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# Multiplication Operators on Some Morrey Spaces

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**Abstract.** The paper aims to discuss some results characterizing various multiplication operators such as compact, invertible and Fredhlom on Morrey and discrete Morrey spaces respectively. Some other relevant results necessary to establish the main results have also been investigated in the sequel.

## 1. Introduction

Morrey spaces were first introduced by C.B. Morrey in relation to the study of the solution of certain elliptic partial differential equations (see [1]). Many operators that are initially studied on Lebesgue spaces $L^p\left(\mathbb{R}^d\right)$ have discrete analogues on $\ell^p\left(\mathbb{Z}^d\right)$ (for instance, see [6], [7], [9], [13], [14], [15], [16]). Some of these operators have also been studied on continuous Morrey spaces $\mathcal{M}_q^p\left(\mathbb{R}^d\right)$ (for example, see [3], [4], [5], [8], [10], [11], [12]). One can refer to [17] for various spaces related to Morrey spaces, and [18], [19] for some relevant results on various multiplication operators. Discrete analogues of Morrey spaces and their generalizations have been studied in [2].

Let $m \in \mathbb{Z}$, $N \in w = \mathbb{N} \cup \{0\}$, and write $S_{m,N} = \{m-N, ..., m, ..., m+N\}$. Then $|S_{m,N}| = 2N+1$, the cardinality of $S_{m,N}$. For $1 \le p \le q < \infty$, the discrete Morrey space $\ell_q^p = \ell_q^p(\mathbb{Z})$ is defined to be the set of all sequences $x = (x_k)_{k\in\mathbb{Z}}$ taking values in $\mathbb{R}$ or $\mathbb{C}$ such that

$$\|x\|_{\ell_q^p} = \sup_{m\in\mathbb{Z}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k|^p\right)^{\frac{1}{p}} < \infty.$$

The discrete Morrey space $\ell_q^p = \ell_q^p(\mathbb{Z})$ is a Banach space under the above norm. We note that when $p = q$, we have $\ell_p^p = \ell^p$, the space of $p$-summable sequences with integer indices.

A multiplication operator is an operator $T_f$ defined on some vector space of functions and whose value at a function $g$ is given by multiplication by a fixed function $f$. That is,

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$$T_f g(x) = f(x) g(x),$$

for all $g$ in the domain of $T_f$, and all $x$ in the domain of $g$.

Let $u : X \longrightarrow \mathbb{C}$ be a function such that $u.f \in \ell_q^p$ for every $f \in \ell_q^p$. Then we can define a multiplication transformation $M_u : \ell_q^p \longrightarrow \ell_q^p$ by

$$M_u f = u.f, \forall f \in \ell_q^p.$$

If $M_u$ is continuous, we call it a multiplication operator induced by $u$.

For $1 \leq p \leq q < \infty$, the Morrey space $\mathcal{M}_q^p = \mathcal{M}_q^p\left(\mathbb{R}^d\right)$ is the set of all $p$-locally integrable functions $f$ on $\mathbb{R}^d$ such that

$$\|f\|_{\mathcal{M}_q^p} = \sup_{a \in \mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left( \int_{B(a,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

Here, $B(a,r)$ denotes the open ball in $\mathbb{R}^d$ centered at $a$ and radius $r > 0$, and $|B(a,r)|$ denotes its Lebesgue measure. The Morrey space $\mathcal{M}_q^p\left(\mathbb{R}^d\right)$ is a Banach space under the above norm. Note that when $p = q$, one can recover the Lebesgue space $L^p\left(\mathbb{R}^d\right)$ as the special case of the Morrey space $\mathcal{M}_q^p\left(\mathbb{R}^d\right)$.

Let $\theta : X \longrightarrow \mathbb{C}$ be a function such that $\theta.f \in \mathcal{M}_q^p$ for every $f \in \mathcal{M}_q^p$. Then we can define a multiplication transformation $M_\theta : \mathcal{M}_q^p \longrightarrow \mathcal{M}_q^p$ by

$$M_\theta f = \theta.f, \forall f \in \mathcal{M}_q^p.$$

If $M_\theta$ is continuous, we call it a multiplication operator induced by $\theta$.

