Filomat 33:17 (2019), 5691–5711
https://doi.org/10.2298/FIL1917691L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Stability and Optimal Control in a Mathematical Model of Online Game Addiction

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Abstract. In this paper, we construct an online game addiction model(including susceptible, infective, professional and quitting compartments). We also consider that the direct transfer from the susceptible individuals to the professional individuals. Some properties of the model are derived by the basic reproduction number R_0 and stability of all kinds of equilibria is obtained. Then we use Pontriagin's maximum principle to solve the optimal control strategy. Finally, Numerical simulations are also conducted in the analytic results.

1. Introduction

In recent years, the online game industry has developed rapidly, and the scale of online users has continued to grow. According to the 43rd China Internet Development Statistics Report issued by China Internet Information Center, in the first half of 2018, the number of online users in China was 829.52 million[1].

On March 13, 2019, the Statistical Classification of Sports Industry (2019) was approved by the 4th Standing Meeting of the National Statistical Bureau of China[2]. E-sports was officially classified as a sport event, coded 020210210.

At the same time, online games also bring some bad effects. People who are weak in self-control often fail to distinguish between virtual reality and reality. They are easy to indulge in games and have many social problems. On April 15, 2007, the General Administration of Press and Publication of China issued the Real Name Certification Scheme for Online Game Anti-addiction System. In this scheme, the cumulative game time is defined as healthy game time within 3 hours per day, fatigue game time within 3-5 hours and unhealthy game time over 5 hours.

On June 19, 2018, the World Health Organization (WHO) officially issued a draft listing of game addiction as a mental disorder, which is expected to be formally introduced to integrate game addiction into the medical system[3]. Online games have a certain attractiveness, and the people who are weak in willpower are easily affected and indulged in it. Therefore, in a sense, the behavior of indulging in online games and infectious diseases have similar transmission mechanism, and online games are contagious.

2010 *Mathematics Subject Classification*. Primary 34Dxx
Keywords. Game addiction model; Stability; Equilibria; Pontriagin's maximum principle; Optimal control.
Received: 16 May 2019; Revised: 21 July 2019; Accepted: 28 August 2019
Communicated by Jelena Manojlović
Research supported by Basic Competence Promotion Project for Young and Middle-aged Teachers in Guangxi, China (2018KY0262, 2019KY0269)
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In the past 30 years, many scholars used biomathematical models to capture the characteristics of many infectious diseases. The dynamic models of infectious diseases have been extensively studied with different theories, tools and methods from different perspectives and emphases. So far, there are many rich theoretical results[4–9] .

Many scholars applied the research methods of infectious diseases to other problems, such as smoking, alcoholism, drug abuse, games and so on. Zaman[10] presented the optimal campaigns in the smoking dynamics which assumed that the giving up smoking model is described by the simplified PLSQ (potential-light-smoker-quit smoker) model. Wang[11] discussed a deterministic SATQ-type mathematical model (including susceptible, alcoholism, treating, and quitting compartments) for the spread of alcoholism with two control strategies to gain insights into this increasingly concerned about health and social phenomenon. Firster[12] studied the optimizing chemotherapy in an HIV model. Weinstein[13] used phenomenology and epidemiology to diagnose game addiction by comparing players with non-players. Jiang[14] discussed the dynamics of game transmission on complex networks, with dividing the total population into four compartments. The threshold of the model and the stability of the positive equilibrium were obtained. Wang[15] studied the dynamic analysis of the mathematical model of online game addiction with age structure.

Optimal control theory and method are more and more widely used in biomathematics. Many biomathematical scientists and technicians devote themselves to this burgeoning field. Khan et al.[19] formulated a dynamical model of asymptomatic carrier zika virus with optimal control strategies, and considered that the asymptomatic carrier individuals have the abilities to take part in the infection generation. The optimal control model and the model without controls are solved numerically by the forward and backward Runge-Kutta method. Cholera-schistosomiasis coinfection dynamics is analyzed in [20], where the authors explored the dynamics of both diseases and their coinfection and provided effective control strategies for disease elimination. Mathematical formulation of hepatitis B virus with optimal control analysis is investigated in [21]. Modelling the effects of heavy alcohol consumption on the transmission dynamics of gonorrhea with optimal control is presented in [22]. Optimal control strategies for dengue transmission in Pakistan is studied in [23], where the authors studied the impact of an imperfect vaccine in the bid to control dengue. And there are still a lot of useful references cited therein.

In this paper, we will formulate a reasonable mathematical model of online game addiction. The fact that our model is reasonable is embodied from the following some aspects.

(1)With the continuous development of e-sports, more and more professionals engaged in e-sports, such as professional contestants, game developers, e-sports media organization planning, e-sports hosts, game commentaries and so on. They have gradually become an important group that can not be ignored. Although they spend more than five hours a day playing games, they are different from the traditional game addicts. Therefore, we set up a storehouse for the professional group of e-sports, which is recorded as P.

(2)In China, America and other countries, many universities have gradually opened e-sports specialty to cultivate talents in e-sports. Since they have changed from susceptible people to professional people directly and do not need to go through the stage of game addiction, we establish a direct transfer from S to P.

(3)After education and treatment from hospitals or psychologists, people addicted may choose to become professionals or quit permanently. Some professionals will opt out at a certain age because of physical reasons. So we set up an quitting warehouse, named Q.

