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A Quarter-symmetric Metric Connection on Almost Contact $B-$metric Manifolds

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Abstract.
The aim of this paper is to study the notion of a quarter-symmetric metric connection on an almost contact $B-$metric manifold $(M, \varphi, \xi, \eta, g)$. We obtain the relation between the Levi-Civita connection and the quarter-symmetric metric connection on $(M, \varphi, \xi, \eta, g)$. We investigate the curvature tensor, Ricci tensor and scalar curvature tensor with respect to the quarter-symmetric metric connection. In case the manifold $(M, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact $B-$metric manifold, we get some formulas. Finally, we give some examples of a quarter-symmetric metric connection.

1. Introduction

The investigations of a quarter-symmetric metric connection in a differentiable manifold with affine connection take a central place in the study of the differential geometry. In 1975, it was defined and studied by Golab[11]. The systematic study of the quarter-symmetric metric connection was continued by [2, 3]. The quarter-symmetric metric connection in Riemannian, Kaehlerian and Sasakian manifolds was studied by [1, 4, 5, 13]. The quarter-symmetric metric connection on Riemannian manifold with an almost contact structure and pseudo-Riemannian manifolds was studied by [12, 14].

A classification of the space of the torsion tensors on almost contact $B-$metric manifolds is made in [8]. According to the classification we determine the class of the torsion tensor of the quarter-symmetric metric connection.

Sasaki-like almost contact $B-$metric manifolds was studied in [7]. We investigate the quarter-symmetric metric connection on Sasaki-like almost contact $B-$metric manifolds.

We organize the present paper as follows: Section 2 contains the basic known results of almost contact $B-$metric manifolds and Sasaki-like almost contact $B-$metric manifolds. The brief results of the quarter-symmetric metric connection on an almost contact $B-$metric manifold are given in Section 3. In Section 4, the properties of the curvature tensors corresponding to the quarter-symmetric metric connection on Sasaki-like almost contact $B-$metric manifolds are investigated. In the last section, we construct some examples of almost contact $B-$metric manifolds equipped with the quarter-symmetric metric connection and verify our results.

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Convention: Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)-$dimensional almost contact $B-$metric manifold.

1. x, y, z, w denote smooth vector fields on M, namely $x, y, z \in \chi(M)$.
2. X, Y, Z, U denote smooth horizontal vector fields on M, namely, $X, Y, Z, U \in \chi(H)$.
3. We will use the $2n-$tuple $\{e_1, \ldots, e_n, e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n\}$ to denote a local orthonormal basis of the horizontal space H.
4. For an orthonormal basis $\{e_0 = \xi, e_1, \ldots, e_n, e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n\}$ we denote $\varepsilon_i = sign(g(e_i, e_i)) = \pm 1$, where $\varepsilon_i = 1$ for $i = 0, 1, \ldots, n$ and $\varepsilon_i = -1$ for $i = n+1, \ldots, 2n$.

2. Almost Contact $B-$metric Manifolds

Let M be $(2n+1)-$dimensional smooth manifold with an almost contact structure (φ, η, ξ) consisting of an endomorphism φ of the tangent bundle, a Reeb vector field ξ, its dual $1-$form η such that the following relations are satisfied:

$$\varphi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0. \tag{1}$$

Then, (M, φ, ξ, η) is called almost contact manifold. Moreover, if the almost contact manifold (M, φ, ξ, η) is endowed with a pseudo-Riemannian metric g of signature $(n+1, n)$ compatible with the almost contact structure in the following way

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \tag{2}$$

then $(M, \varphi, \xi, \eta, g)$ is called almost contact $B-$metric manifold.

$2n-$dimensional contact distribution $H = ker\eta$, induced by the contact $1-$form η, can be considered as the horizontal distribution. The restriction of φ to H is an almost complex structure, and the restriction of g to H is a Norden metric, i.e.,

$$g|_H(\varphi|_H(X), \varphi|_H(Y)) = -g(X, Y)$$

for any $X, Y \in \chi(H)$. Thus, $(H, \varphi|_H)$ can be considered as $2n-$dimensional almost complex manifold with Norden metric.

