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Some Quaternion Matrix Equations Involving ϕ-Hermicity

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Abstract. Let $\mathbb{H}$ be the real quaternion algebra and $\mathbb{H}^{m\times n}$ denote the set of all $m \times n$ matrices over $\mathbb{H}$. For $A \in \mathbb{H}^{m\times n}$, we denote by A_ϕ the $n \times m$ matrix obtained by applying ϕ entrywise to the transposed matrix A^t, where ϕ is a nonstandard involution of $\mathbb{H}$. $A \in \mathbb{H}^{n\times n}$ is said to be ϕ-Hermitian if $A = A_\phi$. In this paper, we construct a simultaneous decomposition of four real quaternion matrices with the same row number (A, B, C, D), where A is ϕ-Hermitian, and B, C, D are general matrices. Using this simultaneous matrix decomposition, we derive necessary and sufficient conditions for the existence of a solution to some real quaternion matrix equations involving ϕ-Hermicity in terms of ranks of the given real quaternion matrices. We also present the general solutions to these real quaternion matrix equations when they are solvable. Finally some numerical examples are presented to illustrate the results of this paper.

1. Introduction

Quaternion matrix equation and its general Hermitian solutions play important roles in dealing with many problems arising from systems and control theory [14]. There have been many papers using different approaches to investigate the real quaternion matrix equations (e.g., [1]-[5], [11]-[13], [15], [16], [20], [21]). For instance, Rodman [14] gave a necessary and sufficient condition for the existence of a unique solution to the Sylvester quaternion matrix equation. Pereira and Vettori [13] considered the stabilities of some quaternionic linear systems and their applications. Futorny et.al. [1] derived the Roth's solvability criteria for the quaternion matrix equations $AX - \widehat{X}B = C$ and $X - A\widehat{X}B = C$.

Solving the real quaternion matrix equations involving ϕ-Hermicity is a new topic in quaternion linear algebra and has attracted more and more attention in recent years. For example, He, Liu and Tam [7] considered mixed pairs of quaternion matrix Sylvester equations involving ϕ-Hermicity. Very recently, He [6] considered the following system of quaternion matrix equations involving ϕ-Hermicity

$$\begin{cases} A_1X_1 + (A_1X_1)_\phi + C_1Y_1(C_1)_\phi + F_1W(F_1)_\phi = E_1, \\ A_2X_2 + (A_2X_2)_\phi + C_2Y_2(C_2)_\phi + F_2W(F_2)_\phi = E_2, \end{cases} \quad Y_1 = (Y_1)_\phi,\ Y_2 = (Y_2)_\phi,\ W = W_\phi. \tag{1}$$

Some necessary and sufficient conditions for the existence of a solution (X, Y, Z) to the system (1) in terms of ranks and Moore-Penrose inverses were presented in [6]. Moreover, the general solution to the system (1) is given when it is solvable.

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In this paper, we consider the following two real quaternion matrix equations involving ϕ-Hermicity:

$$BXB_{\phi} + CYC_{\phi} + DZD_{\phi} = A,\ X = X_{\phi},\ Y = Y_{\phi},\ Z = Z_{\phi}, \tag{2}$$

and

$$BXC + (BXC)_{\phi} + DYD_{\phi} = A,\ Y = Y_{\phi}, \tag{3}$$

where $A = A_{\phi}$, B, C, and D are given real quaternion matrices, X, Y, Z are unknowns. In order to study the above mentioned two equations, we need to construct a simultaneous decomposition for the quaternion matrix array

$$\begin{array}{cc} & \begin{array}{cccc} m & p_1 & p_2 & p_3 \end{array} \\ m & \begin{pmatrix} A & B & C & D \end{pmatrix} \end{array}, \tag{4}$$

where $B \in \mathbb{H}^{m\times p_1}, C \in \mathbb{H}^{m\times p_2}, D \in \mathbb{H}^{m\times p_3}$, and $A \in \mathbb{H}^{m\times m}$ is ϕ-Hermitian. Another goal of this paper is to find invertible quaternion matrices P, T_1, T_2, T_3, such that

$$PAP_{\phi} = S_A, \qquad PBT_1 = S_B, \qquad PCT_2 = S_C, \qquad PDT_3 = S_D, \tag{5}$$

where S_B, S_C, S_D are quasi-diagonal matrices with the finest possible subdivision of matrices, and S_A is ϕ-Hermitian with an appropriate form (see Theorem 3.1 for the definitions in details).

The rest of this paper is organized as follows. In Section 2, we review the definition and properties of ϕ-Hermitian quaternion matrix. We in Section 3 construct a simultaneous decomposition of four real quaternion matrices involving ϕ-Hermicity (4). As applications of this simultaneous decomposition, we in Sections 4 and 5 consider the solvability conditions and general solutions to the systems of real quaternion matrix equations involving ϕ-Hermicity (2) and (3).

2. Preliminaries

In this section, we review some definitions and some known lemmas which are used in this paper.

Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand, respectively, for the real number field and the set of all $m \times n$ matrices over the real quaternion algebra

$$\mathbb{H} = \left\{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \,\middle|\, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

The symbol $r(A)$ stands for the rank of a given real quaternion matrix A. The identity matrix and zero matrix with appropriate sizes are denoted by I and 0, respectively. The set of all $n\times n$ invertible matrix over the quaternion algebra are denoted by $GL_n(\mathbb{H})$.

Rodman [14] presented the definitions of the nonstandard involution ϕ, the resulting real quaternion matrix A_{ϕ}, and the ϕ-Hermitian real quaternion matrix. At first, we review the definition of an involution.

Definition 2.1 (Involution)**.** *[14] A map ϕ: $\mathbb{H} \longrightarrow \mathbb{H}$ is called an antiendomorphism if $\phi(xy) = \phi(y)\phi(x)$ for all $x, y \in \mathbb{H}$, and $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{H}$. An antiendomorphism ϕ is called an involution if $\phi(\phi(x)) = x$ for every $x \in \mathbb{H}$.*

The matrix representation of involutions are given in the following lemma.

Lemma 2.2. *[14] Let ϕ be an antiendomorphism of $\mathbb{H}$. Assume that ϕ does not map $\mathbb{H}$ into zero. Then ϕ is one-to-one and onto $\mathbb{H}$; thus, ϕ is in fact an antiautomorphism. Moreover, ϕ is real linear, and can be represented as a 4×4 real matrix with respect to the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Then ϕ is an involution if and only if*

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \tag{6}$$

where either $T = -I_3$ or T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$.

So we can classify involutions into two classes: the standard involution and the nonstandard involution, as defined below.

Definition 2.3 (Standard Involution). *[14] An involution ϕ is standard if $\phi = \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}$. For $a \in \mathbb{H}$, let a^* denote the standard involution of a.*

Definition 2.4 (Nonstandard Involution). *[14] An involution ϕ is nonstandard if*

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$.

In this paper, we consider only the nonstandard involution. Some examples of nonstandard involutions can be found in [7].

For $A \in \mathbb{H}^{m\times n}$, we denote by A_ϕ [14] the $n \times m$ matrix obtained by applying ϕ entrywise to the transposed matrix A^t, where ϕ is a nonstandard involution. We give some algebraic properties of quaternion matrix nonstandard involution.