(4)Online game addiction is similar to infectious diseases, and it can also be prevented by certain ways, such as propaganda and education, online game supervision and other measures to regulate and control. Therefore, the number of people addicted to games can be controlled by some measures.

Based on the above considerations and inspired by the existing literature, we apply epidemiological dynamics to simulate the spread process of online game addiction according to the characteristics of online games. The population is divided into four compartments, namely, susceptible(S), infective(I), professional(P) and quitting(Q). On the one hand, qualitative analysis and quantitative numerical simulation of the model using classical biomathematics theory and tools are helpful to predict the development of online game addiction. On the other hand, adopting the optimal control strategy can help management to formulate reasonable policies to deal with the problem.

2. The Model Formulation

2.1. System Description

The total population denoted by N(t) is partitioned into four compartments, namely, the susceptible individuals whose gaming time is less than 5 hours per day, denoted by S(t); the infective individuals whose gaming time is more than 5 hours per day and who lack a proper job, denoted by I(t); the professional individuals whose gaming time is more than 5 hours per day and who have a proper job, denoted by P(t); the quitting individuals who quit playing game, denoted by Q(t). Thus, the total population is given by:

$$N(t) = S(t) + I(t) + P(t) + Q(t). \tag{1}$$

The population flow among those compartments is shown in Figure 1.

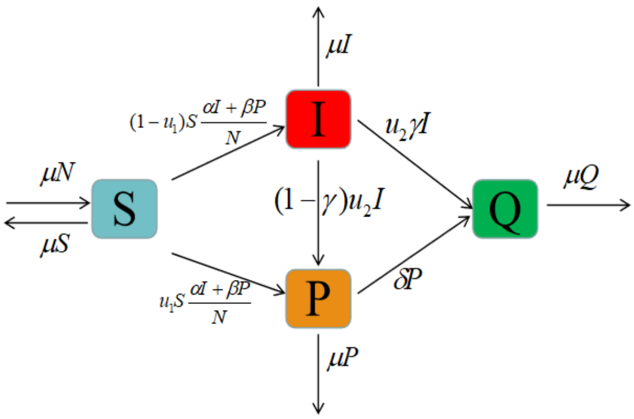

Fig.1. Transfer diagram of model

The transfer diagram leads to the following system of ordinary differential equations:

$$\begin{aligned}
S' &= \mu(N-S) - S\frac{\alpha I + \beta P}{N} \\
I' &= (1-u_1)S\frac{\alpha I + \beta P}{N} - (u_2+\mu)I \\
P' &= u_1 S\frac{\alpha I + \beta P}{N} + (1-\gamma)u_2 I - (\delta+\mu)P \\
Q' &= \gamma u_2 I + \delta P - \mu Q
\end{aligned} \tag{2}$$

where μ is the nature birth rate and death rate; α denotes the transmission coefficient for the addictive individuals ; β is the transmission coefficient for the professional individuals; u_1 represents the probability of people who become professionals directly without indulgence; u_2 represents the proportion of people who are no longer addicted to games; γ denotes the ratio of withdrawal from addiction; δ denotes the quitting rate of P.

2.2. Positivity and Boundedness of Solutions

For system (2), to ensure that the solutions of the system with positive initial conditions remain positive for all $t > 0$, it is necessary to prove that all the state variables are nonnegative. System (2) can be put into the matrix form

$$X' = G(X) \tag{3}$$

where $X = (S, I, P, Q)^T \in R^4$ and G(X) is given by

$$G(X) = \begin{pmatrix} G_1(X) \\ G_2(X) \\ G_3(X) \\ G_4(X) \end{pmatrix} = \begin{pmatrix} \mu(N-S) - S\frac{\alpha I+\beta P}{N} \\ (1-u_1)S\frac{\alpha I+\beta P}{N} - (u_2+\mu)I \\ u_1 S\frac{\alpha I+\beta P}{N} + (1-\gamma)u_2 I - (\delta+\mu)P \\ \gamma u_2 I + \delta P - \mu Q \end{pmatrix}. \tag{4}$$

It is easy to know that

$$G_i(X)|_{X_i(t)=0, X_t\in C_+} \geq 0, \qquad i = 1,2,3,4. \tag{5}$$

Because of $\Sigma_{i=1}^4 G_i(x) = 0$, $N(t)$ is a constant denoted by N. Due to Lemma 2 in [4], all feasible solutions of the system (2) are bounded and belong to the set:

$$\Omega = \{(S, I, P, Q) \in R_+^4 : S + I + P + Q \leq N\}. \tag{6}$$

3. The Basic Reproduction Number and Existence of Disease Equilibrium

3.1. The Basic Reproduction Number

The model has a disease-free equilibrium E_0 given by

$$E_0 = (N, 0, 0, 0). \tag{7}$$

In the following, the basic reproduction number of system (2) will be obtained by the next generation matrix method. Let $x = (I, P, Q, S)^T$, then system(2) can be written as

$$\frac{dx}{dt} = \mathscr{F}(x) - \mathscr{V}(x). \tag{8}$$

where

$$\mathscr{F}(x) = \begin{pmatrix} (1-u_1)S\frac{\alpha I+\beta P}{N} \\ u_1 S\frac{\alpha I+\beta P}{N} \\ 0 \\ 0 \end{pmatrix}, \quad \mathscr{V}(x) = \begin{pmatrix} (u_2+\mu)I \\ (\delta+\mu)P - (1-\gamma)u_2 I \\ \mu Q - \gamma u_2 I - \delta P \\ \mu(S-N) + S\frac{\alpha I+\beta P}{N} \end{pmatrix}.$$