The structure group of the almost contact $B-$metric manifolds is $O(n, \mathbb{C}) \times 1$, that is, $O(n, \mathbb{C}) \times 1$ consists of $(2n+1) \times (2n+1)$ matrices of the following type

$$\begin{pmatrix} A & B & 0_{n\times 1} \\ -B & A & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & 1 \end{pmatrix}, \quad AA^t - BB^t = I_n, \quad AB^t + BA^t = 0_n,$$

where $A, B \in GL(n, \mathbb{R})$ and I_n and 0_n are the unit matrix and zero matrix, respectively.

The fundamental tensor F of type $(0, 3)$ on $(M, \varphi, \xi, \eta, g)$ is determined by

$$F(x, y, z) = g((\nabla_x \varphi)y, z),$$

where ∇ is the Levi-Civita connection of g. Moreover, the tensor F has the following properties:

$$\begin{aligned} &F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi), \\ &(\nabla_x \eta)y = g(\nabla_x \xi, y) = F(x, \varphi y, \xi). \end{aligned} \tag{3}$$

2.1. *Sasaki-like Almost Contact $B-$metric Manifolds*

An almost contact $B-$metric manifold $(M, \varphi, \xi, \eta, g)$ is called a Sasaki-like almost contact $B-$metric manifold if the tensor F satisfies the following conditions:

$$\begin{aligned} &F(X, Y, Z) = F(\xi, Y, Z) = F(\xi, \xi, Z) = 0, \\ &F(X, Y, \xi) = -g(X, Y). \end{aligned}$$

If $(M, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact B-metric manifold, then the following conditions are given in [7]:

a. The covariant derivative $\nabla\varphi$ satisfies the equality

$$(\nabla_x\varphi)y = -g(x,y)\xi - \eta(y)x + 2\eta(x)\eta(y)\xi. \tag{4}$$

b. The manifold M is normal, i.e., $N = 0$, the fundamental $1-$form η is closed, i.e., $d\eta = 0$ and the integral curves of ξ are geodesics, i.e., $\nabla_\xi\xi = 0$.

c. The covariant derivative $\nabla\eta$ satisfies the equality

$$(\nabla_X\eta)Y = -g(X,\varphi Y).$$

d. The $1-$forms θ and θ^* satisfy the equalities $\theta = -2n\,\eta$ and $\theta^* = 0$.

e. $\nabla_\xi X = -\varphi X - [X,\xi]$.

f. $\nabla_x\xi = -\varphi x$.

3. A Quarter-symmetric Metric Connection on Almost Contact $B-$Metric Manifolds

If T is the torsion tensor of a linear connection D given by

$$T(x,y) = D_x y - D_y x - [x,y], \tag{5}$$

then the corresponding tensor of type $(0,3)$ is determined by

$$T(x,y,z) = g(T(x,y),z). \tag{6}$$

It is well-known that any metric connection D is completely determined by its torsion tensor with

$$2g(D_x y - \nabla_x y, z) = T(x,y,z) - T(y,z,x) + T(z,x,y). \tag{7}$$

Definition 3.1. *A linear connection $\widetilde{\nabla}$ on an almost contact $B-$metric manifold is called a quarter-symmetric connection if its torsion tensor T of the connection $\widetilde{\nabla}$ satisfies the condition*

$$T(x,y) = \eta(y)\varphi x - \eta(x)\varphi y. \tag{8}$$

If moreover, the connection $\widetilde{\nabla}$ satisfies the condition

$$(\widetilde{\nabla}_x g)(y,z) = 0, \tag{9}$$

for all $x,y,z \in \chi(M)$, then $\widetilde{\nabla}$ is called a quarter-symmetric metric connection, otherwise it is called a quarter-symmetric non-metric connection.

Let us define a connection $\widetilde{\nabla}_x y$ by the following equation:

$$\begin{aligned} 2g(\widetilde{\nabla}_x y, z) \;=\; & xg(y,z) + yg(z,x) - zg(x,y) + g([x,y],z) \\ & -g([y,z],x) + g([z,x],y) + g(\eta(y)\varphi x - \eta(x)\varphi y, z) \\ & +g(\eta(y)\varphi z - \eta(z)\varphi y, x) + g(\eta(x)\varphi z - \eta(z)\varphi x, y), \end{aligned} \tag{10}$$

where $x,y,z \in \chi(M)$. This connection $\widetilde{\nabla}$ satisfies the following conditions:

$$\begin{aligned} & \widetilde{\nabla}_x(y+z) = \widetilde{\nabla}_x y + \widetilde{\nabla}_x z, \\ & \widetilde{\nabla}_{(x+y)} z = \widetilde{\nabla}_x z + \widetilde{\nabla}_y z, \\ & \widetilde{\nabla}_{fx} y = f\widetilde{\nabla}_x y, \\ & \widetilde{\nabla}_x(fy) = f\widetilde{\nabla}_x y + x(f)y, \end{aligned} \tag{11}$$

for all $x, y, z \in \chi(M)$ and $f \in C^{\infty}(M)$. Therefore, the connection $\widetilde{\nabla}$ determines a linear connection on (M, g). According to (10), we have the following relation:

$$g(\widetilde{\nabla}_x y, z) - g(\widetilde{\nabla}_y x, z) = g([x, y], z) + g(\eta(y)\varphi x - \eta(x)\varphi y, z). \tag{12}$$

Then, we get

$$T(x, y) = \widetilde{\nabla}_x y - \widetilde{\nabla}_y x - [x, y] = \eta(y)\varphi x - \eta(x)\varphi y. \tag{13}$$

Moreover, it can be easily verified that $\widetilde{\nabla}$ is compatible with the metric g on M, i.e.,

$$\widetilde{\nabla} g = 0. \tag{14}$$

$\widetilde{\nabla}$ determines metric connection on $(M, \varphi, \xi, \eta, g)$.

Theorem 3.2. *If $(M, \varphi, \xi, \eta, g)$ is an almost contact $B-$metric manifold, then there exists a unique linear connection $\widetilde{\nabla}$ satisfying the conditions (13) and (14).*

Now we give a relation between the Levi-Civita connection ∇ and the quarter-symmetric metric connection $\widetilde{\nabla}$ on $(M, \varphi, \xi, \eta, g)$. Let

$$\widetilde{\nabla}_x y = \nabla_x y + U(x, y),$$

where $U(x, y)$ is a tensor of type $(1, 2)$. It can be seen that

$$U(x, y) = \frac{1}{2}[T(x, y) + S(x, y) + S(y, x)],$$

where

$$g(S(x, y), z) = g(T(z, x), y).$$

From (13) we get

$$U(x, y) = \eta(y)\varphi x - g(\varphi x, y)\xi. \tag{15}$$

Hence, a quarter-symmetric metric connection $\widetilde{\nabla}$ on $(M, \varphi, \xi, \eta, g)$ is given by

$$\widetilde{\nabla}_x y = \nabla_x y + \eta(y)\varphi x - g(\varphi x, y)\xi. \tag{16}$$

Conversely, it is easy to show that a linear connection $\widetilde{\nabla}$ on $(M, \varphi, \xi, \eta, g)$ defined by (16) determines a quarter-symmetric metric connection.

If T is the torsion tensor of a quarter-symmetric metric connection $\widetilde{\nabla}$, then the corresponding tensor of type $(0, 3)$ is given by

$$T(x, y, z) = \eta(y)g(\varphi x, z) - \eta(x)g(\varphi y, z). \tag{17}$$

In particular, we have

$$T(\varphi x, \varphi y, z) = 0, \text{ and} \tag{18}$$

$$T(x, y, \xi) = 0. \tag{19}$$

The classification of the space of the torsion tensors with respect to almost contact $B-$metric structure is made in [8]. The class of the torsion tensor corresponding to the quarter-symmetric metric connection is determined in the following.

Proposition 3.3. *The torsion T of the quarter-symmetric metric connection $\widetilde{\nabla}$ on $(M, \varphi, \xi, \eta, g)$ belongs to $\mathcal{T}_{10}$.*

Proof. Let $\mathcal{T}$ be a the vector space of all tensors T of type $(0,3)$ over $T_p(M)$ having skew-symmetry by the first two arguments, i.e.,

$$\mathcal{T} = \{T(x,y,z) \in \mathbb{R} | T(x,y,z) = -T(y,x,z), x,y,z \in T_pM\}.$$

Firstly, we have the operator $p_1 : \mathcal{T} \to \mathcal{T}$ by

$$p_1(T)(x,y,z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z) = -g(T(\varphi^2 x, \varphi^2 y), \varphi^2 z).$$

We have the following orthogonal decomposition of $\mathcal{T}$ by the image and the kernel of p_1:

$$W_1 = im(p_1) = \{T \in \mathcal{T} \mid p_1(T) = T\}, \qquad W_2 = ker(p_1) = \{T \in \mathcal{T} \mid p_1(T) = 0\}.$$

