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Best Proximity Points Revisited

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Abstract. In this paper, using the concept of w-distance on a metric space, we prove some new best proximity point results for the mappings of Meir-Keeler type. As an application, we derive some recent best proximity point results of the aforementioned type.

1. Introduction and preliminaries

In 1996 Kada, Suzuki and Takahashi [10] introduced and studied the concept of w-distance in fixed point theory. They gave examples of the w-distance and, among other things, generalized Caristi's fixed point theorem, Ekeland's variational principle and the nonconvex minimization theorem by Takahashi. For more recent, related results on w-distance see [3, 5–8].

Definition 1.1. *Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:*

(P1) $p(x, z) \leq p(x, y) + p(y, z)$, *for any* $x, y, z \in X$,
(P2) *for any* $x \in X$, *function* $p(x, \cdot) : X \to [0, \infty)$ *is lower semicontinuous,*
(P3) *for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $p(z, x) \leq \delta$ *and* $p(z, y) \leq \delta$ *imply* $d(x, y) \leq \epsilon$.

Let us recall that a real-valued function f defined on a metric space X is said to be lower semicontinuous at a point x_0 in X if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} f(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$. The following, very useful lemma is proved in [10].

Lemma 1.2. *Let (X, d) be a metric space and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(i) *If* $p(x_n, y) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $p(x, y) = 0$ *and* $p(x, z) = 0$, *then* $y = z$;
(ii) *if* $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* y_n *converges to* z;
(iii) *if* $p(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence.*
(iv) *if* $p(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

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In 1969, Meir and Keeler [14] have proven the following very interesting fixed point theorem, which has been widely discussed recently due to its peculiar nature as well as many useful applications.

Theorem 1.3. *Let (X, d) be a complete metric space, and let T be a self-mapping on X. Suppose that for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for every $x, y \in X$, the condition*

$$\varepsilon \leq d(Tx, Ty) < \varepsilon + \delta \Rightarrow d(x, y) < \varepsilon$$

holds. Then T has a unique fixed point $x \in X$, and for every $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x.

Let (X, d) be a metric space, A and B two nonempty subsets of X and $T : A \to B$ a non-self-mapping.
The following notations will be used throughout the paper (see e.g. [1, 2, 4, 9, 11–13, 15]):

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$
$$d(y, A) = \inf\{d(x, y) : x \in A\} = d(\{y\}, A)$$
$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}$$

In this paper, firstly we introduce the notions of *MK-p*-proximal contractions by using the concept of w-distance. Then we prove some new best proximity point results for *MK-p*-proximal contractions on complete metric spaces. As an application, we derive the recent best proximity point results due to Jleli et al. [9].

2. Main results

In this section we prove our main results. Among other things, we introduce the notions of *MK-p*-proximal contractions.

Let (X, d) be a metric space, $p : X \times X \to [0, \infty)$ a w-distance on X, and let A and B be two nonempty subsets of X (which need not be equal). We introduce the following notation (see e.g. [13, 15]) :

$$\mathcal{G}_{A,p} = \{g : A \to A : p(x, y) \leq p(gx, gy),\ \forall x, y \in A\}$$
$$\mathcal{T}_{g,p} = \{T : A \to B : p(Tx, Ty) \leq p(Tgx, Tgy),\ \forall x, y \in A\}.$$

Definition 2.1. *A non-self-mapping $T : A \to B$ is said to be an MK-p-proximal contraction of the first kind if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow (p(x, y) < \varepsilon + \delta \Rightarrow p(u, v) < \varepsilon)$$

for every $u, v, x, y \in A$.

Definition 2.2. *A non-self mapping $T : A \to B$ is said to be an MK-p-proximal contraction of the second kind if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow (p(Tx, Ty) < \varepsilon + \delta \Rightarrow p(Tu, Tv) < \varepsilon)$$

for all $u, v, x, y \in A$.

The next two auxiliary statements will be used to prove our main results.

Lemma 2.3. *If $T : A \to B$ is an MK-p-proximal contraction of the first kind, then*

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow p(u, v) \leq p(x, y)$$

for every $u, v, x, y \in A$.