The Jacobian matrices of $\mathscr{F}(x)$ and $\mathscr{V}(x)$ at the disease-free equilibrium E_0 are

$$D\mathscr{F}(E_0) = \begin{pmatrix} F_{3\times3} & 0 \\ 0 & 0 \end{pmatrix}, \quad D\mathscr{V}(E_0) = \begin{pmatrix} V_{3\times3} & 0 \\ \alpha \quad \beta \quad 0 & \mu \end{pmatrix}.$$

where

$$F_{3\times3} = \begin{pmatrix} (1-u_1)\alpha & (1-u_1)\beta & 0 \\ u_1\alpha & u_1\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_{3\times3} = \begin{pmatrix} u_2+\mu & 0 & 0 \\ (\gamma-1)u_2 & \delta+\mu & 0 \\ -\gamma u_2 & -\delta & \mu \end{pmatrix}.$$

The basic reproduction number, denoted by R_0, is given by

$$R_0 = \rho(FV^{-1}) = \frac{(1-u_1)\alpha(\delta+\mu) + \beta u_2(1-\gamma+u_1\gamma) + u_1\beta\mu}{(u_2+\mu)(\delta+\mu)}. \tag{9}$$

3.2. Existence of Disease Equilibrium

The disease equilibrium $E^*(S^*, I^*, P^*, Q^*)$ of system (2) is determined by equations:

$$\mu(N-S) - S\frac{\alpha I + \beta P}{N} = 0 \tag{10}$$

$$(1-u_1)S\frac{\alpha I + \beta P}{N} - (u_2+\mu)I = 0 \tag{11}$$

$$u_1 S\frac{\alpha I + \beta P}{N} + (1-\gamma)u_2 I - (\delta+\mu)P = 0 \tag{12}$$

$$\gamma u_2 I + \delta P - \mu Q = 0 \tag{13}$$

Through equations (10-12), we have

$$S = N - \frac{u_2+\mu}{(1-u_1)\mu}I \tag{14}$$

$$P = \frac{u_1(u_2+\mu) + (1-\gamma)(1-u_1)u_2}{(\delta+\mu)(1-u_1)}I \tag{15}$$

From equations (10),(14),(15), we can get

$$I = \frac{N(1-u_1)\mu}{u_2+\mu}\left(1-\frac{1}{R_0}\right) \tag{16}$$

Theorem 1 In the system (2), there is always a disease-free equilibrium $E_0 = (N, 0, 0, 0)$. When $R_0 > 1$, the system has a unique disease equilibrium $E^* = (S^*, I^*, P^*, Q^*)$, where

$$S^* = \frac{N}{R_0}$$

$$I^* = \frac{N(1-u_1)\mu}{u_2+\mu}\left(1-\frac{1}{R_0}\right)$$

$$P^* = \frac{u_1(u_2+\mu) + (1-\gamma)(1-u_1)u_2}{(\delta+\mu)(1-u_1)}I^*$$

$$Q^* = \frac{\gamma u_2(\delta+\mu)(1-u_1) + \delta u_1(u_2+\mu) + \delta u_2(1-\gamma)(1-u_1)}{\mu(\delta+\mu)(1-u_1)}I^*$$

4. Stability Analysis of Equilibria

We denote a vector $X = (I, P, Q, S)^{\mathrm{T}}$ and

$$f(X) = \begin{pmatrix} (1-u_1)S\frac{\alpha I+\beta P}{N} - (u_2+\mu)I \\ u_1 S\frac{\alpha I+\beta P}{N} + (1-\gamma)u_2 I - (\delta+\mu)P \\ \gamma u_2 I + \delta P - \mu Q \\ \mu(N-S) - S\frac{\alpha I+\beta P}{N} \end{pmatrix}. \tag{17}$$

So the Jacobian matrix of $f(X)$ about vector X is as the following:

$$J = \frac{\partial f(X)}{\partial X} = \begin{pmatrix} (1-u_1)S\frac{\alpha}{N} - (u_2+\mu) & (1-u_1)S\frac{\beta}{N} & 0 & (1-u_1)\frac{\alpha I+\beta P}{N} \\ u_1 S\frac{\alpha}{N} + (1-\gamma)u_2 & u_1 S\frac{\beta}{N} - (\delta+\mu) & 0 & u_1\frac{\alpha I+\beta P}{N} \\ \gamma u_2 & \delta & -\mu & 0 \\ -S\frac{\alpha}{N} & -S\frac{\beta}{N} & 0 & -\mu \end{pmatrix}.$$

Theorem 2 For the system (2), the disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$.