Proposition 2.5. *[14] Let ϕ be a nonstandard involution. Then,*
(1) $(\alpha A + \beta B)_\phi = A_\phi \phi(\alpha) + B_\phi \phi(\beta),\ \alpha, \beta \in \mathbb{H},\ A, B \in \mathbb{H}^{m\times n}$.
(2) $(A\alpha + B\beta)_\phi = \phi(\alpha)A_\phi + \phi(\beta)B_\phi,\ \alpha, \beta \in \mathbb{H},\ A, B \in \mathbb{H}^{m\times n}$.
(3) $(AB)_\phi = B_\phi A_\phi,\ A \in \mathbb{H}^{m\times n},\ B \in \mathbb{H}^{n\times p}$.
(4) $(A_\phi)_\phi = A,\ A \in \mathbb{H}^{m\times n}$.
(5) *If $A \in \mathbb{H}^{n\times n}$ is invertible, then $(A_\phi)^{-1} = (A^{-1})_\phi$.*
(6) $r(A) = r(A_\phi), A \in \mathbb{H}^{m\times n}$.
(7) $I_\phi = I,\ 0_\phi = 0$.

Now we recall the definition of the ϕ-Hermitian matrix.

Definition 2.6 (ϕ-Hermitian). *[14] $A \in \mathbb{H}^{n\times n}$ is said to be ϕ-Hermitian if $A = A_\phi$, where ϕ is a nonstandard involution.*

For $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, a real quaternion matrix $A \in \mathbb{H}^{n\times n}$ is said to be η-Hermitian if $A^{\eta^*} = A$, where $A^{\eta^*} = -\eta A^* \eta$ and A^* stands for the conjugate transpose of A [19]. η-Hermitian matrix is a special case of ϕ-Hermitian, which has applications in statistical signal processing and widely linear modelling ([17]-[19]).

Now we review the decomposition of a ϕ-Hermitian matrix $A \in \mathbb{H}^{n\times n}$.

Lemma 2.7. *Let ϕ be a nonstandard involution. For every ϕ-Hermitian $A \in \mathbb{H}^{n\times n}$, there exists an invertible matrix S such that*

$$SAS_\phi = \begin{pmatrix} 0 & 0 \\ 0 & I_t \end{pmatrix}$$

for a nonnegative integer $t \leq n$. Moreover, t is uniquely determined by A and $t = r(A)$.

The following lemma that is an important tool for obtaining the main result.

Lemma 2.8. *[10], [22] Let $B \in \mathbb{H}^{m\times p_1}, C \in \mathbb{H}^{m\times p_2}$ and $D \in \mathbb{H}^{m\times p_3}$ be given. Then there exist $P_1 \in GL_m(\mathbb{H})$, $W_B \in GL_{p_1}(\mathbb{H})$, $W_C \in GL_{p_2}(\mathbb{H})$, and $W_D \in GL_{p_3}(\mathbb{H})$ such that*

$$P_1 B W_B = \widetilde{S_B}, \qquad P_1 C W_C = \widetilde{S_C}, \qquad P_1 D W_D = \widetilde{S_D},$$

where

$$\widetilde{S_B}=\begin{array}{c}\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\end{array} r(B),\quad \widetilde{S_C}=\begin{array}{cc}\begin{pmatrix} 0 & I & 0\\ 0 & 0 & 0\\ I & 0 & 0\\ 0 & 0 & 0\end{pmatrix} & \begin{array}{c} r_2\\ r(B)-r_2\\ r_1\\ \end{array}\end{array},\quad \widetilde{S_D}=\begin{array}{cc}\begin{pmatrix} 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} & \begin{array}{c} r_6\\ r_2-r_6\\ r_5\\ r_7\\ r(B)-r_2-r_5-r_7\\ r_7\\ r_4-r_7\\ r_1-r_4\\ r_3\\ \end{array}\end{array}, \tag{7}$$

$$r_1=r(B,\ C)-r(B),\ r_2=r(B)+r(C)-r(B,\ C),\ r_3=r(B,\ C,\ D)-r(B,\ C),$$

$$r_4=r(B,\ D)+r(B,\ C)-r(B)-r(B,\ C,\ D),\ r_5=r\begin{pmatrix} D & B & 0\\ D & 0 & C\end{pmatrix}-r(B,\ D)-r(C),$$

$$r_6=r(B)+r(C)+r(D)-r\begin{pmatrix} D & B & 0\\ D & 0 & C\end{pmatrix},\ r_7=r(B,\ C)+r(C,\ D)+r(B,\ D)-r(B,\ C,\ D)-r\begin{pmatrix} D & B & 0\\ D & 0 & C\end{pmatrix}.$$

3. A simultaneous decomposition of four real quaternion matrices (4)

In this section, we establish a simultaneous decomposition of four real quaternion matrices involving ϕ-Hermicity (4).

Theorem 3.1. *Let $A=A_\phi\in\mathbb{H}^{m\times m}, B\in\mathbb{H}^{m\times p_1}, C\in\mathbb{H}^{m\times p_2}$, and $D\in\mathbb{H}^{m\times p_3}$ be given. Then there exist $P\in GL_m(\mathbb{H})$, $T_1\in GL_{p_1}(\mathbb{H})$, $T_2\in GL_{p_2}(\mathbb{H})$, $T_3\in GL_{p_3}(\mathbb{H})$, such that*

$$PAP_\phi=S_A,\qquad PBT_1=S_B,\qquad PCT_2=S_C,\qquad PDT_3=S_D, \tag{8}$$

where

$$S_A=(S_A)_\phi=\begin{pmatrix} A_{11} & \cdots & A_{19} & A_{1,10} & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ (A_{19})_\phi & \cdots & A_{99} & A_{9,10} & 0\\ (A_{1,10})_\phi & \cdots & (A_{9,10})_\phi & 0 & 0\\ 0 & \cdots & 0 & 0 & I_t\end{pmatrix}, \tag{9}$$

$$S_B=\begin{pmatrix} I_{m_1} & 0 & 0 & 0 & 0 & 0\\ 0 & I_{m_2} & 0 & 0 & 0 & 0\\ 0 & 0 & I_{m_3} & 0 & 0 & 0\\ 0 & 0 & 0 & I_{m_4} & 0 & 0\\ 0 & 0 & 0 & 0 & I_{m_5} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix},\ S_C=\begin{pmatrix} 0 & 0 & 0 & I_{m_1} & 0 & 0\\ 0 & 0 & 0 & 0 & I_{m_2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ I_{m_4} & 0 & 0 & 0 & 0 & 0\\ 0 & I_{m_6} & 0 & 0 & 0 & 0\\ 0 & 0 & I_{m_7} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix},\ S_D=\begin{pmatrix} 0 & 0 & 0 & 0 & I_{m_1} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I_{m_3} & 0 & 0\\ 0 & I_{m_4} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I_{m_4} & 0 & 0 & 0 & 0\\ 0 & 0 & I_{m_6} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ I_{m_8} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}, \tag{10}$$

and

$$t=r\begin{pmatrix} A & B & C & D\\ B_\phi & 0 & 0 & 0\\ C_\phi & 0 & 0 & 0\\ D_\phi & 0 & 0 & 0\end{pmatrix}-2r(B,C,D), \tag{11}$$

$$m_1 = r(D) + r(B) + r(C) - r\begin{pmatrix} D & B & 0 \\ D & 0 & C \end{pmatrix}, \tag{12}$$

$$m_2 = r\begin{pmatrix} D & B & 0 \\ D & 0 & C \end{pmatrix} - r(B,\ C) - r(D),\ m_3 = r\begin{pmatrix} D & B & 0 \\ D & 0 & C \end{pmatrix} - r(B,\ D) - r(C), \tag{13}$$

$$m_4 = r(B,\ C) + r(C,\ D) + r(B,\ D) - r(B,\ C,\ D) - r\begin{pmatrix} D & B & 0 \\ D & 0 & C \end{pmatrix}, \tag{14}$$

$$m_5 = r(B,\ C,\ D) - r(C,\ D),\ m_6 = r\begin{pmatrix} D & B & 0 \\ D & 0 & C \end{pmatrix} - r(C,\ D) - r(B), \tag{15}$$

$$m_7 = r(B,\ C,\ D) - r(B,\ D),\ m_8 = r(B,\ C,\ D) - r(B,\ C). \tag{16}$$

Proof. It follows from Lemma 2.8 that there exist four matrices $P_1 \in GL_m(\mathbb{H})$, $W_B \in GL_{p_1}(\mathbb{H})$, $W_C \in GL_{p_2}(\mathbb{H})$, and $W_D \in GL_{p_3}(\mathbb{H})$ such that

$$P_1(B, C, D)\begin{pmatrix} W_B & 0 & 0 \\ 0 & W_C & 0 \\ 0 & 0 & W_D \end{pmatrix} =$$

$$\begin{array}{c} \left(\begin{array}{cccccc cccccc cccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \end{array}
\begin{array}{l} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_4 \\ m_6 \\ m_7 \\ m_8 \\ m - r(B,C,D) \end{array}.$$

