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S-paracompactness and S_2-paracompactness

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Abstract. A topological space X is an *S-paracompact* if there exists a bijective function f from X onto a paracompact space Y such that for every separable subspace A of X the restriction map $f|_A$ from A onto $f(A)$ is a homeomorphism. Moreover, if Y is Hausdorff, then X is called *S_2-paracompact*. We investigate these two properties.

1. Introduction

In this paper, we introduce two new properties in topological spaces which are *S-paracompactness* and *S_2-paracompactness* and our purpose is to investigate these properties. It is useful to introduce the following notations. The order pair will be denoted by $\langle x, y\rangle$. The sets of positive numbers, rational numbers, irrational numbers and real numbers will be denoted by $\mathbb{N}, \mathbb{Q}, \mathbb{P}$ and $\mathbb{R}$ respectively. The closure and the interior of the subset A of X will be denoted respectively by $\overline{A}$ and $\mathrm{int}(A)$. Throughout this paper, a T_1 normal space is called T_4 and a T_1 completely regular space is called Tychonoff space ($T_{3\frac{1}{2}}$). In the definitions of compactness, countable compactness, paracompactness, and local compactness we do not assume T_2. Moreover, in the definitions of Lindelöfness we do not assume regularity. Also, the ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. We denote the first infinite ordinal by ω, the first uncountable ordinal by ω_1, and the successor cardinal of ω_1 by ω_2.

The following definition of the notions of *C-paracompactness* and *C_2-paracompactness* were introduced by A. V. Arhangel'skiǐ (see [8]).

Definition 1.1. *A topological space X is C-paracompact if there exists a bijective function f from X onto a paracompact space Y such that for every compact subspace A of X the restriction map $f|_A$ from A onto $f(A)$ is a homeomorphism. Furthermore, if Y is Hausdorff, then X is called C_2-paracompact.*

2. S-paracompactness and S_2-paracompactness

We introduce the notions of *S-paracompactness* and *S_2-paracompactness* inspired by Definition 1.1 as the following.

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Definition 2.1. *A topological space X is an S-paracompact if there exists a bijective function f from X onto a paracompact space Y such that for every separable subspace A of X the restriction function $f|_A$ from A onto $f(A)$ is a homeomorphism. Moreover, if Y is Hausdorff, then X is called S_2-paracompact.*

From the definition it is clear that any S_2-paracompact is S-paracompact. The next theorem will be used to show that the converse is not necessarily true.

Theorem 2.2. *If X is separable but not Hausdorff, then X cannot be an S_2-paracompact.*

Proof. Let X be any separable non-Hausdorff space. Suppose that X is S_2-paracompact. Then there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that $f|_A : A \longrightarrow f(A)$ is a homeomorphism for all separable subspaces $A \subseteq X$. Since X is separable, then $f : X \longrightarrow Y$ is a homeomorphism. But Y is T_2, then X is T_2 which is a contradiction. □

The following example is an application of Theorem 2.2.

Example 2.3. *Consider the finite complement topology defined on the real numbers, $(\mathbb{R}, \mathcal{CF})$ (see [9, Example 19]). Since $(\mathbb{R}, \mathcal{CF})$ is paracompact being compact, then the identity function* $\mathrm{id} : (\mathbb{R}, \mathcal{CF}) \longrightarrow (\mathbb{R}, \mathcal{CF})$ *shows that it is S-paracompact but not S_2-paracompact being separable but not Hausdorff space.* ■

From Example 2.3, we conclude that any paracompact (T_2 paracompact) space is S-paracompact (S_2-paracompact). For the converse, we have the following counterexample.

Example 2.4. *ω_1 is an S_2-paracompact space which is not paracompact. First, we show that any separable subspace of ω_1 is countable. Suppose that $A \subset \omega_1$ is uncountable which implies that A is unbounded in ω_1. If D is any countable subset of A, then there exists $\alpha < \omega_1$ such that $\alpha = \sup D$. Thus, there exists $\eta \in A$ such that $\alpha < \eta$. The set $((\alpha, \eta] \cap A)$ is a nonempty open subset of A with $((\alpha, \eta] \cap A) \cap D = \emptyset$. Thus, A cannot be separable implying that any separable subspace of ω_1 is countable. Since ω_1 is T_2 locally compact, then there exists a one to one continuous function, say f, onto a Hausdorff compact space Y (see [7]). Let $A \subset \omega_1$ be any separable subspace. Since the closure of any countable set is compact in ω_1, then we get that $f|_{\overline{A}} : \overline{A} \longrightarrow f(\overline{A})$ is a homeomorphism (see [3, 3.1.13]) implying that $f|_A : A \longrightarrow f(A)$ is a homeomorphism.* ■

Here is another example of a Tychonoff space S_2-paracompact that is not paracompact being Hausdorff not normal space.

