.2. Let $H$ be a maximal proper subgraph of a $f$-edge-connected $d$-regular $k$-graph $G$ such that $f \geq 2$ and $q_1(H)$ be the signless Laplacian spectral radius of $H$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Then

$$2d - q_1(H) = 2d \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e = \{v_i,v_{i+1},\ldots,v_n\} \in E(H)} (x_{v_i}^k + \cdots + x_{v_n}^k - kx^e),$$

where $d_i$ is the degree of the vertex $i$ of $H$.

Proof. Similarly, let $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that $H$ is connected since $f \geq 2$. Let $x_u = \max_{v \in V(H)}[x_v]$ and $x_v = \min_{v \in V(H)}[x_v]$. We claim $u \not= u_i$ for any $1 \leq i \leq k$. Indeed, if $u = u_i$ for some $1 \leq i \leq k$, then

$$q_1(H)x_{u_i}^{k-1} = d_{u_i}x_{u_i}^{k-1} + \sum_{e = \{u_i,u_{i+1},\ldots,u_k\} \in E(H)} a_{u_i,u_{i+1},\ldots,u_k}x_{u_{i+1}} \cdots x_{u_k} \leq 2(d - 1)x_{u_i}^{k-1},$$

and thus $q_1(H) \leq 2d - 2$, contradicting the fact that $q_1(H) \geq 2\lambda_1(H) > 2\frac{|E(H)|}{n} = 2d - \frac{2k}{n} > 2d - 2$. We also find that

$$2d - q_1(H) = 2d \sum_{i=1}^{n} x_i^k - d \sum_{i=1}^{n} d_i x_i^k - k \sum_{e \in E(H)} x^e$$

$$= 2 \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e \in E(H)} d_i x_i^k - k \sum_{e \in E(H)} x^e$$

$$= 2 \sum_{i=1}^{n} (d - d_i)x_i^k + \sum_{e = \{v_i,v_{i+1},\ldots,v_n\} \in E(H)} (x_{v_i}^k + \cdots + x_{v_n}^k - kx^e).$$

□
Thus, we have $u$ connecting subgraph of $G$. If $f$, then $H$ be a maximal proper subgraph of $G$, i.e., $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $[u_1, u_2, \ldots, u_k]$. Let $\lambda_1(H)$ be the spectral radius of $H$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{v \in V(H)} \{x_v\}$ and $x_v = \min_{v \in V(H)} \{x_v\}$. By Lemmas 2.2 and 3.1, we have

$$d - \lambda_1(H) > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2)}.$$

Proof. Let $H$ be a maximal proper subgraph of $G$, with $n$ vertices and $m(= \frac{d}{k})$ edges, and $H'$ be a proper subgraph of $G$. If $f, k \geq 2$, then

$$d - \lambda_1(H') > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2)}.$$

By (3.1) and (3.2), we have

$$d - \lambda_1(H) > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2)}(x^k_u - x^k_v)^2.$$

Thus, we have

$$\sum_{w, w' \in E(H)} (x^k_{w'} - x^k_w)^2 \geq \sum_{t=1}^{f - 1} \sum_{w, w' \in E(P_t)} (x^k_{w'} - x^k_w)^2$$

$$\geq \sum_{t=1}^{f - 1} \frac{k}{2r_t} (x^k_{w'} - x^k_w)^2$$

$$\geq \frac{k(f - 1)^2}{\sum_{t=1}^{f - 1} 2r_t} (x^k_{w'} - x^k_w)^2$$

$$\geq \frac{k(f - 1)^2}{2(m - 1)} (x^k_{w'} - x^k_w)^2.$$

By (3.1) and (3.2), we have

$$d - \lambda_1(H) > \frac{k(f - 1)^2}{2(k - 1)(m - 1) + (f - 1)^2)} x^k_u.$$
Theorem 3.5. Let \( G \) be a \( f \)-edge-connected \( d \)-regular \( k \)-graph with \( n \) vertices and \( m(=\frac{dn}{k}) \) edges, and \( H' \) be a proper subgraph of \( G \). If \( f, k \geq 2 \), then

\[
d - \lambda_1(H') > \frac{2k(f - 1)^2}{[4k(m - 1) + (f - 1)^2](\frac{d - 1}{2})\frac{m}{d} + n - 1}.
\]