Proof. Since

$$J(E_0) = \begin{pmatrix} (1-u_1)\alpha - (u_2+\mu) & (1-u_1)\beta & 0 & 0 \\ u_1\alpha + (1-\gamma)u_2 & u_1\beta - (\delta+\mu) & 0 & 0 \\ \gamma u_2 & \delta & -\mu & 0 \\ -\alpha & -\beta & 0 & -\mu \end{pmatrix}.$$

It is known that two of the eigenvalues are $\lambda_1 = \lambda_2 = -\mu < 0$, and λ_3, λ_4 satisfy

$$\begin{aligned} \lambda_3 + \lambda_4 &= (1-u_1)\alpha - (u_2+\mu) + u_1\beta - (\delta+\mu) \\ \lambda_3\lambda_4 &= (u_2+\mu)(\delta+\mu) - \beta u_2(1-(1-u_1)\gamma) - \beta\mu u_1 - (1-u_1)\alpha(\delta+\mu) \end{aligned}$$

Since $R_0 < 1$, $\lambda_3\lambda_4 > 0$.
Though

$$0 < R_0 = \frac{(1-u_1)\alpha}{u_2+\mu} + \frac{\beta u_1}{\delta+\mu} + \frac{\beta u_2(1-\gamma)(1-u_1)}{(\delta+\mu)(u_2+\mu)} < 1$$

Thus,

$$0 < \frac{(1-u_1)\alpha}{u_2+\mu} < 1, \qquad 0 < \frac{\beta u_1}{\delta+\mu} < 1$$

we can get

$$\lambda_3 + \lambda_4 < 0$$

Hence, $\mathrm{Re}(\lambda_3) < 0$, $\mathrm{Re}(\lambda_4) < 0$. The proof is completed. ■

Theorem 3 For the system (2), the disease-free equilibrium E_0 is globally asymptotically stable if $R_0 < 1$.

Proof. Consider the subsystem of (2) as follows:

$$\begin{aligned} I' &= (1-u_1)S\frac{\alpha I + \beta P}{N} - (u_2+\mu)I \\ P' &= u_1 S\frac{\alpha I + \beta P}{N} + (1-\gamma)u_2 I - (\delta+\mu)P \\ Q' &= \gamma u_2 I + \delta P - \mu Q \end{aligned} \tag{18}$$

For $S \le N$,

$$\begin{aligned} \begin{pmatrix} I' \\ P' \\ Q' \end{pmatrix} &\le \begin{pmatrix} ((1-u_1)\alpha - (u_2+\mu))I + (1-u_1)\beta P \\ (u_1\alpha + (1-\gamma)u_2)I + (u_1\beta - (\delta+\mu))P \\ \gamma u_2 I + \delta P - \mu Q \end{pmatrix} \\ &= (F-V)\begin{pmatrix} I \\ P \\ Q \end{pmatrix} \end{aligned}$$

Since the eigenvalues of the matrix $F - V$ all have negative real parts, then system (2) is stable when $R_0 < 1$. So $(I, P, Q) \to (0, 0, 0)$ as $t \to \infty$. By the comparison theorem, it follows that $(I, P, Q) \to (0, 0, 0)$ and $S \to N$ as $t \to \infty$. So E_0 is globally asymptotically stable for $R_0 < 1$. ■

Theorem 4 For the system (2), the disease equilibrium $E^* = (S^*, I^*, P^*, Q^*)$ is globally asymptotically stable if $R_0 > 1$.

Proof. Because $N = S + I + P + Q$ is a constant, we introduce the fractions of them:

$$s = \frac{S}{N}, \quad i = \frac{I}{N}, \quad p = \frac{P}{N}, \quad q = \frac{Q}{N}.$$

with $s + i + p + q = 1$. Thus, the system (2) becomes:

$$\begin{aligned} s' &= \mu(1-s) - s(\alpha i + \beta p) \\ i' &= (1-u_1)s(\alpha i + \beta p) - (u_2 + \mu)i \\ p' &= u_1 s(\alpha i + \beta p) + (1-\gamma)u_2 i - (\delta + \mu)p \\ q' &= \gamma u_2 i + \delta p - \mu q. \end{aligned} \tag{19}$$

And the disease equilibrium $E^* = (S^*, I^*, P^*, Q^*)$ becomes $E^{**} = (s^*, i^*, p^*, q^*)$, where

$$\begin{aligned} s^* &= \frac{1}{R_0} \\ i^* &= \frac{(1-u_1)\mu}{u_2+\mu}(1 - \frac{1}{R_0}) \\ p^* &= \frac{u_1(u_2+\mu) + (1-\gamma)(1-u_1)u_2}{(\delta+\mu)(1-u_1)} i^* \\ q^* &= \frac{\gamma u_2(\delta+\mu)(1-u_1) + \delta u_1(u_2+\mu) + \delta u_2(1-\gamma)(1-u_1)}{\mu(\delta+\mu)(1-u_1)} i^* \end{aligned}$$

We introduce the Lyapunov function V as follows:

$$V = x_1(s - s^* lns) + x_2(i - i^* lni) + x_3(p - p^* lnp) + x_4(q - q^* lnq) \tag{20}$$