Example 2.5. *Recall the modified Dieudonné plank*

$$X = ((\omega_2 + 1) \times (\omega + 1)) \setminus \{\langle \omega_2, \omega \rangle\}.$$

Define $\mathcal{T}$ as the unique topology on X generated by the following neighborhood system: For each $\alpha < \omega_2$ and $n < \omega$, let $\mathcal{B}(\langle \alpha, n \rangle) = \{\{\langle \alpha, n \rangle\}\}$. Let $\mathcal{B}(\langle \alpha, \omega \rangle) = \{\{\alpha\} \times (n, \omega] : n < \omega\}$ for every $\alpha < \omega_2$. For each $n < \omega$, let $\mathcal{B}(\langle \omega_2, n \rangle) = \{(\alpha, \omega_2] \times \{n\} : \alpha < \omega_2\}$.

As mentioned in [5, Example 2], if we define a new topology on X by making each element of the form $\langle \omega_2, n \rangle$ with $n < \omega$ isolated, the modified Dieudonné plank will be a Tychonoff S_2-paracompact space which is not paracompact space. ■

The same technique of the proof of Theorem 2.2 could be used to prove the following theorem.

Theorem 2.6. *If X is separable but not paracompact, then X cannot be an S-paracompact.*

Recall that a space $(X, \mathcal{T})$ is *S-normal* if there exist a normal space Y and a function f such that $f : X \longrightarrow Y$ is a bijection and $f|_B : B \longrightarrow f(B)$ is a homeomorphism for every separable subspaces $B \subseteq X$ (see [5]). Since any T_2 paracompact space is normal, then any S_2-paracompact space is S-normal. However, this relation is not reversible as shown in the following example.

Example 2.7. *Consider the left ray topology defined on* $\mathbb{R}$, $(\mathbb{R}, \mathcal{L})$, *such that* $\mathcal{L} = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$. *It is an example of S-normal space by being a normal space which is not* S_2*-paracompact space since it is separable and not* T_2 *space. In fact, it is not even an S-paracompact because it is separable not a paracompact space.* ■

The following theorem presents a relation between the S_2-paracompactness and compactness.

Theorem 2.8. *Every* T_2 *countably compact separable S-paracompact space is compact.*

Proof. Let X be any T_2 countably compact separable S-paracompact space. Then X is paracompact because the witness function of S-paracompactness is a homeomorphism. Since any countably compact T_2 paracompact space is compact (see [3, 5.1.20]), we get that X is compact. □

Remark 2.9. *It is clear that any separable* S_2*-paracompact is* T_2 *paracompact, hence* T_4. *Thus, the Niemtyzki plane* $\mathbb{L}$ *(see [9, Example 82]), the Sorgenfrey line square* $(\mathbb{R}^2, \mathcal{S})$ *[9, Example 84], and the rational sequence topology* $(\mathbb{R}, \mathcal{RS})$ *(see [9, Example 65]) are examples of Tychonoff separable spaces which are not paracompact because they are Hausdorff non-normal spaces. Hence, they are not S-paracompact spaces. Also, the particular point topology defined on* $\mathbb{R}$, $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$, *is not S-paracompact space by being separable not paracompact space (see [9, Example 10]). Note that since any submetrizable is* C_2*-paracompact,* $(\mathbb{R}^2, \mathcal{S})$ *is an example of a* C_2*-paracompact space which is not* S_2*-paracompact (see [8]).*

A function $f : X \to Y$ witnessing S-paracompactness (S_2-paracompactness) of X need not be continuous, see Example 2.18. But it will be continuous if X is Fréchet. Recall that a space X is called *Fréchet* if for every subset $A \subseteq X$ and for any $x \in \overline{A}$, there exists a sequence $(a_n)_{n \in \mathbb{N}}$ such that $a_n \in A$ for every $n \in \mathbb{N}$ and $a_n \longrightarrow x$ (see [3]).