A bounded linear operator $T : A \longrightarrow A$ (where $A$ is a Banach space) is called compact if $T(B_1)$ has compact closure, where $B_1$ denotes the closed unit ball of $A$.

A bounded linear operator $T : A \longrightarrow A$ is called Fredhlom if $A$ has closed range, dim(ker$A$) and co-dim(ran$A$) are finite.

The sequence $e^n$ is defined as $e^n(k) = \delta_{n_k}$, the Kronecker delta. By $B(A)$, we denote the Banach algebra of bounded linear operators from $A$ into itself.

## 2. Main Results

**Theorem 2.1.** *Let $\theta : \mathbb{Z} \longrightarrow \mathbb{C}$ be a mapping. Then $M_\theta : \ell_q^p \longrightarrow \ell_q^p$ is a bounded operator if and only if $\theta$ is a bounded function.*

*Proof.* Let $\theta$ be a bounded function. Then there exists $M > 0$ such that

$$|\theta_n| \leqslant M, \forall n \in \mathbb{Z}.$$

Let $x = (x_k)_{k \in \mathbb{Z}} \in \ell_q^p$. Then,

$$\begin{aligned}
\|M_\theta x\| &= \sup_{m\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |(\theta x)_k|^p \right)^{\frac{1}{p}} \\
&= \sup_{m\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |\theta_k|^p |x_k|^p \right)^{\frac{1}{p}} \\
&\leqslant M(2N+1)^{\frac{1}{p}} \|x\| \\
&= M^/ \|x\|,
\end{aligned}$$

where $M^/ = M(2N+1)^{\frac{1}{p}}$ .

Thus,
$$\|M_\theta x\| \leqslant M^/ \|x\|, \forall x \in \ell_q^p.$$

Therefore, $M_\theta$ is a bounded operator.

Conversely, we assume that $M_\theta$ is a bounded operator. We are required to proof that $\theta$ is a bounded mapping. Suppose if possible $\theta$ is not a bounded mapping, then for every $n \in \mathbb{Z}$, there exists some $q_n \in \mathbb{Z}$ such that $|\theta_{q_n}| > n$.

Now,

$$\|e^{q_n}\| = \sup_{m=q_n\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}.$$

Let $e^{q_n/} = \frac{e^{q_n}}{\|e^{q_n}\|}$. Then $\|e^{q_n/}\| = 1$.

But

$$\begin{aligned}
\|M_\theta e^{q_n/}\| &= \frac{\|M_\theta e^{q_n}\|}{\|e^{q_n}\|} \\
&= \frac{\sup_{m=q_n\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} |\theta_{q_n}|}{\sup_{m=q_n\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}} \\
&= |\theta_{q_n}| > n,
\end{aligned}$$

which contradicts the boundedness of $M_\theta$.

Hence, $\theta$ must be a bounded function. □

**Example 2.2.** *Let us define $\theta : \mathbb{Z} \longrightarrow \mathbb{C}$ by*

$$\theta(n) = e^{in}, \forall n \in \mathbb{Z}.$$

*Then for every $x \in \ell_q^p$, we have*

$$\begin{aligned}
\|M_\theta x\| &= \sup_{m\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |e^{ik}|^p |x_k|^p \right)^{\frac{1}{p}} \\
&= \sup_{m\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}}
\end{aligned}$$

$= \|x\|.$

*Therefore, $M_\theta$ is a bounded operator.*

**Theorem 2.3.** *$M_\theta$ is an isometry if and only if $|\theta_n| = 1$, for all $n \in \mathbb{Z}$.*

*Proof.* For the necessary part we assume that $M_\theta$ is an isometry. Then for every $x \in \ell_q^p$, we have

$$\|M_\theta x\| = \|x\|.$$

This implies that

$$\sup_{m\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |\theta_k|^p |x_k|^p \right)^{\frac{1}{p}} = \sup_{m\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}}.$$

Thus,

$|\theta_n| = 1$, for all $n \in \mathbb{Z}$.