Let

$$P_1 A (P_1)_\phi = P_1 A_\phi (P_1)_\phi \triangleq \begin{pmatrix} A_{11}^{(1)} & \cdots & A_{1,10}^{(1)} \\ \vdots & \ddots & \vdots \\ (A_{1,10}^{(1)})_\phi & \cdots & A_{10,10}^{(1)} \end{pmatrix},$$

where the symbol $\triangleq$ means "equals by definition". Now we pay attention to the ϕ-Hermitian matrix $A_{10,10}^{(1)}$. By Lemma 2.7, we can find an invertible matrix P_2 such that

$$P_2 A_{10,10}^{(1)} (P_2)_\phi = \begin{pmatrix} 0 & 0 \\ 0 & I_t \end{pmatrix},\ t = r(A_{10,10}^{(1)}).$$

Then we have

$$\begin{pmatrix} I_{r(B,C,D)} & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} A_{11}^{(1)} & \cdots & A_{1,10}^{(1)} \\ \vdots & \ddots & \vdots \\ (A_{1,10}^{(1)})_\phi & \cdots & A_{10,10}^{(1)} \end{pmatrix} \begin{pmatrix} I_{r(B,C,D)} & 0 \\ 0 & P_2 \end{pmatrix}_\phi \triangleq \begin{pmatrix} A_{11}^{(2)} & \cdots & A_{19}^{(2)} & A_{1,10}^{(2)} & A_{1,11}^{(2)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ (A_{19}^{(2)})_\phi & \cdots & A_{99}^{(2)} & A_{9,10}^{(2)} & A_{9,11}^{(2)} \\ (A_{1,10}^{(2)})_\phi & \cdots & (A_{9,10}^{(2)})_\phi & 0 & 0 \\ (A_{1,11}^{(2)})_\phi & \cdots & (A_{9,11}^{(2)})_\phi & 0 & I_t \end{pmatrix},$$

$$\begin{pmatrix} I_{r(B,C,D)} & 0 \\ 0 & P_2 \end{pmatrix} P_1(B,C,D) \begin{pmatrix} W_B & 0 & 0 \\ 0 & W_C & 0 \\ 0 & 0 & W_D \end{pmatrix} =$$

$$\left(\begin{array}{cccccccccccccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{array}{c} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_4 \\ m_6 \\ m_7 \\ m_8 \\ m-r(B,C,D)-t \\ t \end{array}.$$

Let

$$P_3 = \begin{pmatrix} I_{r_{bcd}} & \begin{pmatrix} 0 & -A_{1,11}^{(2)} \\ \vdots & \vdots \\ 0 & -A_{9,11}^{(2)} \end{pmatrix} \\ 0 & I_{m-r_{bcd}} \end{pmatrix}.$$

Then we obtain

$$P_3 \begin{pmatrix} A_{11}^{(2)} & \cdots & A_{19}^{(2)} & A_{1,10}^{(2)} & A_{1,11}^{(2)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ (A_{19}^{(2)})_\phi & \cdots & A_{99}^{(2)} & A_{9,10}^{(2)} & A_{9,11}^{(2)} \\ (A_{1,10}^{(2)})_\phi & \cdots & (A_{9,10}^{(2)})_\phi & 0 & 0 \\ (A_{1,11}^{(2)})_\phi & \cdots & (A_{9,11}^{(2)})_\phi & 0 & I_t \end{pmatrix} (P_3)_\phi \triangleq \begin{pmatrix} A_{11} & \cdots & A_{19} & A_{1,10} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ (A_{19})_\phi & \cdots & A_{99} & A_{9,10} & 0 \\ (A_{1,10})_\phi & \cdots & (A_{9,10})_\phi & 0 & 0 \\ 0 & \cdots & 0 & 0 & I_t \end{pmatrix}.$$

Let

$$P \triangleq P_3 \begin{pmatrix} I_{r(B,C,D)} & 0 \\ 0 & P_2 \end{pmatrix} P_1, \ T_1 = W_C, \ T_2 = W_D, \ T_3 = W_E.$$

Hence, the matrices $P \in GL_m(\mathbb{H})$, $T_1 \in GL_{p_1}(\mathbb{H})$, $T_2 \in GL_{p_2}(\mathbb{H})$, and $T_3 \in GL_{p_3}(\mathbb{H})$ satisfy the equation (8). Now we want to give the expressions of $t, m_1, \ldots, m_8$. It is easy to verify that

$$t = r\begin{pmatrix} A & B & C & D \\ B_\phi & 0 & 0 & 0 \\ C_\phi & 0 & 0 & 0 \\ D_\phi & 0 & 0 & 0 \end{pmatrix} - 2r(B,C,D).$$

It follows from S_A, S_B, S_C, and S_D in (9)-(10) that

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{pmatrix} = \begin{pmatrix} r(B) \\ r(C) \\ r(D) \\ r(B,C) \\ r(B,D) \\ r(C,D) \\ r(B,C,D) \\ r\left(\begin{smallmatrix} D & B & 0 \\ D & 0 & C \end{smallmatrix}\right) - r(B) - r(C) \end{pmatrix}.$$

Solving for $m_i, (i = 1, \ldots, 8)$ gives (12)-(16). □

Remark 3.2. *η-Hermitian is a special case of ϕ-Hermitian, where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. As a special case of Theorem 3.1, we can obtain the simultaneous decomposition of four real quaternion matrices with the same row number (A, B, C, D), where $A \in \mathbb{H}^{m\times m}$ is η-Hermitian, $B \in \mathbb{H}^{m\times p_1}, C \in \mathbb{H}^{m\times p_2}$, and $D \in \mathbb{H}^{m\times p_3}$ are general matrices.*

Let D vanish in Theorem 3.1, then we obtain the simultaneous decomposition of a matrix triplet with the same row numbers

$$(A,\ B,\ C),$$

where A is a ϕ-Hermitian matrix.

Corollary 3.3. *Let $A = A_\phi \in \mathbb{H}^{m\times m}, B \in \mathbb{H}^{m\times p_1}$, and $C \in \mathbb{H}^{m\times p_2}$ be given. Then there exist $P \in GL_m(\mathbb{H})$, $T_1 \in GL_{p_1}(\mathbb{H})$, $T_2 \in GL_{p_2}(\mathbb{H})$, such that*

$$PAP_\phi = S_A, \qquad PBT_1 = S_B, \qquad PCT_2 = S_C,$$

where

$$(S_A,\ S_B,\ S_C) = \begin{array}{c} n_1 \\ n_2 \\ n_3 \\ \\ n_4 \end{array} \left(\begin{array}{ccccc|ccc|ccc} A_{11}^1 & A_{12}^1 & A_{13}^1 & A_{14}^1 & 0 & I & 0 & 0 & I & 0 & 0 \\ (A_{12}^1)_\phi & A_{22}^1 & A_{23}^1 & A_{24}^1 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ (A_{13}^1)_\phi & (A_{23}^1)_\phi & A_{33}^1 & A_{34}^1 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ (A_{14}^1)_\phi & (A_{24}^1)_\phi & (A_{34}^1)_\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

and

$$n_1 = r(B) + r(C) - r(B,\ C),\ n_2 = r(B,\ C) - r(C),\ n_3 = r(B,\ C) - r(B),\ n_4 = r\begin{pmatrix} A & B & C \\ B_\phi & 0 & 0 \\ C_\phi & 0 & 0 \end{pmatrix} - 2r(B,\ C).$$

4. Solvability conditions and general ϕ-Hermitian solution to (2)

In this section, we consider the following real quaternion matrix equation

$$BXB_\phi + CYC_\phi + DZD_\phi = A,\ X = X_\phi,\ Y = Y_\phi,\ Z = Z_\phi, \tag{17}$$

where $A = A_\phi, B, C,$ and D are given real quaternion matrices. We give some solvability conditions for the real quaternion matrix equation (17) to possess a ϕ-Hermitian solution and to present an expression of this ϕ-Hermitian solution when the solvability conditions are met. A numerical example is given to illustrate the main result.