\( \square \)

Theorem 3.4. Let \( G \) be a \( f \)-edge-connected \( d \)-regular \( k \)-graph with \( n \) vertices and \( m(=\frac{dn}{k}) \) edges, and \( H' \) be a proper subgraph of \( G \). If \( f, k \geq 2 \), then

\[
2d - q_1(H') > \frac{2k(f - 1)^2}{[4k(m - 1) + (f - 1)^2](\frac{d - 1}{2})\frac{m}{d} + n - 1}.
\]

Proof. Let \( H \) be a maximal proper subgraph of \( G \), i.e., \( V(H) = V(G) \) and \( H \) differs from \( G \) in a single edge \( \{u_1, u_2, \ldots, u_k\} \). Let \( q_1(H) \) is the signless Laplacian spectral radius of \( H \) with a principal eigenvector \( x \). Let \( x_u = \max_{v \in V(H)} \{x_v\} \) and \( x_v = \min_{v \in V(H)} \{x_v\} \). By Lemmas 2.2 and 3.2, we have

\[
2d - q_1(H) > 2(x_u^2 + x_v^2 + \cdots + x_u^2) + \frac{1}{k-1} \sum_{v \in V(G) \setminus V(H)} (x_v^2 - x_u^2)^2 \geq 2kx_u^2 + \frac{1}{k-1} \sum_{v \in V(G) \setminus V(H)} (x_v^2 - x_u^2)^2. \tag{3.3}
\]

Since \( G \) is a \( f \)-edge-connected \( d \)-regular \( k \)-graph, there are at least \( f - 1 \) edge disjoint paths connecting \( u \) and \( v \) in \( H \). By (3.2) and (3.3), then we have

\[
2d - q_1(H) > 2kx_u^2 + \frac{k(f - 1)^2}{2k(m - 1) + (f - 1)^2}(x_u^2 - x_v^2)^2.
\]

The right hand side of the above inequality is a quadratic function of \( x_u^2 \). By Lemma 2.3, we have

\[
2d - q_1(H) > \frac{2k(f - 1)^2}{4k(m - 1) + (f - 1)^2} x_u^2.
\]

By Lemma 2.6, we have

\[
2d - q_1(H) > \frac{2k(f - 1)^2}{[4k(m - 1) + (f - 1)^2][\frac{d - 1}{2}]\frac{m}{d} + n - 1} = \frac{2k(f - 1)^2}{[4k(m - 1) + (f - 1)^2][\frac{d - 1}{2}]\frac{m}{d} + n - 1}.
\]

Therefore, we have

\[
2d - q_1(H') > \frac{2k(f - 1)^2}{[4k(m - 1) + (f - 1)^2][\frac{d - 1}{2}]\frac{m}{d} + n - 1}.
\]

\( \square \)

Theorem 3.5. Let \( G \) be a \( f \)-connected \( d \)-regular linear \( k \)-graph with \( n \) vertices, and \( H' \) be a proper subgraph of \( G \). If \( f, k \geq 2 \), then

\[
d - \lambda_1(H') > \frac{2k(f - 1)^2}{(2n + d(k^2 - k - 2) + 4)(f - 1)^2 + h'}
\]

where \( h = k(k - 1)(n - k - d + 2)((n + 2(f - 2)^2 - (f - 1)). \)
Proof. Let $H$ be a maximal proper subgraph of $G$, i.e., $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that $H$ is connected since $f \geq 2$. Let $\lambda_1(H)$ be the spectral radius of $H$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{i \in V(H)} |x_i|$ and $x_v = \min_{i \in V(H)} |x_i|$. By the proof of Lemma 3.1, we claim $u \neq u_i$ for $1 \leq i \leq k$. By Lemmas 2.2 and 3.1, we have

$$d - \lambda_1(H') \geq d - \lambda_1(H)$$

$$> x_{u_1}^k + x_{u_2}^k + \cdots + x_{u_k}^k + \frac{1}{k-1} \sum_{v, w_j \in E(H)} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2$$

$$\geq kx_v^k + \frac{1}{k-1} \sum_{v, w_j \in E(H)} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2.$$  \hfill (3.4)