From (18) we obtain $p_1(T) = 0$, namely, $T \in ker(p_1) = W_1^{\perp}$. Now consider the operator $p_2 : W_1^{\perp} \to W_1^{\perp}$ defined by

$$p_2(T)(x,y,z) = \eta(z) T(\varphi^2 x, \varphi^2 y, \xi) = \eta(z) g(T(\varphi^2 x, \varphi^2 y), \xi). \tag{20}$$

Since $p_2 \circ p_2 = p_2$, we have the following decomposition of $W_1^{\perp}$:

$$W_2 = im(p_2) = \{T \in W_1^{\perp} \mid p_2(T) = T\}, \ \ W_2^{\perp} = ker(p_2) = \{T \in W_1^{\perp} \mid p_2(T) = 0\}.$$

According to (19) we get $p_2(T) = 0$, that is, $T \in ker(p_2) = W_2^{\perp}$. We consider the operator $p_3 : W_2^{\perp} \to W_2^{\perp}$ defined by

$$p_3(T)(x,y,z) = \eta(x) T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y) T(\varphi^2 x, \xi, \varphi^2 z).$$

By using the equalities given in (2), (6) and (13), the above equality is written in the form

$$\begin{aligned} p_3(T)(x,y,z) &= \eta(x) T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y) T(\varphi^2 x, \xi, \varphi^2 z) \\ &= \eta(x) g(T(\xi, \varphi^2 y), \varphi^2 z) + \eta(y) g(T(\varphi^2 x, \xi), \varphi^2 z) \\ &= \eta(x) g(\varphi y, \varphi^2 z) + \eta(y) g(-\varphi x, \varphi^2 z) \\ &= -\eta(x) g(y, \varphi z) + \eta(y) g(x, \varphi z) \\ &= -\eta(x) g(\varphi y, z) + \eta(y) g(\varphi x, z) \\ &= g(-\eta(x)\varphi y + \eta(y)\varphi x, z) \\ &= g(T(x,y),z) = T(x,y,z). \end{aligned} \tag{21}$$

Then, $p_3(T) = T$, that is, $T \in im(p_3) = W_3$. The following operators $L_{3,0}$ and $L_{3,1}$ are involutive isometries on W_3:

$$\begin{aligned} &L_{3,0}(T)(x,y,z) = \eta(x) T(\xi, \varphi y, \varphi z) - \eta(y) T(\xi, \varphi x, \varphi z), \\ &L_{3,1}(T)(x,y,z) = \eta(x) T(\xi, \varphi^2 z, \varphi^2 y) - \eta(y) T(\xi, \varphi^2 z, \varphi^2 x). \end{aligned} \tag{22}$$

From (17) we get $L_{3,0}(T) = -T$, namely, $T \in W_3^-$and $L_{3,1}(T) = T$, namely, $T \in W_{3,1}$ where

$$W_3^- = \{T \in W_3 \mid L_{3,0}(T) = -T\}, \ \ W_{3,1} = \{T \in W_3^- \mid L_{3,1}(T) = T\}.$$

The torsion forms t and t^* of T are defined by

$$\begin{aligned} t(x) &= g^{ij} T(x, e_i, e_j), \\ t^*(x) &= g^{ij} T(x, e_i, \varphi e_j), \end{aligned}$$

with respect to the basis $\{\xi, e_1, \ldots, e_{2n}\}$, respectively. By using the torsion tensor T in (17) the torsion forms t and t^* can be easily calculated as $t = 0$, $t^* \neq 0$. Hence, $T \in W_{3,1,2} = \mathcal{T}_{10}$ where

$$W_{3,1,2} = \{T \in W_{3,1} \mid t = 0, t^* \neq 0\}.$$

□

Note that for almost contact $B-$metric manifold (M,φ,ξ,η,g) with respect to the basis $\{\xi,e_1,\dots,e_{2n}\}$ we have the following relation:

$$\widetilde{\nabla}_\xi y = \nabla_\xi y, \qquad \widetilde{\nabla}_x \xi = \nabla_x \xi + \varphi x. \tag{23}$$

Let (M,φ,ξ,η,g) be an almost contact $B-$metric manifold. The curvature tensor of type $(1,3)$ is defined by

$$R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z.$$

If $R(x,y,z,w) = g(R(x,y)z,w)$, then the Ricci tensor Ric, the scalar curvature Scal and * scalar curvature $Scal^*$ are, respectively, defined by

$$\begin{aligned} Ric(x,y) &= \sum_{i=0}^{2n} \varepsilon_i R(e_i,x,y,e_i),\\ Scal &= \sum_{i=0}^{2n} \varepsilon_i Ric(e_i,e_i),\\ Scal^* &= \sum_{i=0}^{2n} \varepsilon_i Ric(e_i,\varphi e_i). \end{aligned} \tag{24}$$

For more details see [6, 7].