Applying the identity $\mu = \mu s^* + s^*(\alpha i^* + \beta p^*)$, the derivative of V is given by

$$\begin{aligned} V' &= x_1(1 - \frac{s^*}{s})s' + x_2(1 - \frac{i^*}{i})i' + x_3(1 - \frac{p^*}{p})p' + x_4(1 - \frac{q^*}{q})q' \\ &= x_1[\mu - \mu s - s(\alpha i + \beta p) - \frac{s^*\mu}{s} + \mu s^* + s^*(\alpha i + \beta p)] \\ &\quad + x_2[(1-u_1)s(\alpha i + \beta p) - (u_2+\mu)i - \frac{i^*}{i}(1-u_1)s(\alpha i + \beta p) + (u_2+\mu)i^*] \\ &\quad + x_3[u_1 s(\alpha i + \beta p) + (1-\gamma)u_2 i - (\delta+\mu)p - \frac{p^*}{p}u_1 s(\alpha i + \beta p) - \frac{p^*}{p}(1-\gamma)u_2 i + p^*(\delta+\mu)] \\ &\quad + x_4[\gamma u_2 i + \delta p - \mu q - \frac{q^*}{q}\gamma u_2 i - \frac{q^*}{q}\delta p + q^*\mu] \end{aligned}$$

$$\begin{aligned} &= x_1[\mu s^* + s^*(\alpha i^* + \beta p^*) - \mu s - s(\alpha i + \beta p) - \frac{s^*\mu s^*}{s} - \frac{s^* s^*}{s}(\alpha i^* + \beta p^*) + \mu s^* + s^*(\alpha i + \beta p)] \\ &\quad + x_2[(1-u_1)s(\alpha i + \beta p) - (u_2+\mu)i - \frac{i^*}{i}(1-u_1)s(\alpha i + \beta p) + (u_2+\mu)i^*] \\ &\quad + x_3[u_1 s(\alpha i + \beta p) + (1-\gamma)u_2 i - (\delta+\mu)p - \frac{p^*}{p}u_1 s(\alpha i + \beta p) - \frac{p^*}{p}(1-\gamma)u_2 i + p^*(\delta+\mu)] \\ &\quad + x_4[\gamma u_2 i + \delta p - \mu q - \frac{q^*}{q}\gamma u_2 i - \frac{q^*}{q}\delta p + q^*\mu] \end{aligned}$$

$$
\begin{aligned}
&= x_1\mu s^*(2-\frac{s}{s^*}-\frac{s^*}{s}) + si[-x_1\alpha + x_2(1-u_1)\alpha + x_3u_1\alpha] + sp[-x_1\beta + x_2(1-u_1)\beta + x_3u_1\beta] \\
&\quad + i[x_1\alpha s^* - x_2(u_2+\mu) + x_3(1-\gamma)u_2 + x_4\gamma u_2] + p[x_1s^*\beta - x_3(\delta+\mu) + x_4\delta] + q(-x_4\mu) \\
&\quad + [x_1s^*(\alpha i^* + \beta p^*) + x_2(u_2+\mu)i^* + x_3(\delta+\mu)p^* + x_4\mu q^*] - [x_1\frac{s^*s^*}{s}(\alpha i^* + \beta p^*) \\
&\quad + x_2\frac{i^*}{i}(1-u_1)s(\alpha i + \beta p) + x_3\frac{p^*}{p}u_1s(\alpha i+\beta p) + x_3\frac{p^*}{p}(1-\gamma)u_2 i + x_4\frac{q^*}{q}\gamma u_2 i + x_4\frac{q^*}{q}\delta p]
\end{aligned}
$$

The positive constants x_1, x_2, x_3, x_4 are chosen such that the coefficients of si, sp, i, p, q are equal to zero, that is,

$$
\begin{aligned}
&-x_1\alpha + x_2(1-u_1)\alpha + x_3u_1\alpha = 0 \\
&-x_1\beta + x_2(1-u_1)\beta + x_3u_1\beta = 0 \\
&x_1\alpha s^* - x_2(u_2+\mu) + x_3(1-\gamma)u_2 + x_4\gamma u_2 = 0 \\
&x_1s^*\beta - x_3(\delta+\mu) + x_4\delta = 0 \\
&-x_4\mu = 0
\end{aligned}
$$

So we have

$$
x_1 = 1, \quad x_2 = \frac{\delta+\mu-u_1s^*\beta}{(1-u_1)(\delta+\mu)}, \quad x_3 = \frac{s^*\beta}{\delta+\mu}, \quad x_4 = 0
$$

Next, we let

$$
\begin{aligned}
V_1' &= x_1s^*(\alpha i^* + \beta p^*) + x_2(u_2+\mu)i^* + x_3(\delta+\mu)p^* + x_4\mu q^* \\
V_2' &= x_1\frac{s^*s^*}{s}(\alpha i^*+\beta p^*) + x_2\frac{i^*}{i}(1-u_1)s(\alpha i+\beta p) + x_3\frac{p^*}{p}u_1s(\alpha i+\beta p) + x_3\frac{p^*}{p}(1-\gamma)u_2 i + x_4\frac{q^*}{q}\gamma u_2 i + x_4\frac{q^*}{q}\delta p
\end{aligned}
$$

Then

$$
V' = x_1\mu s^*(2-\frac{s}{s^*}-\frac{s^*}{s}) + V_1' - V_2'
$$

Due to

$$
x_2(1-u_1)s^*\beta p^* = x_3(1-\gamma)u_2 i^* + x_3u_1s^*\alpha i^* \tag{21}
$$

$$
x_3(1-\gamma)u_2 i^* = \frac{(1-u_1)\beta u_2(1-\gamma)s^*p^*}{u_2(1-(1-u_1)\gamma)+\mu u_1} \tag{22}
$$