Since $G$ is a $f$-connected $d$-regular $k$-graph, by Lemma 2.5, there are at least $f - 1$ vertex disjoint paths $P_1, P_2, \ldots, P_{f-1}$ connecting $u$ and $v$ in $H$. Thus, we have

$$\sum_{i=1}^{f-1} |V(P_i)| \leq n + 2(f - 2).$$

Since $G$ is a linear $k$-graph, we have $|V(P_i)| \geq |E(P_i)| + 1$. Hence, $\frac{2|E(P_i)|}{k} \leq \frac{|V(P_i)||V(P_i)|-1}{2}$. By Lemma 2.1, we have

$$\sum_{v, w_j \in E(H)} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2 \geq \sum_{i=1}^{f-1} \sum_{v, w_j \in E(P_i)} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2$$

$$\geq \sum_{i=1}^{f-1} \frac{k}{2 |E(P_i)|} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2$$

$$\geq \sum_{i=1}^{f-1} \frac{2}{|V(P_i)| - 1} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2$$

$$\geq \frac{2(f - 1)^2}{\sum_{i=1}^{f-1} |V(P_i)| - 1} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2$$

$$\geq \frac{2(f - 1)^2}{(\sum_{i=1}^{f-1} |V(P_i)| - 1)^2} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2$$

$$\geq \frac{2(f - 1)^2}{(n + 2(f - 2))^2} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2.$$  \hfill (3.5)

So by (3.4) and (3.5), we have

$$d - \lambda_1(H') > kx_v^k + \frac{2(f - 1)^2}{(k - 1)(n + 2(f - 2))^2 - (f - 1))} (x_v^{\frac{k}{2}} - x_w^{\frac{k}{2}})^2.$$  \hfill (3.6)

Define

$$C = \frac{2k(f - 1)^2}{(2n + d(k^2 - k - 2) + 4(f - 1)^2 + h'}$$

where $h = k(k - 1)(n - k - d + 2)((n + 2(f - 2))^2 - (f - 1)).$
Let $G$ be a $f$-connected $d$-regular linear $k$-graph with $n$ vertices, and $H$ be a maximal proper subgraph of $G$. From (3.6), we obtain
\[ d - \lambda_1(H') > (x^k_{u_1} + x^k_{u_2} + \cdots + x^k_{u_{d^2}}) + \frac{1}{k-1} \sum_{w,v \in E(H)} (x^k_w - x^k_v)^2 \]
\[ > C + \frac{1}{k-1} \sum_{w,v \in E(H)} (x^k_w - x^k_v)^2 \]
\[ \geq C. \]

**Case 1.** If $\sum_{i=1}^k x^k_{u_i} > C$, then from (3.4), we have
\[ d - \lambda_1(H') > \frac{2}{k-1} x^k_{u_1} + \frac{1}{k-1} \sum_{i=1}^{d^2} (x^k_i - x^k_i)^2 \]
\[ = \frac{1}{k-1} \sum_{i=1}^{d^2} \left( \frac{2}{d^2 - 2} x^k_i + (x^k_i - x^k_i)^2 \right) \]
\[ \geq \frac{1}{k-1} \sum_{i=1}^{d^2} \frac{2}{d^2 - 2} x^k_i \]
\[ \geq \frac{1}{k-1} \frac{2(dk - 1)}{C} \]
\[ = C. \]

**Case 2.** Let $x_{u_i} = \min_{1 \leq i \leq k}[x_{u_i}]$. Since $d_H(u_1) = d - 1$, it is possible to choose at least $d - 2$ distinct vertices $\{v_1, v_2, \ldots, v_{d-2}\}$ from $N_H(u_1)$ such that $u \notin \{v_1, v_2, \ldots, v_{d-2}\}$. If $\sum_{i=1}^{d-2} x^k_i \geq \frac{dk - 1}{2} C$, by (3.4) again and Lemma 2.3, then we have
\[ d - \lambda_1(H') > \frac{2(dk - 1)}{k(k-1)} \frac{2k(f-1)^2}{(n + d(k^2 - k - 1) + 2)(f-1)^2 - (f-1)} \]
\[ = C. \]