The sufficient part is trivial. □

**Theorem 2.4.** *Let $M_\theta \in B\left(\ell_q^p\right)$. Then $M_\theta$ is a compact operator if and only if $\theta_n \to 0$ as $n \to \infty$.*

*Proof.* Suppose that $M_\theta$ is a compact operator. We need to proof that $\theta_n \to 0$ as $n \to \infty$. If not, then there exists $\epsilon > 0$ such that the set $N_\epsilon = \{k \in \mathbb{Z} : |\theta_k| \geqslant \epsilon\}$ is an infinite set. Let $q_1, q_2, \dots, q_n, \dots$ be in $N_\epsilon$.

Let $e^{q_n'} = \frac{e^{q_n}}{\|e^{q_n}\|}$. Then $\{e^{q_n'} : q_n \in N_\epsilon\}$ is an infinite bounded set in $\ell_q^p$.

Now,

$$\|M_\theta e^{q_n'} - M_\theta e^{q_m'}\| = \|\theta e^{q_n'} - \theta e^{q_m'}\|$$

$$\geqslant \epsilon \|e^{q_n'} - e^{q_m'}\|.$$

Thus, the set $\{M_\theta e^{q_n'} : q_n \in N_\epsilon\}$ cannot have a convergent subsequence. This contradicts the compactness of $M_\theta$. Hence, $\theta_n \to 0$ as $n \to \infty$.

Conversely, suppose that $\theta_n \to 0$ as $n \to \infty$. Then for every $\epsilon > 0$, $N_\epsilon = \{n \in \mathbb{Z} : |\theta_n| \geqslant \epsilon\}$ is a finite set. Then $\ell_q^p(N_\epsilon)$ is a finite dimensional space for every $\epsilon > 0$. So, $M_\theta|\ell_q^p(N_\epsilon)$ is a compact operator. For each $n \in \mathbb{Z}$, define $\theta_n : \mathbb{Z} \longrightarrow \mathbb{C}$ by

$$\theta_n(m) = \begin{cases} \theta(m), \forall m \in N_{\frac{1}{n}} \\ 0, \forall m \notin N_{\frac{1}{n}}. \end{cases}$$

Clearly, $M_{\theta_n}$ is a compact operator as the space $\ell_q^p\left(N_{\frac{1}{n}}\right)$ is a finite dimensional space for each $n \in \mathbb{Z}$.

Now,

$$\begin{aligned} \|(M_{\theta_n} - M_\theta)x\| &= \sup_{m\in\mathbb{Z},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |\theta_n(k)x_k - \theta(k)x_k|^p \right)^{\frac{1}{p}} \\ &= \sup_{m\in N_{\frac{1}{n}},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |\theta_n(k)x_k - \theta(k)x_k|^p \right)^{\frac{1}{p}} + \sup_{m\in N^c_{\frac{1}{n}},N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |\theta_n(k)x_k - \theta(k)x_k|^p \right)^{\frac{1}{p}} \end{aligned}$$

$$= \sup_{m\in N^c_{\frac{1}{n}}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |\theta(k) x_k|^p \right)^{\frac{1}{p}}$$
$$< \frac{1}{n} \sup_{m\in N^c_{\frac{1}{n}}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}}$$
$$\leq \frac{1}{n}\|x\|.$$

This means that $\|(M_{\theta_n} - M_\theta)x\| < \frac{1}{n}\|x\|$. Therefore, $\|M_{\theta_n} - M_\theta\| < \frac{1}{n}$ and $M_\theta$ is a limit of compact operators and hence $M_\theta$ is a compact operator. □

**Theorem 2.5.** *Let $M_\theta \in B\left(\ell^p_q\right)$. Then $M_\theta$ has closed range if and only if $\theta$ is bounded away from zero on $\mathbb{Z}\setminus ker\theta = S$.*

*Proof.* Let $\theta$ be bounded away from zero on $S$. Then there exists $\epsilon > 0$, such that $|\theta_k| \geq \epsilon \forall k \in S$. We are required to prove that range of $M_\theta$ is closed. Let $y$ be a limit point of ran$M_\theta$. Then there exists a sequence $\{y^{(n)}\}$ in ran$M_\theta$ such that $y^{(n)} \to y$, where $y^{(n)} = M_\theta x^{(n)}$, for some $x^{(n)} = \{x^{(n)}_k\}$ in $\ell^p_q$. Clearly, the sequence $\{M_\theta x^{(n)}\}$ is a Cauchy sequence.