So

$$
\begin{aligned}
V_1' &= s^*(\alpha i^*+\beta p^*) + x_2(u_2+\mu)i^* + x_3(\delta+\mu)p^* \\
&= [x_2(1-u_1)+x_3u_1]s^*(\alpha i^*+\beta p^*) + x_2(1-u_1)s^*(\alpha i^*+\beta p^*) + x_3u_1s^*(\alpha i^*+\beta p^*) + x_3(1-\gamma)u_2 i^* \\
&= 2x_2(1-u_1)s^*\alpha i^* + 2x_3u_1s^*\beta p^* + 3x_3(1-\gamma)u_2 i^* + 4x_3u_1s^*\alpha i^* \\
&= V_{11}' + V_{12}' + V_{13}' + V_{14}'
\end{aligned}
$$

where

$$
V_{11}' = 2x_2(1-u_1)s^*\alpha i^*, \quad V_{12}' = 2x_3u_1s^*\beta p^*, \quad V_{13}' = 3x_3(1-\gamma)u_2 i^*, \quad V_{14}' = 4x_3u_1s^*\alpha i^*
$$

and

$$\begin{aligned}
V_2' &= x_2(1-u_1)\frac{(s^*)^2}{s}\alpha i^* + x_3u_1\frac{(s^*)^2}{s}\alpha i^* + x_2(1-u_1)\frac{(s^*)^2}{s}\beta p^* + x_3u_1\frac{(s^*)^2}{s}\beta p^* + x_2(1-u_1)\alpha i^* s \\
&\quad + x_2(1-u_1)\beta\frac{i^*}{i}sp + x_3u_1\alpha\frac{p^*}{p}si + x_3u_1\beta p^* s + x_3(1-\gamma)u_2\frac{p^*}{p}i \\
&= x_2(1-u_1)\frac{(s^*)^2}{s}\alpha i^* + x_3u_1\frac{(s^*)^2}{s}\alpha i^* + \theta x_2(1-u_1)\frac{(s^*)^2}{s}\beta p^* + (1-\theta)x_2(1-u_1)\frac{(s^*)^2}{s}\beta p^* \\
&\quad + x_3u_1\frac{(s^*)^2}{s}\beta p^* + x_2(1-u_1)\alpha i^* s + \theta x_2(1-u_1)\beta\frac{i^*}{i}sp + (1-\theta)x_2(1-u_1)\beta\frac{i^*}{i}sp \\
&\quad + x_3u_1\alpha\frac{p^*}{p}si + x_3u_1\beta p^* s + x_3(1-\gamma)u_2\frac{p^*}{p}i
\end{aligned}$$

where

$$\theta = \frac{x_2(1-u_1)(\alpha i^* + \beta p^*) - \alpha i^*}{x_2(1-u_1)\beta p^*}, \quad 1-\theta = \frac{x_3u_1\alpha i^*}{x_2(1-u_1)\beta p^*}$$

So we have

$$\begin{aligned}
V_2' &= [x_2(1-u_1)\frac{(s^*)^2}{s}\alpha i^* + x_2(1-u_1)\alpha i^* s] + [x_3u_1\frac{(s^*)^2}{s}\beta p^* + x_3u_1\beta p^* s] \\
&\quad + [\theta x_2(1-u_1)\frac{(s^*)^2}{s}\beta p^* + \theta x_2(1-u_1)\beta\frac{i^*}{i}sp + x_3(1-\gamma)u_2\frac{p^*}{p}i] \\
&\quad + [x_3u_1\frac{(s^*)^2}{s}\alpha i^* + (1-\theta)x_2(1-u_1)\frac{(s^*)^2}{s}\beta p^* + (1-\theta)x_2(1-u_1)\beta\frac{i^*}{i}sp + x_3u_1\alpha\frac{p^*}{p}si] \\
&= V_{21}' + V_{22}' + V_{23}' + V_{24}'
\end{aligned}$$

Using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
V_{21}' &\geq 2[x_2(1-u_1)\frac{(s^*)^2}{s}\alpha i^* x_2(1-u_1)\alpha i^* s]^{\frac{1}{2}} = 2x_2(1-u_1)s^*\alpha i^* = V_{11}' \\
V_{22}' &\geq 2[x_3u_1\frac{(s^*)^2}{s}\beta p^* x_3u_1\beta p^* s]^{\frac{1}{2}} = 2x_3u_1s^*\beta p^* = V_{12}'
\end{aligned}$$

Due to

$$x_3 = \frac{s^*\beta}{\delta+\mu}, \quad i^* = \frac{(\delta+\mu)(1-u_1)}{u_1(u_2+\mu) + (1-\gamma)(1-u_1)u_2}p^*$$