**Case 3.** Since $G$ is a linear $k$-graph, we have $v_i \neq u_i$ for $1 \leq i \leq d - 2$, $2 \leq i \leq k$. If $\sum_{i=1}^k x^k_{u_i} \leq C$ and $\sum_{i=1}^{d-2} x^k_i < \frac{dk - 1}{2} C$, then
\[ x^k_{u_i} \geq \frac{1 - \sum_{i=1}^k x^k_{u_i} - \sum_{i=1}^{d-2} x^k_i}{n - k - (d - 2)} \]
\[ > \frac{1}{n - k - d + 2} (1 - \frac{dk - 1}{2} C) = \frac{1}{n - k - d + 2} (1 - \frac{dk - d + 2}{2} C), \]
and from (3.6) and Lemma 2.3, we obtain
\[ d - \lambda_1(H') > \frac{2k(f-1)^2}{k(k-1)(n + 2(f-2))^2 - (f-1)} \]
\[ x^k_{u_i} = C. \]

\[ \square \]

**Theorem 3.6.** Let $G$ be a $f$-connected $d$-regular linear $k$-graph with $n$ vertices, and $H'$ be a proper subgraph of $G$. If $f, k \geq 2$, then
\[ 2d - q_1(H') > \frac{2k(f-1)^2}{(n + d(k^2 - k - 1) + 2)(f-1)^2 + h'}, \]
where $h = k(k-1)(n - k - d + 2)((n + 2(f-2))^2 - (f-1))$.

**Proof.** Let $H$ be a maximal proper subgraph of $G$, i.e., $V(H) = V(G)$ and $H$ differs from $G$ in a single edge $\{u_1, u_2, \ldots, u_k\}$. We know that $H$ is connected since $f \geq 2$. Let $q_1(H)$ be the signless Laplacian spectral radius of $H$ with a principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{v \in V(H)}[x_v]$ and $x_v = \min_{v \in V(H)}[x_v]$. By Lemmas 2.2 and 3.2, we have
\[ 2d - q_1(H') \geq 2d - q_1(H) > 2(x^k_{u_1} + x^k_{u_2} + \cdots + x^k_{u_k}) + \frac{1}{k-1} \sum_{w,v \in E(H)} (x^k_w - x^k_v)^2. \]
By (3.5) and (3.7), similarly, we have
\[
2d - q_1(H') > 2kx^k + \frac{2(f - 1)^2}{(k - 1)((n + 2)(f - 2)^2 - (f - 1))}(x^1_t - x^1_u)^2. \tag{3.8}
\]
Define
\[
C = \frac{2k(f - 1)^2}{(n + d(k^2 - k - 1) + 2)(f - 1)^2 + h'},
\]
where \( h = (k - 1)(n - k - d + 2)((n + 2)(f - 2)^2 - (f - 1)). \)

Case 1. If \( \sum_{i=1}^{k} x^k_{u_i} > \frac{C}{2}, \) then from (3.7), we have
\[
2d - q_1(H') > 2(x^k_{u_1} + x^k_{u_2} + \cdots + x^k_{u_{2k}}) + \frac{1}{k - 1} \sum_{v_i, v_j \in E(I)} (x_{u_{2i}}^1 - x_{u_{2i}}^1)^2
\]
\[
> 2\frac{C}{2} + \frac{1}{k - 1} \sum_{v_i, v_j \in E(I)} (x_{u_{2i}}^1 - x_{u_{2i}}^1)^2
\]
\[
\geq C.
\]

Case 2. Let \( x_{u_i} = \min_{1 \leq i \leq k} \{x_{u_i}\}. \) Since \( d_H(u_1) = d - 1, \) it is possible to choose at least \( d - 2 \) distance vertices \( \{v_1, v_2, \ldots, v_{d-2}\} \) from \( N_H(u_1) \) such that \( u \notin \{v_1, v_2, \ldots, v_{d-2}\}. \) If \( \sum_{i=1}^{d-2} x^k_{v_i} \geq \frac{d(k-1)}{2} C, \) by (3.7) again and Lemma 2.3, then we have
\[
2d - q_1(H') > \frac{2}{k - 1} x^k_{u_1} + \frac{1}{k - 1} \sum_{i=1}^{d-2} (x_{v_i}^1 - x_{u_i}^1)^2
\]
\[
= \frac{1}{k - 1} \sum_{i=1}^{d-2} \left( \frac{2}{d - 2} x^k_{u_1} + (x_{v_i}^1 - x_{u_i}^1)^2 \right)
\]
\[
\geq \frac{1}{k - 1} \sum_{i=1}^{d-2} \frac{2}{d - 2} \left( 1 + \frac{2}{d - 2} x^k_{v_i} \right)
\]
\[
\geq \frac{1}{k - 1} \frac{2(d(k - 1))}{(d - 2)} C
\]
\[
= C.
\]