Proof. Take $\varepsilon = p(x, y) + \lambda$ and $\delta = \delta(\varepsilon) > 0$, where $\lambda > 0$ is arbitrary. Then the inequality

$$p(x, y) < p(x, y) + \lambda + \delta$$

is true, which implies

$$p(u, v) < \varepsilon = p(x, y) + \lambda \tag{1}$$

since T is an MK-p-proximal contraction of the first kind. Taking $\lambda \to 0$ in (1) yields $p(u, v) \le p(x, y)$. □

Lemma 2.4. *If $T : A \to B$ is an MK-p-proximal contraction of the second kind, then*

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow p(Tu, Tv) \le p(Tx, Ty)$$

for every $u, v, x, y \in A$.

Proof. Take $\varepsilon = p(Tx, Ty) + \lambda$ and $\delta = \delta(\varepsilon) > 0$, where $\lambda > 0$ is arbitrary. Then the inequality

$$p(Tx, Ty) < p(Tx, Ty) + \lambda + \delta$$

is true, which implies

$$p(Tu, Tv) < \varepsilon = p(Tx, Ty) + \lambda \tag{2}$$

since T is an MK-p-proximal contraction of the second kind. Taking $\lambda \to 0$ in (2) yields $p(Tu, Tv) \le p(Tx, Ty)$. □

Now we state and prove our main results.

Theorem 2.5. *Let A and B be two nonempty subsets of a complete metric space (X, d) with a w-distance p, such that A_0 is nonempty and closed. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

1. *T is an MK-p-proximal contraction of the first kind;*
2. *$g \in \mathcal{G}_{A,p}$;*
3. *$A_0 \subseteq g(A_0)$;*
4. *$T(A_0) \subseteq B_0$.*

Then there exists a unique element $x \in A_0$ such that $d(gx, Tx) = d(A, B)$ and $p(x, x) = 0$. Moreover, for any initial $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$ there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B).$$

Similarly, for $x_1 \in A_0$ there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B).$$

Continuing this process, for any $x_n \in A_0$ we can find $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Tx_n) = d(A, B).$$

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$p(x_{n_0}, x_{n_0+1}) = 0, \tag{3}$$

by Lemma 2.3 we have $p(gx_{n_0+1}, gx_{n_0+2}) = 0$, so that

$$\begin{aligned} p(x_{n_0}, x_{n_0+2}) &\leq p(x_{n_0}, x_{n_0+1}) + p(x_{n_0+1}, x_{n_0+2}) \\ &\leq p(x_{n_0}, x_{n_0+1}) + p(gx_{n_0+1}, gx_{n_0+2}) = 0 \end{aligned}$$

implies $p(gx_{n_0}, gx_{n_0+2}) = 0$. Since $g \in \mathcal{G}_{A,p}$, we get

$$p(x_{n_0}, x_{n_0+2}) = 0. \tag{4}$$

But by Lemma 1.2 (i), (3) and (4) imply that $x_{n_0+2} = x_{n_0+1}$, so $d(gx_{n_0+2}, Tx_{n_0+1}) = d(gx_{n_0+1}, Tx_{n_0+1}) = d(A, B)$.

Hence, we can assume that $p(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is an MK-proximal contraction of the first kind, using Lemma 2.3 we obtain

$$p(gx_{n+1}, gx_{n+2}) \leq p(x_n, x_{n+1}) \leq p(gx_n, gx_{n+1}) \tag{5}$$

for any $n \in \mathbb{N} \cup \{0\}$, which means that the sequence $\{p(gx_n, gx_{n+1})\} \subseteq (0, \infty)$ is decreasing. Hence, there exists $r \geq 0$ such that

$$p(gx_n, gx_{n+1}) \to r \text{ as } n \to \infty. \tag{6}$$

Suppose that $r > 0$. Now choose $\delta = \delta(r) > 0$, so by (6) there exists $m \in \mathbb{N} \cup \{0\}$ such that $p(x_m, x_{m+1}) \leq p(gx_m, gx_{m+1}) < r+\delta$. Since T is an MK-p-proximal contraction of the first kind, then we get $p(gx_{m+1}, gx_{m+2}) < r$, a contradiction. Hence, we have $r = 0$ which implies that