Now,

$$\|M_\theta x^{(n)} - M_\theta x^{(m)}\| = \sup_{m\in\mathbb{Z}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |\theta_k x^{(n)}_k - \theta_k x^{(m)}_k|^p \right)^{\frac{1}{p}}$$
$$= \sup_{m\in\mathbb{Z}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S, k\in S_{m,N}} |\theta_k|^p |x^{(n)}_k - x^{(m)}_k|^p \right)^{\frac{1}{p}}$$
$$\geq \epsilon \sup_{m\in\mathbb{Z}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S, k\in S_{m,N}} |x^{(n)}_k - x^{(m)}_k|^p \right)^{\frac{1}{p}}$$
$$= \epsilon \sup_{m\in\mathbb{Z}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left( \sum_{k\in S_{m,N}} |x^{(n)^\sim}_k - x^{(m)^\sim}_k|^p \right)^{\frac{1}{p}}$$
$$= \epsilon \|x^{(n)^\sim} - x^{(m)^\sim}\|,$$

where

$$x^{(n)^\sim}_k = \begin{cases} x^{(n)}_k, \text{ if } k \in S \\ 0, \text{ if } k \notin S. \end{cases}$$

Therefore, $\{x^{(n)^\sim}\}$ is a Cauchy sequence in $\ell^p_q$. But $\ell^p_q$ is a Banach space. So, there exists $x \in \ell^p_q$ such that $x^{(n)^\sim} \to x$ as $n \to \infty$. In view of continuity of $M_\theta$, $M_\theta x^{(n)^\sim} \to M_\theta x$. But $M_\theta x^{(n)} = M_\theta x^{(n)^\sim} \to y$. Therefore, $M_\theta x = y$. Hence, $y \in$ ran$M_\theta$. This implies $M_\theta$ has closed range.

Conversely, suppose that $M_\theta$ has closed range. Then $M_\theta$ is bounded away from zero on $(\ker M_\theta)^\perp = \ell^p_q(\mathbb{Z} \setminus \ker\theta)$. That is, there exists $\epsilon > 0$ such that

$$\|M_\theta x\| \geq \epsilon\|x\|, \forall x \in \ell^p_q(\mathbb{Z} \setminus \ker\theta). \tag{1}$$

Let $B = \{k \in \mathbb{Z} \setminus \ker\theta : |\theta_k| < \frac{\epsilon}{2}\}$. If $B \neq \phi$, then for $r_0 \in B$, we have

$$\|M_\theta e^{r_0}\| = \sup_{m=r_0\in\mathbb{Z}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} |\theta_{r_0}|$$
$$< \frac{\epsilon}{2} \sup_{m=r_0\in\mathbb{Z}, N\in w} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}$$

$< \epsilon \| e^{r_0} \|.$

So,

$\| M_\theta e^{r_0} \| < \epsilon \| e^{r_0} \|$, which contradicts (1).

Thus, $B = \phi$ and this proves the theorem. □

**Theorem 2.6.** *Let $\theta : \mathbb{Z} \longrightarrow \mathbb{C}$ be a mapping. Then $M_\theta : \ell_q^p \longrightarrow \ell_q^p$ is invertible if and only if there exist $k > 0$ and $K > 0$ such that $k < \theta_n < K$, for all $n \in \mathbb{Z}$.*

*Proof.* We first assume that the condition i.e, there exist $k > 0$ and $K > 0$ such that $k < \theta_n < K$, for all $n \in \mathbb{Z}$ holds. Define $\alpha : \mathbb{Z} \longrightarrow \mathbb{C}$ by $\alpha_n = \frac{1}{\theta_n}$. Then by Theorem 2.1, $M_\theta$ and $M_\alpha$ are both bounded linear operators. Also, $M_\theta M_\alpha = M_\alpha M_\theta = I$. Hence, $M_\alpha$ is the inverse of $M_\theta$.