Case 3. Since \( G \) is a linear \( k \)-graph, we have \( v_i \neq u_i, \) for \( 1 \leq t \leq d - 2, \) \( 2 \leq i \leq k. \) If \( \sum_{i=1}^{k} x^k_{u_i} \leq \frac{C}{2} \) and \( \sum_{i=1}^{d-2} x^k_{v_i} \leq \frac{d(k-1)}{2} C, \) then
\[
x^k_u \geq \frac{1 - \sum_{i=1}^{k} x^k_{u_i} - \sum_{i=1}^{d-2} x^k_{v_i}}{n - k - (d - 2)} > \frac{\frac{d(k-1)}{2} C}{n - k - d + 2 (1 - \frac{C}{2})} = \frac{\frac{d(k-1)}{2} C}{n - k - d + 2 (1 - \frac{dk - d + 1}{2} \frac{C}{2})}.
\]
and from (3.8) and Lemma 2.3, we obtain
\[
2d - q_1(H') > \frac{2k(f - 1)^2}{k(k - 1)((n + 2)(f - 2)^2 - (f - 1)) + (f - 1)^2 x^k_u} = C.
\]
\[\square\]
4. The signless Laplacian spectral radius of connected nonregular (linear) k-graphs

In this section, we mainly study the upper bounds of the (signless Laplacian) spectral radius of a f-(edge)-connected nonregular k-graph G with maximum degree δ, respectively.

**Theorem 4.1.** Let G be a nonregular f-edge-connected k-graph with n vertices, m edges, minimum degree δ and maximum degree δ. Then

\[ 2\Delta - q_1(G) > 2k(n\Delta - km) \frac{f^2}{[4m(k - 1)(n\Delta - km) + kf^2](\frac{k}{\delta})^{\frac{1}{m(n - 1)}} + n - 1}. \]

**Proof.** Let \( q_1(G) \) be the signless Laplacian spectral radius of G with the principal eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \). Let \( x_u = \max_{v \in V(G)}(x_i) \) and \( x_v = \min_{v \in V(G)}(x_i) \). We also find that

\[ 2\Delta - q_1(G) = 2\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{e \in E(G)} d_e x_i^2 - k \sum_{e \in E(G)} x^e \]

\[ = 2 \sum_{i=1}^{n} (\Delta - d_i) x_i^2 + \sum_{i=1}^{n} \sum_{e \in E(G)} d_e x_i^2 - k \sum_{e \in E(G)} x^e \]

\[ = 2 \sum_{i=1}^{n} (\Delta - d_i) x_i^2 + \sum_{e \in \{u_1, u_2, \ldots, u_k\} \in E(G)} x^e, \]

where \( d_i \) is the degree of the vertex \( i \). By Lemma 2.2, we have

\[ 2\Delta - q_1(G) > 2(n\Delta - km)x_v^2 + \frac{1}{k - 1} \sum_{u,v \in E(G)} (x_u^2 - x_v^2)^2. \] (4.1)

Let \( P_i : u = u_0, e_1, u_1, \ldots, u_n = v \) be a path from \( u \) to \( v \). By Lemma 2.1, we have

\[ \sum_{u,v \in E(P_i)} (x_u^2 - x_v^2)^2 \geq \frac{k}{2r_i} (x_u^2 - x_v^2)^2. \]

Since \( G \) is f-edge-connected, similar to (3.2), we have

\[ \sum_{u,v \in E(G)} (x_u^2 - x_v^2)^2 \geq \sum_{i=1}^{f} \frac{k}{2r_i} (x_u^2 - x_v^2)^2 \geq \frac{k f^2}{2m} (x_u^2 - x_v^2)^2. \] (4.2)