Theorem 2.10. *If* X *is an S-paracompact (*S_2*-paracompact) space and Fréchet such that* $f : X \longrightarrow Y$ *is a witness of S-paracompactness (*S_2*-paracompactness) of* X*, then* f *is continuous.*

Proof. Assume that X is S-paracompact and Fréchet. Let $f : X \longrightarrow Y$ witnesses S-paracompactness of X. Let $A \subseteq X$ and pick $y \in f(\overline{A})$ implying that there exists a unique $x \in X$ such that $f(x) = y$ and $x \in \overline{A}$. Since X is Fréchet, then there exists a sequence $(a_n) \subseteq A$ such that $a_n \longrightarrow x$. The subspace $B = \{x, a_n : n \in \mathbb{N}\}$ of X is separable by being countable, thus $f|_B : B \longrightarrow f(B)$ is a homeomorphism. Now, let $W \subseteq Y$ be any open neighborhood of y. Then $W \cap f(B)$ is open in the subspace $f(B)$ containing y. Since $f(\{a_n : n \in \mathbb{N}\}) \subseteq f(B) \cap f(A)$ and $W \cap f(B) \neq \emptyset$, $W \cap f(A) \neq \emptyset$. Hence $y \in \overline{f(A)}$, thus $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore, f is continuous. □

Theorem 2.11. *S-paracompactness (*S_2*-paracompactness) is a topological property.*

Proof. Let X be an S-paracompact space and let Z be any topological space such that X is homeomorphic to Z. Let f be the function witnessing S-paracompactness of X onto a paracompact space Y and $g : X \longrightarrow Z$ be a homeomorphism. Then $f \circ g^{-1} : Z \longrightarrow Y$ will be the witness of S-paracompactness of Z. □

Taking a compactification of the first three Tychonoff spaces mentioned in Remark 2.9 will show that S-paracompactness (S_2-paracompactness) is not hereditary. Moreover, as shown in Example 2.12 below, S-paracompactness (S_2-paracompactness) is not a multiplicative property.

Example 2.12. *The Sorgenfrey line* $(\mathbb{R}, \mathcal{S})$ *is an* S_2*-paracompact by being a* T_2 *paracompact space. However, as mentioned in Remark 2.9,* $(\mathbb{R}^2, \mathcal{S})$ *is not an* S_2*-paracompact.* ■

Here is a case when a product of two S_2-paracompact spaces will be an S_2-paracompact.

Theorem 2.13. *The Cartesian product of two* S_2*-paracompact spaces is* S_2*-paracompact in case that at least one of them is countably compact and Fréchet.*

Proof. Let X and Z be S_2-paracompact such that X is countably compact and Fréchet. Let Y and $f : X \longrightarrow Y$ be witnesses of S_2-paracompactness of X. Then f is continuous by Theorem 2.10 implying that Y is countably compact. Hence, Y is compact. Let Y' and $f' : Z \longrightarrow Y'$ be witnesses of S_2-paracompactness of Z. Consider the function $g := f \times f' : X \times Z \longrightarrow Y \times Y'$. Observe that $Y \times Y'$ is T_2 paracompact (see [3, 5.1.36]). Now, let D be any separable subspace of $X \times Z$. Therefore, $p_1(D) \subseteq X$ and $p_2(D) \subseteq Z$ are both separable subspaces of X and Z respectively being continuous images of a separable subspace $D \subseteq X \times Z$. Then using the fact that countable product of separable spaces is separable, $p_1(D) \times p_2(D)$ is separable in $X \times Z$. Thus, as $D \subseteq p_1(D) \times p_2(D)$, we get that $g\,|_D : D \longrightarrow g(D)$ is a homeomorphism. □

As an application of Theorem 2.13, consider the following topological space: $\omega_1 \times I^\kappa$, where κ is an uncountable ordinal (see [9, Example 106]). Observe that ω_1 is an S_2-paracompact, Fréchet, and countably compact. Moreover, I^κ is S_2-paracompact by being T_2 compact. By Theorem 2.13, we get that $\omega_1 \times I^\kappa$ is an S_2-paracompact. Observe that $\omega_1 \times I^\kappa$ is not paracompact because it is T_2 non-normal since I^κ is not of countable tightness.