Theorem 4.1. *Let $A = A_\phi \in \mathbb{H}^{m\times m}, B \in \mathbb{H}^{m\times p_1}, C \in \mathbb{H}^{m\times p_2}$, and $D \in \mathbb{H}^{m\times p_3}$ be given. Then the real quaternion matrix equation (17) has a ϕ-Hermitian solution (X, Y, Z) if and only if the ranks satisfy:*

$$r(A, B, C, D) = r(B, C, D),\ r\begin{pmatrix} A & B & C \\ D_\phi & 0 & 0 \end{pmatrix} = r(B, C) + r(D), \tag{18}$$

$$r\begin{pmatrix} A & B & D \\ C_\phi & 0 & 0 \end{pmatrix} = r(B, D) + r(C),\ r\begin{pmatrix} A & C & D \\ B_\phi & 0 & 0 \end{pmatrix} = r(C, D) + r(B), \tag{19}$$

$$r\begin{pmatrix} 0 & D_\phi & D_\phi & 0 & 0 \\ D & -A & 0 & 0 & B \\ D & 0 & A & C & 0 \\ 0 & C_\phi & 0 & 0 & 0 \\ 0 & 0 & B_\phi & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} D & B & 0 \\ D & 0 & C \end{pmatrix}. \tag{20}$$

In this case, the general ϕ-Hermitian solution to (17) can be expressed as

$$X = T_1\widehat{X}(T_1)_\phi, \quad Y = T_2\widehat{Y}(T_2)_\phi, \quad Z = T_3\widehat{Z}(T_3)_\phi,$$

where

$$\widehat{X} = \widehat{X}_\phi = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & A_{15} & X_{16} \\ (X_{12})_\phi & X_{22} & A_{23} & A_{24} & A_{25} & X_{26} \\ (X_{13})_\phi & (A_{23})_\phi & X_{33} & A_{34} - A_{36} & A_{35} & X_{36} \\ (X_{14})_\phi & (A_{24})_\phi & (A_{34} - A_{36})_\phi & A_{44} - A_{46} & A_{45} & X_{46} \\ (A_{15})_\phi & (A_{25})_\phi & (A_{35})_\phi & (A_{45})_\phi & A_{55} & X_{56} \\ (X_{16})_\phi & (X_{26})_\phi & (X_{36})_\phi & (X_{46})_\phi & (X_{56})_\phi & X_{66} \end{pmatrix}, \tag{21}$$

$$\widehat{Y} = \widehat{Y}_\phi = \begin{pmatrix} A_{66} - A_{46} & A_{67} - A_{47} & A_{68} & (A_{16} - A_{14} + X_{14})_\phi & (A_{26})_\phi & Y_{16} \\ (A_{67} - A_{47})_\phi & Y_{22} & A_{78} & Y_{24} & (A_{27})_\phi & Y_{26} \\ (A_{68})_\phi & (A_{78})_\phi & A_{88} & (A_{18})_\phi & (A_{28})_\phi & Y_{36} \\ A_{16} - A_{14} + X_{14} & (Y_{24})_\phi & A_{18} & Y_{44} & A_{12} - X_{12} & Y_{46} \\ A_{26} & A_{27} & A_{28} & (A_{12} - X_{12})_\phi & A_{22} - X_{22} & Y_{56} \\ (Y_{16})_\phi & (Y_{26})_\phi & (Y_{36})_\phi & (Y_{46})_\phi & (Y_{56})_\phi & Y_{66} \end{pmatrix}, \tag{22}$$

$$\widehat{Z} = \widehat{Z}_\phi = \begin{pmatrix} A_{99} & (A_{69})_\phi & (A_{79})_\phi & (A_{39})_\phi & (A_{19})_\phi & Z_{16} \\ A_{69} & A_{46} & A_{47} & (A_{36})_\phi & (A_{14} - X_{14})_\phi & Z_{26} \\ A_{79} & (A_{47})_\phi & A_{77} - Y_{22} & (A_{37})_\phi & (A_{17})_\phi - Y_{24} & Z_{36} \\ A_{39} & A_{36} & A_{37} & A_{33} - X_{33} & (A_{13} - X_{13})_\phi & Z_{46} \\ A_{19} & A_{14} - X_{14} & A_{17} - (Y_{24})_\phi & A_{13} - X_{13} & Z_{55} & Z_{56} \\ (Z_{16})_\phi & (Z_{26})_\phi & (Z_{36})_\phi & (Z_{46})_\phi & (Z_{56})_\phi & Z_{66} \end{pmatrix}, \tag{23}$$

in which $X_{11}, X_{22}, X_{33}, X_{66}, Y_{22}, Y_{44}, Y_{66}, Z_{55}$, and Z_{66} are arbitrary ϕ-Hermitian matrices over $\mathbb{H}$ with appropriate sizes, the remaining X_{ij}, Y_{ij}, Z_{ij} are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Proof. Observe that the dimensions of the coefficient matrices A, B, C, and D in the real quaternion matrix equation (17) have the same number of rows. Hence, the coefficient matrices A, B, C, D can be arranged in the following matrix array

$$\begin{pmatrix} A & B & C & D \end{pmatrix}.$$

It follows from Theorem 3.1 that there exist $P \in GL_m(\mathbb{H}),\ T_1 \in GL_{p_1}(\mathbb{H}),\ T_2 \in GL_{p_2}(\mathbb{H}),\ T_3 \in GL_{p_3}(\mathbb{H})$, such that

$$PAP_\phi = S_A, \qquad PBT_1 = S_B, \qquad PCT_2 = S_C, \qquad PDT_3 = S_D,$$

where S_A, S_B, S_C, and S_D are given in (9) and (10). Hence the matrix equation (17) is equivalent to the matrix equation

$$P^{-1}S_B[T_1^{-1}X(T_1)_\phi^{-1}](S_B)_\phi P_\phi^{-1} + P^{-1}S_C[T_2^{-1}Y(T_2)_\phi^{-1}](S_C)_\phi P_\phi^{-1} + P^{-1}S_D[T_3^{-1}Z(T_3)_\phi^{-1}](S_D)_\phi P_\phi^{-1} = P^{-1}S_A P_\phi^{-1},$$

i.e.,

$$S_B[T_1^{-1}X(T_1)_\phi^{-1}](S_B)_\phi + S_C[T_2^{-1}Y(T_2)_\phi^{-1}](S_C)_\phi + S_D[T_3^{-1}Z(T_3)_\phi^{-1}](S_D)_\phi = S_A. \tag{24}$$