Next, we assume that $M_\theta$ is invertible. Then $\text{ran} M_\theta = \ell_q^p$. So, $\text{ran} M_\theta$ is closed. This implies there exists $\epsilon > 0$ such that $|\theta_n| \geq \epsilon$, $\forall n \in \mathbb{Z} \setminus \ker\theta$, by Theorem 2.5. Now, if $\theta_{m_0} = 0$, for some $m_0 \in \mathbb{Z}$, then $e^{m_0} \in \ker M_\theta$, which contradicts the fact that $M_\theta$ is one-one. Thus, $\ker\theta$ is the empty set. Hence, $|\theta_n| \geq \epsilon$, $\forall n \in \mathbb{Z}$. Since $M_\theta$ is bounded, so there exists $K > 0$ such that $|\theta_n| \leq K$, $\forall n \in \mathbb{Z}$, using Theorem 2.1. Hence, $\epsilon \leq |\theta_n| \leq K$, $\forall n \in \mathbb{Z}$. □

**Theorem 2.7.** *Let $M_\theta : \ell_q^p \longrightarrow \ell_q^p$ be a bounded operator. Then $M_\theta$ is Fredholm operator if and only if*

($a$) $\ker\theta$ *is a finite subset of* $\mathbb{Z}$.

($b$) $|\theta_n| \geq \epsilon, \forall n \in \mathbb{Z} \setminus \ker\theta$.

*Proof.* Suppose $M_\theta$ is Fredholm operator. Then $M_\theta$ has closed range. Therefore, condition ($b$) is satisfied from Theorem 2.5.

Next, if $\ker\theta$ is an infinite subset of $\mathbb{Z}$, then $e^n \in \ker M_\theta$, for all $n \in \ker\theta$. But $e^n$'s are linearly independent. This means that $\ker M_\theta$ is an infinite dimensional, which is absurd as $M_\theta$ is a Fredholm operator. Hence $\ker\theta$ must be a finite subset of $\mathbb{Z}$.

Conversely, we assume that the conditions ($a$), ($b$) are fulfilled. Condition ($a$) states that $\dim(\ker M_\theta)$ and co-dim(ran $M_\theta$) are finite. Also, from condition ($b$), we have $\text{ran} M_\theta$ is closed by using Theorem 2.5. Hence, $M_\theta$ is a Fredholm operator. □

**Theorem 2.8.** *Let $u : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a p-locally integrable function. Then $M_u : \mathcal{M}_q^p \longrightarrow \mathcal{M}_q^p$ is a bounded operator if u is a bounded function.*

*Proof.* Suppose $u$ is a bounded $p$-locally integrable function. Then for every $f \in \mathcal{M}_q^p\left(\mathbb{R}^d\right)$, we have

$$\begin{aligned}
\|M_u f\|_{\mathcal{M}_q^p} &= \sup_{a \in \mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left( \int_{B(a,r)} |(uf)(y)|^p dy \right)^{\frac{1}{p}} \\
&= \sup_{a \in \mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left( \int_{B(a,r)} |u(y) f(y)|^p dy \right)^{\frac{1}{p}} \\
&\leq \|u\|_\infty \sup_{a \in \mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left( \int_{B(a,r)} |f(y)|^p dy \right)^{\frac{1}{p}} \\
&= \|u\|_\infty \|f\|_{\mathcal{M}_q^p}.
\end{aligned}$$

Thus,

$$\|M_u f\|_{\mathcal{M}_q^p} \leq \|u\|_\infty \|f\|_{\mathcal{M}_q^p},$$

which means that $M_u$ is a bounded operator. □

**Theorem 2.9.** *Let $M_u$ be a compact operator, for each $\epsilon > 0$, define $A_\epsilon(u) = \left\{x \in \mathbb{R}^d : |u(x)| \geq \epsilon\right\}$, and $\mathcal{M}_q^p(A_\epsilon(u)) = \left\{f\chi_{A_\epsilon(u)} : f \in \mathcal{M}_q^p\left(\mathbb{R}^d\right)\right\}$. Then $\mathcal{M}_q^p(A_\epsilon(u))$ is a closed invariant subspace of $\mathcal{M}_q^p\left(\mathbb{R}^d\right)$ under $M_u$. Moreover, $M_u|\mathcal{M}_q^p(A_\epsilon(u))$ is a compact operator.*

*Proof.* Let $h, s \in \mathcal{M}_q^p(A_\epsilon(u))$ and $\alpha, \beta \in \mathbb{R}$. Then

$h = f\chi_{A_\epsilon(u)}$ and $s = g\chi_{A_\epsilon(u)}$,
for some $f, g \in \mathcal{M}_q^p\left(\mathbb{R}^d\right)$.