Theorem 2.14. *S-paracompactness (S_2-paracompactness) is an additive property.*

Proof. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of S_2-paracompact spaces. Hence, for each $\alpha \in \Lambda$ there exist a paracompact space Y_α and a bijection $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_\alpha\,|_{A_\alpha} : A_\alpha \longrightarrow f_\alpha(A_\alpha)$ is a homeomorphism for each separable subspace A_α of X_α. Since paracompactness is an additive property (see [3, 5.1.30.]), $\oplus_{\alpha\in\Lambda} Y_\alpha$ is paracompact. Define $f : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ as follows: for each $x \in \oplus_{\alpha\in\Lambda} X_\alpha$, there exists a unique $\gamma \in \Lambda$ such that $x \in X_\gamma$ then $f(x) = f_\gamma(x)$. Let A be any separable subspace of $\oplus_{\alpha\in\Lambda} X_\alpha$. Write $A = \cup_{\alpha\in\Lambda^*}(A \cap X_\alpha)$ where $\Lambda^* = \{\alpha \in \Lambda : A \cap X_\alpha \neq \emptyset\}$. Since A is separable, then Λ^* is countable and $A \cap X_\alpha$ is separable in X_α for all $\alpha \in \Lambda^*$. Therefore, $f_\alpha\,|_{A\cap X_\alpha} : A \cap X_\alpha \longrightarrow f_\alpha(A \cap X_\alpha)$ is a homeomorphism for each $\alpha \in \Lambda^*$ implying that $f\,|_A : A \longrightarrow f(A)$ is a homeomorphism. □

The following result gives a relation between S_2-paracompactness and metrizability.

Theorem 2.15. *Every second countable S_2-paracompact space is metrizable.*

Proof. Let $(X, \mathcal{T})$ be an S_2-paracompact second countable space which yields that X is separable S_2-paracompact. Then X is T_4 implying that X is regular. Since any second countable T_3 space is metrizable [3, 4.2.9], we get that X is metrizable. □

Corollary 2.16. *Every T_2 second countable S-paracompact space is metrizable.*

The converse of Theorem 2.15 is not true in general. For example, the discrete topology defined on $\mathbb{R}$ is metrizable and S_2-paracompact but not second countable.

Recall that a topological space X is called *P-space* if X is T_1 and every G_δ-set is open (see [6]). In the following theorem, we will show that the class of P-spaces is S_2-paracompact.

Theorem 2.17. *Every P-space is S_2-paracompact.*

Proof. Let $(X, \mathcal{T})$ be a P-space. If X is countable, then it is discrete [6] implying that X is S_2-paracompact. Assume now that X is uncountable. Let $A \subseteq X$ be an arbitrary uncountable subset of X and let $D \subset A$ be any countable subset of A. Then D is closed set in A and $A \setminus D$ is a non-empty open set in A with $(A \setminus D) \cap D = \emptyset$. Hence, D cannot be dense in A implying that A cannot be separable. Thus, we conclude that any separable subspace of X must be countable and hence any separable subspace of X is discrete. Take the identity map $\mathrm{id} : (X, \mathcal{T}) \longrightarrow (X, \mathcal{D})$. Then $\mathrm{id}\,|_A : A \longrightarrow f(A)$ is a homeomorphism for all separable subspaces A. Therefore, $(X, \mathcal{T})$ is S_2-paracompact. □

Note that $(\mathbb{R}, \mathcal{U})$, ω_1, and the modified Dieudonné plank are examples of S_2-paracompact spaces which are not P-space.

Example 2.18. *An application of Theorem 2.17, consider* $(\mathbb{R}, \mathcal{CC})$, *where* $\mathcal{CC}$ *is the countable complement topology defined on* $\mathbb{R}$ *(see [9, Example 20]). Since* $(\mathbb{R}, \mathcal{CC})$ *is P-space, then by Theorem 2.17,* $(\mathbb{R}, \mathcal{CC})$ *is* S_2*-paracompact . Note that the function witnessing the* S_2*-paracompactness here is the identity taken from* $\mathcal{CC}$ *to the discrete topology defined on* $\mathbb{R}$*, write* $(\mathbb{R}, \mathcal{D})$*. However,* $\mathrm{id} : (\mathbb{R}, \mathcal{CC}) \longrightarrow (\mathbb{R}, \mathcal{D})$ *is not continuous.* ■

Recall that X is *locally separable* if each element has a separable open neighborhood (see [3, 4.4.F]). Next theorem describes the relation between S_2-paracompactness, locally separability and the Lindelöfness property. Theorem 2.2 and Theorem 2.6 give the following statement.