Let the matrices

$$\widehat{X} = T_1^{-1}X(T_1)_\phi^{-1} = \begin{pmatrix} X_{11} & \cdots & X_{16} \\ \vdots & \ddots & \vdots \\ (X_{16})_\phi & \cdots & X_{66} \end{pmatrix} = \widehat{X}_\phi, \tag{25}$$

$$\widehat{Y} = T_2^{-1}Y(T_2)_\phi^{-1} = \begin{pmatrix} Y_{11} & \cdots & Y_{16} \\ \vdots & \ddots & \vdots \\ (Y_{16})_\phi & \cdots & Y_{66} \end{pmatrix} = \widehat{Y}_\phi, \tag{26}$$

$$\widehat{Z} = T_3^{-1}Z(T_3)_\phi^{-1} = \begin{pmatrix} Z_{11} & \cdots & Z_{16} \\ \vdots & \ddots & \vdots \\ (Z_{16})_\phi & \cdots & Z_{66} \end{pmatrix} = \widehat{Z}_\phi, \tag{27}$$

be partitioned in accordance with (24). Substituting $\widehat{X}$, $\widehat{Y}$, and $\widehat{Z}$ of (25)-(27) into (24) yields

$$\begin{pmatrix} X_{11}+Y_{44}+Z_{55} & X_{12}+Y_{45} & X_{13}+(Z_{45})_\phi & X_{14}+(Z_{25})_\phi & X_{15} & (Y_{14}+Z_{25})_\phi & (Y_{24}+Z_{35})_\phi & (Y_{34})_\phi & (Z_{15})_\phi & 0 & 0 \\ (X_{12}+Y_{45})_\phi & X_{22}+Y_{55} & X_{23} & X_{24} & X_{25} & (Y_{15})_\phi & (Y_{25})_\phi & (Y_{35})_\phi & 0 & 0 & 0 \\ (X_{13})_\phi+Z_{45} & (X_{23})_\phi & X_{33}+Z_{44} & X_{34}+(Z_{24})_\phi & X_{35} & (Z_{24})_\phi & (Z_{34})_\phi & 0 & (Z_{14})_\phi & 0 & 0 \\ (X_{14})_\phi+Z_{25} & (X_{24})_\phi & (X_{34})_\phi+Z_{24} & X_{44}+Z_{22} & X_{45} & Z_{22} & Z_{23} & 0 & (Z_{12})_\phi & 0 & 0 \\ (X_{15})_\phi & (X_{25})_\phi & (X_{35})_\phi & (X_{45})_\phi & X_{55} & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_{14}+Z_{25} & Y_{15} & Z_{24} & Z_{22} & 0 & Y_{11}+Z_{22} & Y_{12}+Z_{23} & Y_{13} & (Z_{12})_\phi & 0 & 0 \\ Y_{24}+Z_{35} & Y_{25} & Z_{34} & (Z_{23})_\phi & 0 & (Y_{12}+Z_{23})_\phi & Y_{22}+Z_{33} & Y_{23} & (Z_{13})_\phi & 0 & 0 \\ Y_{34} & Y_{35} & 0 & 0 & 0 & (Y_{13})_\phi & (Y_{23})_\phi & Y_{33} & 0 & 0 & 0 \\ Z_{15} & 0 & Z_{14} & Z_{12} & 0 & Z_{12} & Z_{13} & 0 & Z_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & \cdots & A_{19} & A_{1,10} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ (A_{19})_\phi & \cdots & A_{99} & A_{9,10} & 0 \\ (A_{1,10})_\phi & \cdots & (A_{9,10})_\phi & 0 & 0 \\ 0 & \cdots & 0 & 0 & I_t \end{pmatrix}. \tag{28}$$

If the equation (17) has a ϕ-Hermitian solution (X, Y, Z), then by (28), we obtain that

$$t = 0,\ A_{49} = A_{69},\ A_{46} = (A_{46})_\phi,\ \big((A_{1,10})_\phi,\ \cdots,\ (A_{9,10})_\phi\big) = 0, \tag{29}$$

$$A_{29} = 0,\ A_{38} = 0,\ A_{48} = 0,\ A_{56} = 0,\ A_{57} = 0,\ A_{58} = 0,\ A_{59} = 0,\ A_{89} = 0, \tag{30}$$

and

$$X_{11} + Y_{44} + Z_{55} = A_{11},\ X_{12} + Y_{45} = A_{12},\ X_{13} + Z_{54} = A_{13},\ X_{14} + Z_{52} = A_{14},\ X_{15} = A_{15},$$

$$Y_{41} + Z_{52} = A_{16},\ Y_{42} + Z_{53} = A_{17},\ Y_{43} = A_{18},\ Z_{51} = A_{19}, X_{21} + Y_{54} = A_{21},\ X_{22} + Y_{55} = A_{22},$$

$$X_{23} = A_{23},\ X_{24} = A_{24},\ X_{25} = A_{25},\ Y_{51} = A_{26},\ Y_{52} = A_{27},\ Y_{53} = A_{28},\ X_{31} + Z_{45} = A_{31},$$

$$X_{32} = A_{32},\ X_{33} + Z_{44} = A_{33},\ X_{34} + Z_{42} = A_{34},\ X_{35} = A_{35},\ Z_{42} = A_{36},\ Z_{43} = A_{37},\ Z_{41} = A_{39},$$

$$X_{41} + Z_{25} = A_{41},\ X_{42} = A_{42},\ X_{43} + Z_{24} = A_{43},\ X_{44} + Z_{22} = A_{44},\ X_{45} = A_{45},\ Z_{22} = A_{46},$$

$$Z_{23} = A_{47},\ Z_{21} = A_{49},\ X_{51} = A_{51},\ X_{52} = A_{52},\ X_{53} = A_{53},\ X_{54} = A_{54},\ X_{55} = A_{55},$$

$$Y_{14} + Z_{25} = A_{61},\ Y_{15} = A_{62},\ Z_{24} = A_{63},\ Z_{22} = A_{64},\ Y_{11} + Z_{22} = A_{66},\ Y_{12} + Z_{23} = A_{67},$$

$$Y_{13} = A_{68},\ Z_{21} = A_{69},\ Y_{24} + Z_{35} = A_{71},\ Y_{25} = A_{72},\ Z_{34} = A_{73},\ Z_{32} = A_{74},\ Y_{21} + Z_{32} = A_{76},$$

$$Y_{22} + Z_{33} = A_{77},\ Y_{23} = A_{78},\ Z_{31} = A_{79},\ Y_{34} = A_{81},\ Y_{35} = A_{82},\ Y_{31} = A_{86},\ Y_{32} = A_{87},$$

$$Y_{33} = A_{88},\ Z_{15} = A_{91},\ Z_{14} = A_{93},\ Z_{12} = A_{94},\ Z_{12} = A_{96},\ Z_{13} = A_{97},\ Z_{11} = A_{99}.$$

Hence, the general ϕ-Hermitian solution (X, Y, Z) can be expressed as (21)-(23) by (28).

Conversely, assume that the equalities in (29) and (30) hold, then by (25)-(28), it can be verified that the matrices having the forms of (21)-(23) form a ϕ-Hermitian solution of (24), i.e., (17).