Now,

$$\begin{aligned}\alpha h + \beta s &= \alpha\left(f\chi_{A_\epsilon(u)}\right) + \beta\left(g\chi_{A_\epsilon(u)}\right)\\ &= (\alpha f + \beta g)\,\chi_{A_\epsilon(u)} \in \mathcal{M}_q^p(A_\epsilon(u)).\end{aligned}$$

So,

$\mathcal{M}_q^p(A_\epsilon(u))$ is a subspace of $\mathcal{M}_q^p\left(\mathbb{R}^d\right)$.

Next, for all $h \in \mathcal{M}_q^p(A_\epsilon(u))$, we have

$M_u h = uh = u\left(f\chi_{A_\epsilon(u)}\right) = (uf)\,\chi_{A_\epsilon(u)}$,
where $uf \in \mathcal{M}_q^p\left(\mathbb{R}^d\right)$.

Therefore, $M_u h \in \mathcal{M}_q^p(A_\epsilon(u))$.

Thus, $\mathcal{M}_q^p(A_\epsilon(u))$ is an invariant subspace of $\mathcal{M}_q^p\left(\mathbb{R}^d\right)$ under $M_u$.

Next, we claim that $\mathcal{M}_q^p(A_\epsilon(u))$ is a closed set.

Let $y$ be a function belonging to the closure of $\mathcal{M}_q^p(A_\epsilon(u))$, then there exists a sequence $\{y_n\}$ in $\mathcal{M}_q^p(A_\epsilon(u))$ such that $y_n \to y$ in $\mathcal{M}_q^p\left(\mathbb{R}^d\right)$. Note that

$y = y\chi_{A_\epsilon(u)} + y\chi_{A_\epsilon^c(u)}$.

Next, we want to show that $y\chi_{A_\epsilon^c(u)} = 0$.

For a given $\epsilon_1 > 0$, there exists $n_0 \in \mathbb{Z}$ such that

$$\begin{aligned}\|y\chi_{A_\epsilon^c(u)}\| &= \|\left(y - y_{n_0} + y_{n_0}\right)\chi_{A_\epsilon^c(u)}\|\\ &= \|\left(y - y_{n_0}\right)\chi_{A_\epsilon^c(u)} + y_{n_0}\chi_{A_\epsilon^c(u)}\|\end{aligned}$$

$$= \| (y - y_{n_0}) \chi_{A_\epsilon^c(u)} \|$$
$$\leq \|y - y_{n_0}\|$$
$$< \epsilon_1,$$

as $y_n \to y$.

Thus, $y\chi_{A_\epsilon^c(u)} = 0$, which means that $y = y\chi_{A_\epsilon(u)} \in \mathcal{M}_q^p (A_\epsilon (u))$.

This completes the proof. □

**Proposition 2.10.** *$M_u$ is one-one on $\mathcal{M}_q^p (supp (u))$, where $supp (u) = \{x \in \mathbb{R}^d : u (x) \neq 0\}$.*

*Proof.* Let $Y = \mathcal{M}_q^p (\text{supp} (u))$

$$= \left\{ f\chi_{\text{supp}(u)} : f \in \mathcal{M}_q^p \left(\mathbb{R}^d\right)\right\}.$$

Assume that $M_u (f^\sim) = 0$, for some $f^\sim = f\chi_{\text{supp}(u)} \in Y$.

Then,

$$M_u \left(f\chi_{\text{supp}(u)}\right) = 0$$

So,

$$uf\chi_{\text{supp}(u)} = 0$$
$$\left(uf\chi_{\text{supp}(u)}\right)(x) = 0, \forall x \in \text{supp} (u)$$
$$f (x) \chi_{\text{supp}(u)} (x) = 0, \forall x \in \text{supp} (u)$$
$$\left(f\chi_{\text{supp}(u)}\right)(x) = 0, \forall x \in \mathbb{R}^d$$
$$f\chi_{\text{supp}(u)} = 0.$$

Thus,

$$f^\sim = 0.$$

Hence, $M_u$ is injective. □

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