$$\lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0. \tag{7}$$

Let $\varepsilon > 0$ be arbitrary. Without loss of generality, we can assume that $\delta = \delta(\varepsilon) < \varepsilon$. By (7) there exists $N = N(\varepsilon) \in \mathbb{N} \cup \{0\}$ such that

$$p(gx_n, gx_{n+1}) < \delta \text{ for all } n \geq N. \tag{8}$$

We will show that for all $n \geq N$ and every $k \in \mathbb{N}$

$$p(gx_n, gx_{n+k}) < \varepsilon + \delta \tag{9}$$

by induction with respect to k. Fix $n \geq N$. By (8), (9) holds for $k = 1$. Suppose that (9) holds for some $k \in \mathbb{N}$, i.e.

$$p(x_n, x_{n+k}) \leq p(gx_n, gx_{n+k}) < \varepsilon + \delta.$$

But then

$$p(gx_{n+1}, gx_{n+k+1}) < \varepsilon$$

since T is an MK-p-proximal contraction of the first kind. Thus by (8),

$$p(gx_n, gx_{n+k+1}) \leq p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_{n+k+1}) < \delta + \varepsilon.$$

In (9) $\delta < \varepsilon$, so we have

$$p(gx_n, gx_{n+k}) < 2\varepsilon \tag{10}$$

for all $n \geq N$ and $k \in \mathbb{N}$. In other words, we have proven that

$$\lim_{n\to\infty} \sup_{m>n} p(gx_n, gx_m) = 0,$$

so by Lemma 1.2 (iii), $\{gx_n\}$ is a Cauchy sequence in A_0. Since (X, d) is a complete metric space and A_0 is a closed subset of X, there exists $\lim_{n\to\infty} gx_n = gx$ for some $x \in A_0$. Since $gx_n \in A_0$ for all $n \in \mathbb{N} \cup \{0\}$ and A_0

is closed, we also have $gx \in A_0$. On the other hand, since $gx \in A_0$ and $T(A_0) \subseteq B_0$, for x there exists $z \in A_0$ such that $d(z, Tx) = d(A, B)$.

Let us prove that $z = gx$.

From (10), for any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N} \cup \{0\}$ such that for a fixed $n \geq N$ we obtain

$$p(gx_n, gx) \leq \liminf_{k\to\infty} p(gx_n, gx_{n+k}) < 2\varepsilon$$

which implies that

$$\lim_{n\to\infty} p(gx_n, gx) = 0. \tag{11}$$

Since T is an MK-p-proximal contraction of the first kind, by Lemma 2.3 we have

$$p(gx_{n+1}, z) \leq p(x_n, x) \leq p(gx_n, gx)$$

for any $n \geq N$, which combined with (11) yields

$$\lim_{n\to\infty} p(gx_{n+1}, z) = 0. \tag{12}$$

Finally, from (11) and (12) we conclude that $z = gx$ by Lemma 1.2 (i). Since $d(z, Tx) = d(A, B)$ we get $d(gx, Tx) = d(A, B)$.

To prove the uniqueness, let y be in A_0 such that

$$d(gy, Ty) = d(A, B).$$

Assume that $p(gx, gy) \geq p(x, y) > 0$. Take $\varepsilon = p(x, y)$ and $\delta = \delta(\varepsilon)$, so that $p(x, y) < \varepsilon + \delta$. Since T is an MK-p-proximal contraction of the first kind, we obtain $p(gx, gy) < \varepsilon = p(x, y)$ which is a contradiction. Hence

$$p(x, y) = 0 \tag{13}$$

and symmetrically, we can show that also $p(y, x) = 0$, which implies $p(x, x) \leq p(x, y) + p(y, x) = 0$, i.e.

$$p(x, x) = 0. \tag{14}$$

By Lemma 1.2 (i), from (13) and (14) we conclude that $x = y$.

By a similar argument we prove $p(x, x) = 0$. □

Analogously to the Theorem 2.5, we can prove the following best proximity point result for MK-p-proximal contractions of the second kind.