It is well-known that the manifold (M,φ,ξ,η,g) is called Einstein if the Ricci tensor Ric is proportional to the metric tensor g, i.e. $Ric = \lambda g$, $\lambda \in \mathbb{R}$. Moreover, the manifold M is called an $\eta-$complex-Einstein manifold if the Ricci tensor Ric satisfies the condition

$$Ric = \lambda g + \mu \widetilde{g} + \nu \eta \otimes \eta, \tag{25}$$

where $\lambda,\mu,\nu \in \mathbb{R}$ and $\widetilde{g}(x,y) = g(x,\varphi y) + \eta(x)\eta(y)$. If $\mu = 0$, we call M an $\eta-$Einstein manifold.

The relation between curvature tensors with respect to the Levi-Civita connection and the quarter-symmetric metric connection on almost contact $B-$metric manifold (M,φ,ξ,η,g) is given by

$$\begin{aligned} \widetilde{R}(x,y)z \;=\;& R(x,y)z + \eta(z)(\nabla_x(\varphi y) - \nabla_y(\varphi x) - \varphi[x,y])\\ & - g(z,\varphi y + \nabla_y\xi)\varphi x + g(z,\varphi x + \nabla_x\xi)\varphi y\\ & - g(\nabla_x(\varphi y) - \nabla_y(\varphi x) - \varphi[x,y],z)\xi - g(\varphi y,z)\nabla_x\xi + g(\varphi x,z)\nabla_y\xi, \end{aligned} \tag{26}$$

where $\nabla_x(\varphi y) - \nabla_y(\varphi x) - \varphi[x,y] = (\nabla_x\varphi)y - (\nabla_y\varphi)x$.

When the structures are $\nabla-$parallel, i.e. $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = 0$, the almost contact $B-$metric manifold belongs to the class $\mathcal{F}_0$. If the almost contact $B-$metric manifold is in the special class $\mathcal{F}_0$, then the relation between the curvature tensors $\widetilde{R}$ and R is given by

$$\widetilde{R}(x,y)z \;=\; R(x,y)z - g(z,\varphi y)\varphi x + g(z,\varphi x)\varphi y. \tag{27}$$

4. A Quarter-symmetric Metric Connection on Sasaki-like Almost Contact $B-$Metric Manifolds

There is considerable interest in natural connections having some additional geometric or algebraic properties about their torsion[9]. In this section we show that the quarter-symmetric metric connection $\widetilde{\nabla}$ on Sasaki-like almost contact $B-$metric manifolds is a natural connection and investigate curvature properties on these manifolds.

Theorem 4.1. *The quarter-symmetric metric connection $\widetilde{\nabla}$ on Sasaki-like almost contact $B-$metric manifolds is a natural connection, i.e.* $\widetilde{\nabla}\varphi = \widetilde{\nabla}\xi = \widetilde{\nabla}\eta = \widetilde{\nabla}g = 0$.

Proof. By using (4) we obtain the following:

$$\begin{aligned}(\widetilde{\nabla}_x\varphi)y &= \widetilde{\nabla}_x(\varphi y) - \varphi(\widetilde{\nabla}_x y)\\ &= \nabla_x(\varphi y) - g(\varphi x, \varphi y)\xi - \varphi(\nabla_x y) - \eta(y)\varphi^2 x\\ &= (\nabla_x\varphi)y + g(x, y)\xi - \eta(x)\eta(y)\xi + \eta(y)x - \eta(x)\eta(y)\xi\\ &= 0.\end{aligned} \tag{28}$$

The equality $\widetilde{\nabla}\xi = \widetilde{\nabla}\eta = \widetilde{\nabla}g = 0$ follow immediately from (14), (28) and second part of (3).
□

The following Proposition given in [7] gives some properties of Sasaki-like almost contact $B-$metric manifolds with Levi-Civita connection.