We now show that (18)-(20) $\Longleftrightarrow$ (29) and (30). From S_A, S_B, S_C, and S_D in Theorem 3.1, we can infer that
We now show that (18)-(20) $\Longleftrightarrow$ (29) and (30). From S_A, S_B, S_C, and S_D in Theorem 3.1, we can infer that

$$r(A,\ B,\ C,\ D) = r(B,\ C,\ D) \Longleftrightarrow \big((A_{1,10})_\phi,\ \cdots,\ (A_{9,10})_\phi\big) = 0,\ t = 0,$$

$$r\begin{pmatrix} A & B & C \\ D_\phi & 0 & 0 \end{pmatrix} = r(B, C) + r(D) \Longleftrightarrow A_{29} = 0,\ A_{89} = 0,\ A_{49} = A_{69},\ t = 0,$$

$$r\begin{pmatrix} A & B & D \\ C_\phi & 0 & 0 \end{pmatrix} = r(B, D) + r(C) \Longleftrightarrow A_{38} = 0,\ A_{48} = 0,\ A_{58} = 0,\ A_{89} = 0,\ t = 0,$$

$$r\begin{pmatrix} A & C & D \\ B_\phi & 0 & 0 \end{pmatrix} = r(C, D) + r(B) \Longleftrightarrow A_{56} = 0,\ A_{57} = 0,\ A_{58} = 0,\ A_{59} = 0,\ t = 0,$$

$$r\begin{pmatrix} 0 & D_\phi & D_\phi & 0 & 0 \\ D & -A & 0 & 0 & B \\ D & 0 & A & C & 0 \\ 0 & C_\phi & 0 & 0 & 0 \\ 0 & 0 & B_\phi & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} D & B & 0 \\ D & 0 & C \end{pmatrix} \Longleftrightarrow A_{46} = (A_{46})_\phi,\ t = 0.$$

□

Now we present an example to illustrate Theorem 4.1.

Example 4.2. *Given the real quaternion matrices:*

$$B = \begin{pmatrix} \mathbf{i}+\mathbf{j}+\mathbf{k} & 1 & 1+\mathbf{i}+\mathbf{j}-\mathbf{k} \\ -1-\mathbf{j}+\mathbf{k} & \mathbf{i} & -1+\mathbf{i}+\mathbf{j}+\mathbf{k} \end{pmatrix},\ C = \begin{pmatrix} 1 & 2\mathbf{i}+\mathbf{j} & -1+\mathbf{k} \\ \mathbf{i}+\mathbf{k} & 1+\mathbf{i}+\mathbf{j}-\mathbf{k} & 0 \end{pmatrix},$$

$$D=\begin{pmatrix}\mathbf{j}+2\mathbf{k} & \mathbf{i}+\mathbf{k} & \mathbf{j}\\ -2\mathbf{j}+\mathbf{k} & -1-\mathbf{j} & \mathbf{k}\end{pmatrix},\ A=A_{\phi}=\begin{pmatrix}-1+5\mathbf{i}-20\mathbf{k} & -25-2\mathbf{i}-17\mathbf{j}-5\mathbf{k}\\ -25-2\mathbf{i}+17\mathbf{j}-5\mathbf{k} & -9-18\mathbf{i}-14\mathbf{k}\end{pmatrix},$$

we consider the ϕ-Hermitian solution to the real quaternion matrix equation (17), where $\phi(a)=a^{\mathbf{j}*}=-\mathbf{j}a^{*}\mathbf{j}$ for $a\in\mathbb{H}$. Check that

$$r(A,B,C,D)=r(B,C,D)=2,\ r\begin{pmatrix}A & B & C\\ D_{\phi} & 0 & 0\end{pmatrix}=r(B,C)+r(D)=3,$$

$$r\begin{pmatrix}A & B & D\\ C_{\phi} & 0 & 0\end{pmatrix}=r(B,D)+r(C)=3,\ r\begin{pmatrix}A & C & D\\ B_{\phi} & 0 & 0\end{pmatrix}=r(C,D)+r(B)=3,$$

$$r\begin{pmatrix}0 & D_{\phi} & D_{\phi} & 0 & 0\\ D & -A & 0 & 0 & B\\ D & 0 & A & C & 0\\ 0 & C_{\phi} & 0 & 0 & 0\\ 0 & 0 & B_{\phi} & 0 & 0\end{pmatrix}=2r\begin{pmatrix}D & B & 0\\ D & 0 & C\end{pmatrix}=6.$$

All the rank equalities in (18)-(20) hold. Hence, the real quaternion matrix equation (17) has a ϕ-Hermitian solution (X,Y,Z). Note that

$$X=X_{\phi}=\begin{pmatrix}1 & \mathbf{i}+\mathbf{k} & 0\\ \mathbf{i}+\mathbf{k} & 1+\mathbf{i} & 1-\mathbf{k}\\ 0 & 1-\mathbf{k} & 0\end{pmatrix},\ Y=Y_{\phi}=\begin{pmatrix}0 & 1+\mathbf{i} & \mathbf{k}\\ 1+\mathbf{i} & \mathbf{i} & 2\mathbf{k}\\ \mathbf{k} & 2\mathbf{k} & 1\end{pmatrix},\ Z=Z_{\phi}=\begin{pmatrix}\mathbf{i} & \mathbf{i}-\mathbf{k} & \mathbf{k}\\ \mathbf{i}-\mathbf{k} & \mathbf{i} & 1\\ \mathbf{k} & 1 & 1\end{pmatrix}$$

satisfy the real quaternion matrix equation (17).

5. The solution to (3) with Y being ϕ-Hermitian

We now turn attention to the following real quaternion matrix

$$BXC+(BXC)_{\phi}+DYD_{\phi}=A,\ Y=Y_{\phi}, \tag{31}$$

where $A=A_{\phi}, B, C,$ and D are given real quaternion matrices. We derive necessary and sufficient conditions for (31) in terms of ranks of the coefficient matrices. We also give the general solution to this real quaternion matrix equation. A numerical example is also given to illustrate the main result.

Theorem 5.1. *Let $A=A_{\phi}\in\mathbb{H}^{m\times m}, B\in\mathbb{H}^{m\times p_1}, C\in\mathbb{H}^{p_2\times m}$, and $D\in\mathbb{H}^{m\times p_3}$ be given. Then the real quaternion matrix equation (31) has a solution (X,Y), where Y is ϕ-Hermitian, if and only if the ranks satisfy:*

$$r(A,\ B,\ C_{\phi},\ D)=r(B,\ C_{\phi},\ D),\ r\begin{pmatrix}A & B & C_{\phi}\\ D_{\phi} & 0 & 0\end{pmatrix}=r(B,\ C_{\phi})+r(D), \tag{32}$$

$$r\begin{pmatrix}A & B & D\\ B_{\phi} & 0 & 0\end{pmatrix}=r(B,\ D)+r(B),\ r\begin{pmatrix}A & C_{\phi} & D\\ C & 0 & 0\end{pmatrix}=r(C_{\phi},\ D)+r(C), \tag{33}$$

$$r\begin{pmatrix}A & 0 & B & 0 & D\\ 0 & -A & 0 & C_{\phi} & D\\ B_{\phi} & 0 & 0 & 0 & 0\\ 0 & C & 0 & 0 & 0\\ D_{\phi} & D_{\phi} & 0 & 0 & 0\end{pmatrix}=2r\begin{pmatrix}B & 0 & D\\ 0 & C_{\phi} & D\end{pmatrix}. \tag{34}$$