Theorem 2.19. *If* X *is Lindelöf, locally separable and* S_2*-paracompact, then* X *is* T_2 *paracompact, and hence is* T_4*.*

Note that S_2-paracompactness is essential in Theorem 2.19. For example $(\mathbb{R}, \mathcal{CF})$ is locally separable and Lindelöf but neither T_2 nor normal. Observe that $(\mathbb{R}, \mathcal{CF})$ is not an S_2-paracompact.

Let X be a topological space. Recall that the G_δ-extension X_δ of X is the topology on the same underlying set X generated by the family of all G_δ-subsets of X (see [2]). If $(X, \mathcal{T})$ is T_1 first countable space, then any singleton is a G_δ-set. Therefore, the G_δ-extension of any T_1 first countable space is S_2-paracompact being a discrete space. The converse is not true in general. As an example, consider the three Tychonoff spaces mentioned in 2.9.

3. Invariance

In this section we will discuss the invariance of S-paracompactness (S_2-paracompactness) under different mappings. The following examples will prove that S-paracompactness (S_2-paracompactness) is neither invariant, inverse invariant nor open invariant.

Example 3.1. *The identity function* $\mathrm{id} : (\mathbb{R}, \mathcal{U}) \longrightarrow (\mathbb{R}, \mathcal{L})$ *is a continuous bijective function. As shown in Example 2.7,* $(\mathbb{R}, \mathcal{L})$ *is not S-paracompact unlikely to* $(\mathbb{R}, \mathcal{U})$ *which is* T_2 *paracompact. Hence, S-paracompactness (*S_2*-paracompactness) is not invariant.*

On the other hand, the identity function $\mathrm{id} : (\mathbb{R}^2, \mathcal{S}) \longrightarrow (\mathbb{R}^2, \mathcal{U})$ *is a bijective continuous function. Since* $(\mathbb{R}^2, \mathcal{S})$ *is not S-paracompact, we get that S-paracompactness is not inverse invariant. In addition,* $p : \mathbb{L} \longrightarrow (\mathbb{R}, \mathcal{U})$ *such that* $p(\langle x, y\rangle) = x$ *is an example showing that S-paracompactness (*S_2*-paracompactness) is not inverse open invariant.* ■

S-paracompactness (S_2-paracompactness) is not open invariant as shown in the following example.

Example 3.2. *Consider* $(\mathbb{R}, \mathcal{U})$*, the usual topology defined on the set of real numbers. Then the Alexandroff Duplicate of the usual topology is defined as follows:*

$$A(\mathbb{R}) = \mathbb{R} \cup \mathbb{R}',$$

where $\mathbb{R}' = \mathbb{R} \times \{1\} = \{\langle y, 1\rangle = y' : y \in \mathbb{R}\}$ *such that the basic open neighborhood for every* $x \in \mathbb{R}$ *has the form* $U \cup (U' \setminus \{x'\})$ *where* $x \in U \in \mathcal{U}$ *and* $U' = \{\langle y, 1\rangle : y \in U\}$ *and the basic open set for every* $x' \in \mathbb{R}'$ *is* $\{x'\}$*. The space* $A(\mathbb{R})$ *is* S_2*-paracompact begin* T_2 *paracompact (see [1, 4]). Now let* $i = \sqrt{-1} \notin \mathbb{R}$*. Consider the closed extension* (X, τ) *of* $(\mathbb{R}, \mathcal{U})$ *where* $X = \mathbb{R} \cup \{i\}$ *and* $\tau \subseteq \mathcal{P}(X)$ *is defined as follows:*

$$\tau = \{\emptyset\} \cup \{W \cup \{i\} : W \in \mathcal{U}\}.$$

The space (X, τ) *is not* S_2*-paracompact since it is neither* T_2 *nor paracompact but separable as* $\{i\}$ *is a countable dense subset of* X*. Define* $f : A(\mathbb{R}) \longrightarrow X$ *by:*

$$f(x) = \begin{cases} x & ; x \in \mathbb{R} \\ i & ; x \in \mathbb{R}' \end{cases}$$

Then f *is continuous, open, and surjective.* ■

Since any continuous open surjective function is a quotient, we conclude the following:

Corollary 3.3. *S-paracompactness (S_2-paracompactness) is not preserved under quotient maps.*

We do not have any result yet regarding the closed invariant. We also do not have an answer to the following problem.

Problem 3.4. *If X is S-paracompact (S_2-paracompact), is then its Alexandroff duplicate $A(X)$ S-paracompact (S_2-paracompact)?*

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