Proposition 4.2. *On a Sasaki-like almost contact $B-$metric manifold $(M, \varphi, \xi, \eta, g)$ the following formulas hold:*

a. $R(x, y)\xi = \eta(y)x - \eta(x)y.$
b. $[X, \xi] \in H.$
c. $\nabla_\xi X = -\varphi X - [X, \xi] \in H.$
d. $R(\xi, X)\xi = -X.$
e. $Ric(y, \xi) = 2n\eta(y).$
f. $Ric(\xi, \xi) = 2n.$

If $(M, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact $B-$metric manifold, then we have the following relation:

$$\begin{aligned}\widetilde{R}(x, y)z = {}& R(x, y)z - \eta(y)\eta(z)x + \eta(x)\eta(z)y + \eta(y)g(x, z)\xi - \eta(x)g(y, z)\xi\\ &+g(\varphi y, z)\varphi x - g(\varphi x, z)\varphi y.\end{aligned} \tag{29}$$

Set $z = \xi$ into (29) and use (1) to obtain

$$\widetilde{R}(x, y)\xi = R(x, y)\xi - \eta(y)x + \eta(x)y. \tag{30}$$

Set $x = \xi$ and $y \to x$ into (30) to get

$$\widetilde{R}(\xi, x)\xi = R(\xi, x)\xi - \eta(x)\xi + x.$$

Moreover, for any $X \in \chi(H)$ the above equality implies by Proposition (4.2)(d)

$$\widetilde{R}(\xi, X)\xi = 0.$$

By virtue of the identity in the Proposition (4.2)(f) we have

$$\widetilde{Ric}(\xi, \xi) = Ric(\xi, \xi) - 2n = 0. \tag{31}$$

5. Examples

In this section we construct a number of examples of almost contact $B-$metric manifolds with the quarter-symmetric metric connection.

5.1. Example 1

Consider the real connected 3-dimensional Lie group L with the left invariant vector fields $\{\xi = e_0, e_1, e_2\}$. Let the non-zero commutators of the corresponding Lie algebra be

$$[\xi, e_1] = \alpha e_2, \ \ [\xi, e_2] = -\alpha e_1,$$

where $\alpha \in \mathbb{R}$. Let an almost contact $B-$metric structure be defined by

$$\begin{array}{l} g(e_0, e_0) = g(e_1, e_1) = -g(e_2, e_2) = 1, \\ \varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(\xi) = 0, \quad \eta(\xi) = 1. \end{array} \tag{32}$$

It can be easily shown that L is an almost contact $B-$metric manifold. The non-zero components of the Levi-Civita connection ∇ and the quarter-symmetric metric connection $\widetilde{\nabla}$ is respectively given by

$$\nabla_{e_1}\xi = -\alpha e_2, \quad \nabla_{e_2}\xi = \alpha e_1, \quad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\alpha\xi, \tag{33}$$

$$\begin{array}{ll} \widetilde{\nabla}_{e_1}\xi = (1-\alpha)e_2, & \widetilde{\nabla}_{e_2}\xi = (\alpha-1)e_1, \\ \widetilde{\nabla}_{e_1}e_2 = (1-\alpha)\xi, & \widetilde{\nabla}_{e_2}e_1 = (1-\alpha)\xi. \end{array} \tag{34}$$

The non-zero components of the curvature tensors R and $\widetilde{R}$ corresponding to the connections ∇ and $\widetilde{\nabla}$ are given by

$$\begin{array}{lll} R_{010} = -\alpha^2 e_1, & R_{020} = -\alpha^2 e_2, & R_{011} = \alpha^2\xi, \\ R_{022} = -\alpha^2\xi, & R_{121} = \alpha^2 e_2, & R_{122} = \alpha^2 e_1, \end{array} \tag{35}$$

$$\begin{array}{lll} \widetilde{R}_{010} = (\alpha-\alpha^2)e_1, & \widetilde{R}_{020} = (\alpha-\alpha^2)e_2, & \widetilde{R}_{011} = (\alpha^2-\alpha)\xi, \\ \widetilde{R}_{022} = (\alpha-\alpha^2)\xi, & \widetilde{R}_{121} = (\alpha-1)^2 e_2, & \widetilde{R}_{122} = (\alpha-1)^2 e_1. \end{array} \tag{36}$$