In this case, the general solution to (31) can be expressed as

$$X = T_1\widehat{X}(T_2)_\phi, \quad Y = T_3\widehat{Y}(T_3)_\phi,$$

where

$$\widehat{X} = \begin{pmatrix} X_{11} & X_{12} & A_{18} & X_{14} & A_{12} - (X_{24})_\phi & X_{16} \\ A_{26} & A_{27} & A_{28} & X_{24} & \frac{1}{2}A_{22} + Z & X_{26} \\ A_{36} - A_{34} & X_{32} & A_{38} & X_{34} & (A_{23})_\phi & X_{36} \\ A_{46} - A_{44} & A_{47} - A_{67} & A_{48} & (A_{14} - A_{16} + X_{11})_\phi & (A_{24})_\phi & X_{46} \\ A_{56} & A_{57} & A_{58} & (A_{15})_\phi & (A_{25})_\phi & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}, \tag{35}$$

$$\widehat{Y} = \begin{pmatrix} A_{99} & (A_{49})_\phi & (A_{79})_\phi & (A_{39})_\phi & (A_{19})_\phi & Y_{16} \\ A_{49} & A_{44} & A_{67} & (A_{34})_\phi & (A_{14})_\phi - X_{44} & Y_{26} \\ A_{79} & (A_{67})_\phi & A_{77} & (A_{37} - X_{32})_\phi & (A_{17} - X_{12})_\phi & Y_{36} \\ A_{39} & A_{34} & A_{37} - X_{32} & A_{33} & (A_{13})_\phi - X_{34} & Y_{46} \\ A_{19} & A_{14} - (X_{44})_\phi & A_{17} - X_{12} & A_{13} - (X_{34})_\phi & A_{11} - X_{14} - (X_{14})_\phi & Y_{56} \\ (Y_{16})_\phi & (Y_{26})_\phi & (Y_{36})_\phi & (Y_{46})_\phi & (Y_{56})_\phi & Y_{66} \end{pmatrix}, \tag{36}$$

in which Y_{66} and Z are arbitrary ϕ-Hermitian matrices and ϕ-skewhermitian ($Z + Z_\phi = 0$) matrices over $\mathbb{H}$, respectively, the remaining X_{ij} and Y_{ij} are arbitrary matrices over $\mathbb{H}$.

Proof. Note that the dimensions of the coefficient matrices A, B, C_ϕ, and D in real quaternion matrix equation (31) have the same number of rows. Hence, the coefficient matrices A, B, C, D can be arranged in the following matrix array

$$\begin{pmatrix} A & B & C_\phi & D \end{pmatrix}.$$

It follows from Theorem 3.1 that there exist $P \in GL_m(\mathbb{H})$, $T_1 \in GL_{p_1}(\mathbb{H})$, $T_2 \in GL_{p_2}(\mathbb{H})$, $T_3 \in GL_{p_3}(\mathbb{H})$, such that

$$PAP_\phi = S_A, \qquad PBT_1 = S_B, \qquad PC_\phi T_2 = S_C, \qquad PDT_3 = S_D,$$

where S_A, S_B, S_C, and S_D are given in (9) and (10). Hence the real quaternion matrix equation (31) is equivalent to the real quaternion matrix equation

$$P^{-1}S_B[T_1^{-1}X(T_2)_\phi^{-1}](S_C)_\phi P_\phi^{-1} + P^{-1}S_C[T_2^{-1}X_\phi(T_1)_\phi^{-1}](S_B)_\phi P_\phi^{-1} + P^{-1}S_D[T_3Y(T_3)_\phi](S_D)_\phi P_\phi^{-1} = P^{-1}S_AP_\phi^{-1},$$

i.e.,

$$S_B[T_1^{-1}X(T_2)_\phi^{-1}](S_C)_\phi + S_C[T_2^{-1}X_\phi(T_1)_\phi^{-1}](S_B)_\phi + S_D[T_3Y(T_3)_\phi](S_D)_\phi = S_A. \tag{37}$$

Let the matrices

$$\widehat{X} = T_1^{-1}X(T_2)_\phi^{-1} = \begin{pmatrix} X_{11} & \cdots & X_{16} \\ \vdots & \ddots & \vdots \\ X_{61} & \cdots & X_{66} \end{pmatrix}, \tag{38}$$

$$\widehat{Y} = T_3^{-1}Y(T_3)_\phi^{-1} = \begin{pmatrix} Y_{11} & \cdots & Y_{16} \\ \vdots & \ddots & \vdots \\ (Y_{16})_\phi & \cdots & Y_{66} \end{pmatrix} = \widehat{Y}_\phi, \tag{39}$$

be partitioned in accordance with (37). Substituting $\widehat{X}$ and $\widehat{Y}$ of (38) and (39) into (37) yields

$$\begin{pmatrix} X_{14}+(X_{14})_\phi+Y_{55} & X_{15}+(X_{24})_\phi & (X_{34}+Y_{45})_\phi & (X_{44}+Y_{25})_\phi & (X_{54})_\phi & X_{11}+(Y_{25})_\phi & X_{12}+(Y_{35})_\phi & X_{13} & (Y_{15})_\phi & 0 & 0 \\ X_{24}+(X_{15})_\phi & X_{25}+(X_{25})_\phi & (X_{35})_\phi & (X_{45})_\phi & (X_{55})_\phi & X_{21} & X_{22} & X_{23} & 0 & 0 & 0 \\ X_{34}+Y_{45} & X_{35} & Y_{44} & (Y_{24})_\phi & 0 & X_{31}+(Y_{24})_\phi & X_{32}+(Y_{34})_\phi & X_{33} & (Y_{14})_\phi & 0 & 0 \\ X_{44}+Y_{25} & X_{45} & Y_{24} & Y_{22} & 0 & X_{41}+Y_{22} & X_{42}+Y_{23} & X_{43} & (Y_{12})_\phi & 0 & 0 \\ X_{54} & X_{55} & 0 & 0 & 0 & X_{51} & X_{52} & X_{53} & 0 & 0 & 0 \\ (X_{11})_\phi+Y_{25} & (X_{21})_\phi & (X_{31})_\phi+Y_{24} & (X_{41})_\phi+Y_{22} & (X_{51})_\phi & Y_{22} & Y_{23} & 0 & (Y_{12})_\phi & 0 & 0 \\ (X_{12})_\phi+Y_{35} & (X_{22})_\phi & (X_{32})_\phi+Y_{34} & (X_{42}+Y_{23})_\phi & (X_{52})_\phi & (Y_{23})_\phi & Y_{33} & 0 & (Y_{13})_\phi & 0 & 0 \\ (X_{13})_\phi & (X_{23})_\phi & (X_{33})_\phi & (X_{43})_\phi & (X_{53})_\phi & 0 & 0 & 0 & 0 & 0 & 0 \\ Y_{15} & 0 & Y_{14} & Y_{12} & 0 & Y_{12} & Y_{13} & 0 & Y_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & \cdots & A_{19} & A_{1,10} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ (A_{19})_\phi & \cdots & A_{99} & A_{9,10} & 0 \\ (A_{1,10})_\phi & \cdots & (A_{9,10})_\phi & 0 & 0 \\ 0 & \cdots & 0 & 0 & I_t \end{pmatrix}. \tag{40}$$

If the equation (31) has a solution (X, Y), then by (40), we obtain that

$$t = 0, \quad \left((A_{1,10})_\phi, \ \cdots, \ (A_{9,10})_\phi\right) = 0, \quad A_{44} = A_{66}, \quad A_{49} = A_{69}, \tag{41}$$

$$A_{29} = 0, \ A_{59} = 0, \ A_{89} = 0, \ A_{68} = 0, \ A_{78} = 0, \ A_{88} = 0, \ A_{35} = 0, \ A_{45} = 0, \ A_{55} = 0, \tag{42}$$

and

$$X_{14} + (X_{14})_\phi + Y_{55} = A_{11}, \ X_{15} + (X_{24})_\phi = A_{12}, \ (X_{34} + Y_{45})_\phi = A_{13}, \ (X_{44} + Y_{25})_\phi = A_{14},$$

$$(X_{54})_\phi = A_{15}, \ X_{11} + (Y_{25})_\phi = A_{16}, \ X_{12} + (Y_{35})_\phi = A_{17}, \ X_{13} = A_{18}, \ (Y_{15})_\phi = A_{19},$$

$$X_{25} + (X_{25})_\phi = A_{22}, \ (X_{35})_\phi = A_{23}, \ (X_{45})_\phi = A_{24}, \ (X_{55})_\phi = A_{25}, \ X_{21} = A_{26}, \ X_{22} = A_{27},$$

$$X_{23} = A_{28}, \ Y_{44} = A_{33}, \ (Y_{24})_\phi = A_{34}, \ X_{31} + (Y_{24})_\phi = A_{36}, \ X_{32} + (Y_{34})_\phi = A_{37}, \ X_{33} = A_{38},$$

$$(Y_{14})_\phi = A_{39}, \ Y_{22} = A_{44}, \ X_{41} + Y_{22} = A_{46}, \ X_{42} + Y_{23} = A_{47}, \ X_{43} = A_{48}, \ (Y_{12})_\phi = A_{49}, \ X_{51} = A_{56},$$

$$X_{52} = A_{57}, \ X_{53} = A_{58}, \ Y_{22} = A_{66}, \ Y_{23} = A_{67}, \ (Y_{12})_\phi = A_{69}, \ Y_{33} = A_{77}, \ (Y_{13})_\phi = A_{79}, \ Y_{11} = A_{99}.$$

Hence, the general solution (X, Y) can be expressed as (35) and (36) by (40).