By (4.1) and (4.2), we have

\[ 2\Delta - q_1(G) > 2(n\Delta - km)x_v^2 + \frac{f^2}{2m(k - 1)} (x_u^2 - x_v^2)^2. \]

The right hand side of the above inequality is a quadratic function of \( x_u^2 \). By Lemma 2.3, we have

\[ 2\Delta - q_1(G) > \frac{2k(n\Delta - km)f^2}{4m(k - 1)(n\Delta - km) + kf^2} x_u^2. \]

By Lemma 2.6, we have

\[ 2\Delta - q_1(G) > \frac{2k(n\Delta - km)f^2}{[4m(k - 1)(n\Delta - km) + kf^2](\frac{k}{\delta})^{\frac{1}{m(n - 1)}} + n - 1}. \]

\[ \square \]
Theorem 4.2. Let $G$ be a nonregular $f$-connected linear $k$-graph with $n$ vertices, $m$ edges and maximum degree $\Delta$. Then
\[
2\Delta - q_1(G) > \frac{2(n\Delta - km) f^2}{(n + 2k - 1)(n\Delta - km) + (k - 2)(f - 1))^2 + h'}
\]
where $h = (n - f)(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f)$.

Proof. Let $q_1(G)$ be the signless Laplacian spectral radius of $G$ with the principal eigenvector $x = (x_1, x_2, \ldots, x_n)^T$. Let $x_u = \max_{v \in V(G)}|x_v|$ and $x_v = \min_{v \in V(G)}|x_v|$. Consider the following two cases:

Case 1. Suppose $d_u \leq \Delta - 1$. Since $Qx^{k-1} = q_1 x^{k-1}$, we have
\[
q_1(G)x_u^{k-1} = d_u x_u^{k-1} + \sum_{e=\{u,v_1,\ldots,v_{k-1}\} \in E(G)} x_{v_1}x_{v_2} \cdots x_{v_{k-1}} \leq 2(\Delta - 1)x_u^{k-1}.
\]
Thus, we have $q_1(G) \leq 2\Delta - 2$. Consequently,
\[
2\Delta - q_1(G) \geq 2 > \frac{2(n\Delta - km) f^2}{(n + 2k - 1)(n\Delta - km) + (k - 2)(f - 1))^2 + h'}
\]
where $h = (n - f)(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f)$.

Case 2. Suppose $d_u = \Delta$. Since $G$ is a $f$-connected $k$-graph, there are at least $f$ vertex disjoint paths $P_1, P_2, \ldots, P_f$ connecting $u$ and $v$ in $G$. By Lemma 2.5, we have
\[
\sum_{i=1}^f |V(P_i)| \leq n + 2(f - 1).
\]
Thus, we have
\[
2\Delta - q_1(G) = 2\Delta \sum_{i=1}^n x_i^k - \sum_{i=1}^n d_i x_i^k - k \sum_{e \in E(G)} x_e = 2 \sum_{i=1}^n (\Delta - d_i) x_i^k + \sum_{i=1}^n d_i x_i^k - k \sum_{e \in E(G)} x_e = 2 \sum_{i=1}^n (\Delta - d_i) x_i^k + \sum_{e=\{v_1,v_2,\ldots,v_k\} \in E(G)} x_{v_1}^k + \cdots + x_{v_k}^k - kx_e,
\]
where $d_i$ is the degree of the vertex $i$. By Lemma 2.2, we have
\[
2\Delta - q_1(G) > 2(n\Delta - km)x_u^k + \frac{1}{k - 1} \sum_{v \in V(G)} (x_v^k - x_u^k)^2. \tag{4.6}
\]
Similar to the proof of (3.5), we have
\[
\sum_{v \in V(G)} (x_v^k - x_u^k)^2 > \frac{2f^2}{(n + 2f - 2)^2 - f} (x_u^k - x_v^k)^2. \tag{4.7}
\]
By (4.6), (4.7) and Lemma 2.3, we have
\[
2\Delta - q_1(G) > 2(n\Delta - km)x_u^k + \frac{2f^2}{(k - 1)((n + 2f - 2)^2 - f)} (x_u^k - x_v^k)^2 \geq \frac{2(n\Delta - km) f^2}{(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f) + f^2 x_u^k}. \tag{4.8}
\]
Define
\[
C = \frac{2(n\Delta - km) f^2}{(n + 2(k - 1)(n\Delta - km) + (k - 2)(f - 1))^2 + h'}
\]
where \(h = (n - f)(k - 1)(n\Delta - km)((n + 2f - 2)^2 - f)\).