Moreover, the non-zero-components of Ricci tensors Ric and $\widetilde{Ric}$ can be easily calculated as

$$\begin{array}{l} Ric_{00} = 2\alpha^2, \ \ Ric_{11} = Ric_{22} = 0, \\ \widetilde{Ric}_{00} = 2(\alpha^2-\alpha), \ \ \widetilde{Ric}_{11} = \alpha - 1, \ \ \widetilde{Ric}_{22} = 1-\alpha. \end{array} \tag{37}$$

By using above components of R and $\widetilde{R}$ the scalar curvatures $Scal$, $\widetilde{Scal}$ and $\widetilde{Scal^*}$ can be easily calculated as

$$Scal = 2\alpha^2, \quad \widetilde{Scal} = 2\alpha^2 - 2, \quad \widetilde{Scal^*} = 0.$$

The Ricci tensor $\widetilde{Ric}$ satisfies the condition

$$Ric = \widetilde{Ric} = (\alpha-1)g + (2\alpha^2 - 3\alpha + 1)\eta\otimes\eta.$$

Then, the manifold M is an $\eta-$Einstein manifold.

In particular, in case of $\alpha = 1$ we get $\widetilde{R} = 0$. Namely, L has a flat quarter-symmetric metric connection.

5.2. Example 2

In [10] a real connected Lie group L as a manifold from the class $\mathcal{F}_5$ is introduced. Now we consider this example. In this case, L is a $3-$dimensional real connected Lie group and its associated Lie algebra with a global basis $\{\xi = e_0, e_1, e_2\}$ of the left invariant vector fields on L is defined by

$$[\xi, e_1] = \alpha e_1, \qquad [\xi, e_2] = \alpha e_2, \qquad [e_1, e_2] = 0,$$

where $\lambda \in \mathbb{R}$. Let an almost $B-$metric contact structure be defined by (32). Then, $(L, \varphi, \xi, \eta, g)$ is a $3-$dimensional almost contact $B-$metric manifold.

By using the Kozsul formula, we get the non-zero covariant derivatives of e_i with respect to the Levi-Civita connection ∇ and the quarter-symmetric metric connection $\widetilde{\nabla}$ as follows:

$$\nabla_{e_1}e_1 = -\nabla_{e_2}e_2 = \alpha\xi,\ \nabla_{e_1}\xi = -\alpha e_1,\ \nabla_{e_2}\xi = -\alpha e_2, \tag{38}$$

$$\begin{aligned}&\widetilde{\nabla}_{e_1}e_1 = -\widetilde{\nabla}_{e_2}e_2 = \alpha\xi,\ \widetilde{\nabla}_{e_1}\xi = -\alpha e_1 + e_2,\ \widetilde{\nabla}_{e_2}\xi = -\alpha e_2 - e_1,\\ &\widetilde{\nabla}_{e_1}e_2 = \widetilde{\nabla}_{e_2}e_1 = \xi.\end{aligned} \tag{39}$$

The non-zero components of the curvature tensor R and $\widetilde{R}$ are respectively given by

$$R_{010} = R_{122} = \alpha^2 e_1,\ R_{020} = R_{121} = \alpha^2 e_2,\ R_{022} = -R_{011} = \alpha^2\xi, \tag{40}$$

$$\begin{aligned}&\widetilde{R}_{010} = \alpha^2 e_1 - \alpha e_2,\ \widetilde{R}_{020} = \alpha^2 e_2 + \alpha e_1,\ \widetilde{R}_{121} = (\alpha^2+1)e_2,\\ &\widetilde{R}_{021} = \widetilde{R}_{012} = -\alpha\xi,\ \widetilde{R}_{011} = -\widetilde{R}_{022} = -\alpha^2\xi,\ \widetilde{R}_{122} = (\alpha^2+1)e_1.\end{aligned} \tag{41}$$

The non-zero components of the Ricci tensors Ric and $\widetilde{Ric}$ with respect to the connections ∇ and $\widetilde{\nabla}$ can be easily calculated as follows:

$$\begin{aligned}&Ric_{00} = Ric_{11} = -Ric_{22} = -2\alpha^2,\\ &\widetilde{Ric}_{00} = -2\alpha^2,\ \widetilde{Ric}_{22} = -\widetilde{Ric}_{11} = 2\alpha^2+1,\ \widetilde{Ric}_{12} = -\alpha.\end{aligned} \tag{42}$$