Theorem 2.6. *Let (X, d) be a complete metric space with w-distance p, and let A and B be two nonempty subsets of X such that A_0 is nonempty and closed. Assume that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

1. *T is an MK-p-proximal contraction of the second kind;*
2. *$T \in \mathcal{T}_{g,p}$;*
3. *$A_0 \subseteq g(A_0)$;*
4. *$T(A_0) \subseteq B_0$.*

Then there exists a point $x \in A_0$ such that $d(gx, Tx) = d(A, B)$, and $p(Tx,Tx)=0$. Moreover, for any $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ which converges to x such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

If, additionally, T is an injective mapping on A, then the best proximity point x described in the previous paragraph is unique.

Proof. Analogously to the proof of Theorem 2.5, we conclude that for an arbitrary $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$, which converges to the point $x \in A_0$ such that

$$d(gx, Tx) = d(A, B)$$

and

$$p(Tx, Tx) = 0. \tag{15}$$

Now suppose that $y \in A_0$ is another point such that

$$d(gy, Ty) = d(A, B).$$

Again, using the similar reasoning to that in the proof of Theorem 2.5, we obtain that

$$p(Tx, Ty) = 0. \tag{16}$$

By (15) and (16) and using the Lemma 1.2 we have $Tx = Ty$, which implies that $x = y$, since T is an injective mapping on A. □

3. Conclusions

In this section, we prove the results of Jleli et al. [9] under weaker assumptions as consequences of our main results. To this end, let us first recall the notions of *MK*-proximal contractions introduced by Jleli et al. [9]:

Definition 3.1. *A non-self-mappping $T : A \to B$ is said to be an MK-proximal contraction of the first kind if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow (\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(u, v) < \varepsilon)$$

for every $u, v, x, y \in A$.

Definition 3.2. *A non-self-mappping $T : A \to B$ is said to be an MK-proximal contraction of the second kind if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \Rightarrow (\varepsilon \leq d(Tx, Ty) < \varepsilon + \delta \Rightarrow d(Tu, Tv) < \varepsilon)$$

for every $u, v, x, y \in A$.

Now we prove the first main result of Jleli et al. ([9, Theorem 3.1]):

Theorem 3.3. *Let A and B be closed subsets of a complete metric space (X, d) such that A_0 is nonvoid and the pair (A, B) satisfies the weakly P-property. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

(a) T is an MK-proximal contraction of the first and second kinds;

(b) $T(A_0) \subseteq B_0$;

(c) g is an isometry (i.e. $d(gx, gy) = d(x, y)$ for all $x, y \in X$);

(d) $A_0 \subseteq g(A_0)$;

(e) T preserves the isometric distance with respect to g (i.e. $d(Tgx, Tgy) = d(Tx, Ty)$ for all $x, y \in X$).

Then, there exists a unique element $x^* \in A$ such that $d(gx^*, Tx^*) = d(A, B)$. Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by $d(gx_{n+1}, Tx_n) = d(A, B)$ converges to x^*.

Proof. Notice that all the conditions of Theorem 2.5 are satisfied if we take $p = d$. Hence, the same conculsion holds. □

Analogously, we can also show the second main result of Jleli et al. ([9, Theorem 3.5]):

Theorem 3.4. *Let A and B be closed subsets of a complete metric space (X, d) such that A_0 is nonvoid, the pair (A, B) satisfies the weakly P-property, and B is approximatively compact with respect to A. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

(a) T is an MK-proximal contraction of the first kind;

(b) $T(A_0) \subseteq B_0$;

(c) g is an isometry;

(d) $A_0 \subseteq g(A_0)$.

Then, there exists a unique element $x^* \in A$ such that $d(gx^*, Tx^*) = d(A, B)$. Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by $d(gx_{n+1}, Tx_n) = d(A, B)$ converges to x^*.

Remark 3.5. *Let A and B be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weakly P-property ([1, 2, 9]) if and only if*

$$\left.\begin{aligned} d(x_1, y) = d(A, B) \\ d(x_2, y) = d(A, B) \end{aligned}\right\} \Rightarrow x_1 = x_2.$$

Also, B is said to be approximatively compact ([9]) with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x, y_n) \to d(x, B)$ for some $x \in A$ has a convergent subsequence. Notice that Jleli et al. [9] make use of the weakly P-property and the approximative compactness property in order to furnish their results. Thus we have shown here that these conditions are redundant. Moreover, our main results hold in a more general setting, while the proofs are significantly simpler.

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