**Case 2.1.** Suppose \(f = 1\), we have
\[
2\Delta - q_1(G) > \frac{2(n\Delta - km)}{(k - 1)(n\Delta - km)(n^2 - 1) + 1} x_v^k
\]
and
\[
C = \frac{2(n\Delta - km)}{(n + 2(k - 1)(n\Delta - km)) + h'}
\]
where \(h = (n - 1)(k - 1)(n\Delta - km)(n^2 - 1)\).

**Case 2.1.1.** If \(x_v^k \geq \frac{C}{2(n\Delta - km)}\), then from (4.6) and (4.7), we obtain
\[
2\Delta - q_1(G) > 2(n\Delta - km) \frac{C}{2(n\Delta - km)} + \frac{2f^2}{(k - 1)((n + 2f - 2)^2 - f)} (x_v^k - x_v^i)^2 > C.
\]

**Case 2.1.2.** If \(x_v^k < \frac{C}{2(n\Delta - km)}\), then since \(\sum_{i=1}^{n} x_i^k = 1\), we have
\[
x_u^k \geq \frac{1 - x_v^k}{n - 1} > \frac{1}{n - 1} \left(1 - \frac{C}{2(n\Delta - km)}\right).
\]
Thus, by (4.9), we have
\[
2\Delta - q_1(G) > C.
\]

**Case 2.2.** Suppose \(f \geq 2\).

**Case 2.2.1.** If \(x_v^k \geq \frac{C}{2(n\Delta - km)}\), then the result can be obtained using a similar argument of the case 2.1.1.

**Case 2.2.2.** Since \(G\) is a \(f\)-connected linear \(k\)-graph, we have \(d_v \geq f\). We can choose at least \(f - 1\) vertices from \(N_G(v)\), denoted by \(\{v_1, v_2, \ldots, v_{f-1}\}\), such that \(u \notin \{v_1, v_2, \ldots, v_{f-1}\}\). If \(\sum_{i=1}^{f-1} x_v^k > C(k - 1)(1 + \frac{f - 1}{2(n\Delta - km)})\), by (4.6), we have
\[
2\Delta - q_1(G) > 2(n\Delta - km)x_v^k + \frac{1}{k - 1} \sum_{i=1}^{f-1} (x_v^k - x_v^i)^2 \\
\geq 2(n\Delta - km)x_v^k + \frac{1}{k - 1} \sum_{i=1}^{f-1} (x_v^k - x_v^i)^2.
\]
Similar to the proof of the case 2 of Theorem 3.6, we have
\[
2\Delta - q_1(G) > C.
\]

**Case 2.2.3.** If \(x_v^k < \frac{C}{2(n\Delta - km)}\) and \(\sum_{i=1}^{f-1} x_v^k \leq C(k - 1)(1 + \frac{f - 1}{2(n\Delta - km)})\), by \(\sum_{i=1}^{n} x_i^k = 1\), then we have
\[
x_u^k \geq \frac{1}{n - f} (1 - x_v^k) \sum_{i=1}^{f-1} x_v^i > \frac{1}{n - f} (1 - \frac{2(k - 1)(n\Delta - km) + (k - 1)(f - 1) + 1}{2(n\Delta - km)} C).
\]
Thus, by (4.8), we have
\[
2\Delta - q_1(G) > C.
\]
\(\square\)
Theorem 4.3. Let $G$ be a nonregular $f$-connected linear $k$-graph with $n$ vertices, $m$ edges and maximum degree $\Delta$. Then

$$\Delta - \lambda_1(G) > \frac{2(n\Delta - km)}{2(n + (k-1)(n\Delta - km) + (k-2)(f-1)) f^2 + h'},$$

where $h = (n-f)(k-1)(n\Delta - km)((n+2f-2)^2 - f)$.

Proof. The result can be obtained by using a similar argument of Theorem 4.2. □

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