Let $K = u_1\mu + (1-(1-u_1)\gamma)u_2$, so

$$\begin{aligned}
V_{23}' &\geq 3(\beta^2u_2)^{\frac{1}{3}}[\theta^2x_2^2x_3(1-u_1)^2(s^*)^2(1-\gamma)(p^*)^2i^*]^{\frac{1}{3}} \\
&= 3(\beta^2u_2)^{\frac{1}{3}}\frac{s^*(1-u_1)p^*}{K}\{\beta(1-\gamma)[\frac{K-u_1(u_2+\mu)}{(1-u_1)}]^2\}^{\frac{1}{3}} \\
&= 3\frac{s^*(1-u_1)p^*\beta u_2(1-\gamma)}{K} \\
&= 3x_3(1-\gamma)u_2i^* \\
&= V_{13}' \\
V_{24}' &\geq 4s^*[x_3(1-\theta)u_1x_2(1-u_1)\beta\alpha i^* p^*]^{\frac{1}{2}} \\
&= 4s^*x_3u_1\alpha i^* \\
&= V_{14}'
\end{aligned}$$

Hence,

$$\begin{aligned} V' &= x_1\mu s^*(2 - \frac{s}{s^*} - \frac{s^*}{s}) + V_1' - V_2' \\ &\le 0 \end{aligned}$$

$V' = 0$ if and only if $s = s^*, i = i^*, p = p^*, q = q^*$. According to LaSalle's invariance principle, we can derive the conclusion that the disease equilibrium $E^* = (S^*, I^*, P^*, Q^*)$ is globally asymptotically stable. ■

5. Optimal Control Problem

5.1. The Existence of Optimal Control

In order to investigate an effective campaign to control the problem of online game addiction, we will reconsider the system (2) which pursue the goals of the minimized infective compartment and use two variables to reduce it. the system (2) is:

$$\begin{aligned} S' &= \mu(N - S) - S\frac{\alpha I + \beta P}{N} \\ I' &= (1 - u_1)S\frac{\alpha I + \beta P}{N} - (u_2 + \mu)I \\ P' &= u_1 S\frac{\alpha I + \beta P}{N} + (1 - \gamma)u_2 I - (\delta + \mu)P \\ Q' &= \gamma u_2 I + \delta P - \mu Q \end{aligned}$$

$u_1(t)$ is used to limit the proportion of the susceptible individual who contact with infective individual, usually by propaganda and education. The control variable $u_2(t)$ is used to limit the infective individual turn into others compartments, and what we want is the fewer infective people, the better . In view of this, our optimal control problem to minimize the objective function is given by

$$J(u_1, u_2) = \int_0^{t_f} [I(t) + \frac{c_1}{2}u_1^2(t) + \frac{c_2}{2}u_2^2(t)]\, dt \tag{23}$$

With initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad P(0) = P_0, \quad Q(0) = Q_0. \tag{24}$$

Here,$u_i(t) \in (0, 1)$, for all $t \in [0, t_f]$, $i = 1, 2$. c_i are weight factors(positive constants) that adjust the intensity of two different control measures.

Theorem 5 There exists an optimal control pair $u^* = (u_1^*, u_2^*) \in U$ such that

$$J(u_1^*, u_2^*) = min\ J(u_1, u_2), \qquad u_1(t), u_2(t) \in U \tag{25}$$

subjects to the control system (2) with initial conditions (24).

Proof. To prove the existence of an optimal control, according to the classic literature[18], we have to show the following.

(1) The control and state variables are nonnegative values.

(2) The control set U is convex and closed.

(3) The integrand of the objective function is concave on U.

(4)The right side of the state system is bounded by linear functions in the state and control variables.

(5) There exist constants $d_1, d_2 > 0$ and $\alpha > 1$ such that the integrand $L(t; u_1, u_2) \triangleq I(t) + \frac{c_1}{2}u_1^2(t) + \frac{c_2}{2}u_2^2(t)$ of the objective function satisfies

$$L(t; u_1, u_2) \ge d_1(|u_1|^2 + |u_2|^2)^{\alpha/2} - d_2 \tag{26}$$

The statements (1), (2) and (3) are obviously satisfied, we only need to test the latter two. Since

$$S' \leq \mu N, \quad I' \leq (1-u_1)S\frac{\alpha I+\beta P}{N}, \quad P' \leq u_1 S\frac{\alpha I+\beta P}{N} + (1-\gamma)u_2 I, \quad Q' \leq \gamma u_2 I + \delta P$$

the fourth condition is set up. As the last condition,

$$L(t; u_1, u_2) \geq d_1(|u_1|^2 + |u_2|^2)^{\alpha/2} - d_2$$

is also true, when we choose $\alpha = 2, d_1 = min\{c_1/2, c_2/2\}$. Then the proof is completed. ∎

5.2. The Characterization of Optimal Control

We have just known the existence of the optimal control pairs. Next we begin by defining an augmented Hamiltonian H with penalty terms for the control constraints as follows:

$$\begin{aligned} H = {} & I(t) + \frac{c_1}{2}u_1^2(t) + \frac{c_2}{2}u_2^2(t) + \lambda_1[\mu(N-S) - S\frac{\alpha I+\beta P}{N}] + \lambda_2[(1-u_1)S\frac{\alpha I+\beta P}{N} - (u_2+\mu)I] \\ & + \lambda_3[u_1 S\frac{\alpha I+\beta P}{N} + (1-\gamma)u_2 I - (\delta+\mu)P] + \lambda_4[u_2\gamma I + \delta P - \mu Q] \\ & - w_{11}u_1(t) - w_{12}(1-u_1(t)) - w_{21}u_2(t) - w_{22}(1-u_2(t)) \end{aligned}$$

where $w_{ij}(t) \geq 0$ are the penalty multipliers satisfying

$$w_{11}(t)u_1^*(t) = w_{12}(1-u_1^*(t)) = 0, \quad w_{21}(t)u_2^*(t) = w_{22}(1-u_2^*(t)) = 0$$