The scalar curvatures $Scal$ and $\widetilde{Scal}$ with respect to ∇ and $\widetilde{\nabla}$ are computed by $Scal = -6\alpha^2$ and $\widetilde{Scal} = -6\alpha^2-2$, respectively. $Scal$ and $\widetilde{Scal}$ are negative for all $\alpha \in \mathbb{R}$. While * scalar curvature $Scal^*$ with respect to ∇ is zero, $*$ scalar curvature $\widetilde{Scal}^*$ with respect to $\widetilde{\nabla}$ is -2α. The Ricci tensor Ric with respect to the Levi-Civita connection ∇ satisfies the condition

$$Ric = -2\alpha^2 g.$$

Then, the Lie group L is an Einstein manifold. The Ricci tensor $\widetilde{Ric}$ with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ satisfies the condition

$$\widetilde{Ric} = (-2\alpha^2-1)g + \alpha\widetilde{g} + (1-\alpha)\eta\otimes\eta.$$

Then, the Lie group L with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$ is an η complex-Einstein manifold.

5.3. Example 3

Let us consider the Lie group G of dimension 5 with a basis of left-invariant vector fields $\{\xi = e_0, e_1, e_2, e_3, e_4\}$ defined by the commutators

$$[\xi, e_1] = \lambda e_2 + \mu e_4 + e_3, \qquad [\xi, e_2] = -\lambda e_1 - \mu e_3 + e_4,$$

$$[\xi, e_3] = -e_1 - \mu e_2 + \lambda e_4, \qquad [\xi, e_4] = \mu e_1 - e_2 - \lambda e_3,$$

where $\lambda, \mu \in \mathbb{R}$ [7]. Define an invariant almost contact $B-$metric structure on G by

$$\begin{aligned}&g(e_1,e_1) = g(e_2,e_2) = -g(e_3,e_3) = -g(e_4,e_4) = g(\xi,\xi) = 1,\\ &\varphi(e_1) = e_3,\quad \varphi(e_2) = e_4,\quad \varphi(e_3) = -e_1,\quad \varphi(e_4) = -e_2,\\ &\varphi(\xi) = 0,\quad \eta(\xi) = 1.\end{aligned} \tag{43}$$

By using the Koszul formula the non-zero connection $1-$forms of the Levi-Civita connection ∇ are calculated in [7] as follows:

$$\begin{aligned}&\nabla_\xi e_1 = \lambda e_2 + \mu e_4,\ \nabla_\xi e_2 = -\lambda e_1 - \mu e_3,\\ &\nabla_\xi e_3 = -\mu e_2 + \lambda e_4,\ \nabla_\xi e_4 = \mu e_1 - \lambda e_3,\\ &\nabla_{e_1}\xi = -e_3, \nabla_{e_2}\xi = -e_4, \nabla_{e_3}\xi = e_1, \nabla_{e_4}\xi = e_2,\\ &\nabla_{e_1}e_3 = \nabla_{e_2}e_4 = \nabla_{e_3}e_1 = \nabla_{e_4}e_2 = -\xi.\end{aligned} \tag{44}$$

It can be easily checked that the constructed manifold $(G, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact $B-$metric manifold. A quarter-symmetric metric connection $\widetilde{\nabla}$ on $(G, \varphi, \xi, \eta, g)$ is given by (20). The non-zero connection $1-$forms of the quarter-symmetric metric connection $\widetilde{\nabla}$ can be calculated as follows:

$$\begin{aligned} &\widetilde{\nabla}_{\xi} e_1 = \lambda e_2 + \mu e_4, \ \ \widetilde{\nabla}_{\xi} e_2 = -\lambda e_1 - \mu e_3, \\ &\widetilde{\nabla}_{\xi} e_3 = -\mu e_2 + \lambda e_4, \ \ \widetilde{\nabla}_{\xi} e_4 = \mu e_1 - \lambda e_3. \end{aligned} \tag{45}$$

Then, we get $\widetilde{R}(e_i, e_j)e_k = 0$ for $i, j, k = 0, \ldots 4$. That is, $\widetilde{R} = 0$. Hence, the manifold $(G, \varphi, \xi, \eta, g)$ has a flat quarter-symmetric metric connection. In particular, if we take $\lambda = 0$ and $\mu = 0$, then it can be verified that all covariant derivatives $\widetilde{\nabla}_{e_i} e_j$ are zero, that is, $\widetilde{\nabla} = 0$. The curvature tensor R with respect to the Levi-Civita connection ∇ is not zero but, $\widetilde{R} = 0$.

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