Conversely, assume that the equalities in (41) and (42) hold. Then by (38)-(40), it can be verified that the matrices having the forms of (35) and (36) form a solution of (40), i.e., (31).

We now want to prove that (32)-(34) $\Longleftrightarrow$ (41) and (42). From S_A, S_B, S_C, and S_D in Theorem 3.1, we can infer that

$$r(A, B, C_\phi, D) = r(B, C_\phi, D) \Longleftrightarrow \left((A_{1,10})_\phi, \ \cdots, \ (A_{9,10})_\phi\right) = 0, \ t = 0,$$

$$r\begin{pmatrix} A & B & C_\phi \\ D_\phi & 0 & 0 \end{pmatrix} = r(B, C_\phi) + r(D) \Longleftrightarrow A_{29} = 0, \ A_{89} = 0, \ A_{49} = A_{69}, \ t = 0,$$

$$r\begin{pmatrix} A & B & D \\ B_\phi & 0 & 0 \end{pmatrix} = r(B,D) + r(B) \Longleftrightarrow A_{68} = 0,\ A_{78} = 0,\ A_{88} = 0,\ A_{89} = 0,\ t = 0,$$

$$r\begin{pmatrix} A & C_\phi & D \\ C & 0 & 0 \end{pmatrix} = r(C_\phi,D) + r(C) \Longleftrightarrow A_{35} = 0,\ A_{45} = 0,\ A_{55} = 0,\ A_{59} = 0,\ t = 0,$$

$$r\begin{pmatrix} A & 0 & B & 0 & D \\ 0 & -A & 0 & C_\phi & D \\ B_\phi & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & 0 \\ D_\phi & D_\phi & 0 & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} B & 0 & D \\ 0 & C_\phi & D \end{pmatrix} \Longleftrightarrow A_{44} = A_{66} = 0,\ t = 0.$$

□

Next we give an example to illustrate Theorem 5.1.

Example 5.2. *Let*

$$B = \begin{pmatrix} 1+\mathbf{j} & \mathbf{i}+\mathbf{k} & 1+2\mathbf{i}+\mathbf{j} & -1-\mathbf{k} \\ \mathbf{i}-\mathbf{j} & -1-\mathbf{k} & -2+\mathbf{i}-\mathbf{j} & -\mathbf{i}+\mathbf{k} \end{pmatrix},\ C = \begin{pmatrix} \mathbf{i}+\mathbf{j} & -2+\mathbf{k} \\ 1+2\mathbf{j} & 2\mathbf{i}+2\mathbf{k} \\ -\mathbf{i}+\mathbf{j}+\mathbf{k} & 2-\mathbf{j}+\mathbf{k} \\ \mathbf{j} & \mathbf{k} \end{pmatrix},$$

$$D = \begin{pmatrix} \mathbf{i}+\mathbf{j} & 1+3\mathbf{i} & 1+\mathbf{k} \\ -1+\mathbf{k} & -3+\mathbf{i} & \mathbf{i}-\mathbf{j} \end{pmatrix},\ A = A_\phi = \begin{pmatrix} -16-6\mathbf{j}+34\mathbf{k} & 9+17\mathbf{i}-31\mathbf{j}-3\mathbf{k} \\ 9-17\mathbf{i}-31\mathbf{j}-3\mathbf{k} & -30+12\mathbf{j}-16\mathbf{k} \end{pmatrix}.$$

Now we consider the real quaternion matrix equation (31), where $\phi(a) = a^{\mathbf{i}*} = -\mathbf{i}a^*\mathbf{i}$ for $a \in \mathbb{H}$. Check that

$$r(A,B,C_\phi,D) = r(B,C_\phi,D) = 2,\ r\begin{pmatrix} A & B & C_\phi \\ D_\phi & 0 & 0 \end{pmatrix} = r(B,C_\phi) + r(D) = 3,$$

$$r\begin{pmatrix} A & B & D \\ B^{\mathbf{i}*} & 0 & 0 \end{pmatrix} = r(B,D) + r(B) = 4,\ r\begin{pmatrix} A & C_\phi & D \\ C & 0 & 0 \end{pmatrix} = r(C_\phi,D) + r(C) = 4,$$

$$r\begin{pmatrix} A & 0 & B & 0 & D \\ 0 & -A & 0 & C_\phi & D \\ B_\phi & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & 0 \\ D_\phi & D_\phi & 0 & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} B & 0 & D \\ 0 & C_\phi & D \end{pmatrix} = 8.$$

All the rank equalities in (32)-(34) hold. Hence, the real quaternion matrix equation (31) has a solution. It is easy to show that

$$X = \begin{pmatrix} 2+\mathbf{i}+\mathbf{k} & 1+\mathbf{i}+\mathbf{j} & 1 & \mathbf{i}+\mathbf{k} \\ -1+\mathbf{k} & -\mathbf{i}+\mathbf{k} & \mathbf{j} & 1 \\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & 1 & 1+\mathbf{j} & 1+\mathbf{i}+\mathbf{k} \\ \mathbf{i}+\mathbf{j}+2\mathbf{k} & 1-\mathbf{i}+\mathbf{k} & 1+2\mathbf{j} & 2+\mathbf{i}+\mathbf{k} \end{pmatrix},\ Y = Y_\phi = \begin{pmatrix} 1+\mathbf{j} & 1+\mathbf{i} & \mathbf{j} \\ 1-\mathbf{i} & \mathbf{k} & \mathbf{i} \\ \mathbf{j} & -\mathbf{i} & \mathbf{j} \end{pmatrix}$$

satisfy the real quaternion matrix equation (31).

Remark 5.3. *The research on the system of quaternion matrix equations involving η-Hermicity has attracted more and more attentions in recent years (e.g. [8], [9], [23]-[25]). As special cases of the quaternion matrix equations (2) and (3), we can derive some necessary and sufficient conditions for the existence of a solution to the following four quaternion matrix equations involving η-Hermicity for $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$:*

$$BXB^{\eta*} + CYC^{\eta*} + DZD^{\eta*} = A,\ X = X^{\eta*},\ Y = Y^{\eta*},\ Z = Z^{\eta*},$$
$$BXC + (BXC)^{\eta*} + DYD^{\eta*} = A,\ Y = Y^{\eta*},$$

where $A = A^{\eta*}, B, C,$ and D are given quaternion matrices.

6. Conclusion

We have derived a simultaneous decomposition of four quaternion matrices with the same row number (A, B, C, D), where $A = A_\phi \in \mathbb{H}^{m\times m}, B \in \mathbb{H}^{m\times p_1}, C \in \mathbb{H}^{m\times p_2}, D \in \mathbb{H}^{m\times p_3}$, ϕ is a nonstandard involution of $\mathbb{H}$. As applications of this simultaneous decomposition, we have presented necessary and sufficient conditions for the existences and the general solutions to the quaternion matrix equations involving ϕ-Hermicity (2) and (3). Some numerical examples are presented to illustrate the results.

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