Theorem 6 Given optimal control pairs (u_1^*, u_2^*) and solutions $S(t), I(t), P(t), Q(t)$ of the corresponding state system (23), there exist adjoint variables $\lambda_i, i = 1, 2, 3, 4$, satisfying

$$\begin{aligned} \lambda_1' &= \lambda_1\mu + \lambda_1\frac{\alpha I+\beta P}{N} - \lambda_2(1-u_1)\frac{\alpha I+\beta P}{N} - \lambda_3 u_1\frac{\alpha I+\beta P}{N} \\ \lambda_2' &= -1 + \lambda_1(t)\frac{S(t)\alpha}{N} + \lambda_2(1-u_1)\frac{S\alpha}{N} + \lambda_2(u_2+\mu) - \lambda_3 u_1\frac{S\alpha}{N} - \lambda_3(1-\gamma)u_2 - \lambda_4\gamma u_2 \\ \lambda_3' &= \lambda_1\frac{S\beta}{N} - \lambda_2(1-u_1)\frac{S\beta}{N} - \lambda_3 u_1\frac{S\beta}{N} + \lambda_3(\delta+\mu) - \lambda_4\delta \\ \lambda_4' &= \mu\lambda_4 \end{aligned} \tag{27}$$

with the terminal conditions

$$\lambda_i(t_f) = 0, \qquad i = 1, 2, 3, 4. \tag{28}$$

and (u_1^*, u_2^*) are represented by

$$u_1^* = \min\{1, \max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\} \tag{29}$$

$$u_2^* = \min\{1, \max\{0, \frac{1}{c_2}[(\lambda_2-\lambda_3)I + (\lambda_3-\lambda_4)\gamma I]\}\} \tag{30}$$

Proof. According to Pontryagin Maximum Principle, we differentiate the Hamiltonian operator H, with respect to state. The adjoint system can be written as

$$\lambda_1' = -\frac{\partial H}{\partial S}, \qquad \lambda_2' = -\frac{\partial H}{\partial I}, \qquad \lambda_3' = -\frac{\partial H}{\partial P}, \qquad \lambda_4' = -\frac{\partial H}{\partial Q}$$

and the terminal condition of adjoint equations is given by

$$\lambda_i(t_f) = 0, \qquad i = 1, 2, 3, 4.$$

Let

$$\frac{\partial H}{\partial u_1} = c_1 u_1(t) - \lambda_2 S\frac{\alpha I + \beta P}{N} + \lambda_3 S\frac{\alpha I + \beta P}{N} - w_{11} + w_{12} = 0$$
$$\frac{\partial H}{\partial u_2} = c_2 u_2(t) - \lambda_2 I + \lambda_3(1-\gamma)I + \lambda_4\gamma I - w_{21} + w_{22} = 0$$

By solving the optimal control, we obtain

$$u_1^* = \frac{1}{c_1}[(\lambda_2 - \lambda_3)S\frac{\alpha I + \beta P}{N} + w_{11} - w_{12}] \tag{31}$$
$$u_2^* = \frac{1}{c_2}[(\lambda_2 - \lambda_3)I + (\lambda_3 - \lambda_4)\gamma I + w_{21} - w_{22}] \tag{32}$$

In order to express the optimal control without w_{11} and w_{12}, the main aim is to standardise, take $u_1^*(t)$ as an example.

(1) On the set $\{t|0 < u_1^*(t) < 1\}$, let $w_{11}(t) = w_{12}(t) = 0$. So the optimal control is

$$u_1^* = \frac{1}{c_1}[(\lambda_2 - \lambda_3)S\frac{\alpha I + \beta P}{N}]. \tag{33}$$

(2) On the set $\{t|u_1^*(t) = 1\}$, let $w_{11}(t) = 0$. So

$$1 = u_1^*(t) = \frac{1}{c_1}[(\lambda_2 - \lambda_3)S\frac{\alpha I + \beta P}{N} - w_{12}]. \tag{34}$$

This implies that

$$\frac{1}{c_1}[(\lambda_2 - \lambda_3)S\frac{\alpha I + \beta P}{N}] \geq 1 \qquad since\ w_{12} \geq 0 \tag{35}$$

(3) On the set $\{t|u_1^*(t) = 0\}$, let $w_{12}(t) = 0$. So

$$0 = u_1^*(t) = \frac{1}{c_1}[(\lambda_2 - \lambda_3)S\frac{\alpha I + \beta P}{N} + w_{11}]. \tag{36}$$

This implies that

$$\frac{1}{c_1}[(\lambda_2 - \lambda_3)S\frac{\alpha I + \beta P}{N}] \leq 0 \qquad since\ w_{11} \geq 0 \tag{37}$$

To summarize,

$$u_1^*(t) = min\{1, max\{0, \frac{1}{c_1}(\lambda_2 - \lambda_3)S\frac{\alpha I + \beta P}{N}\}\}$$

Using the similar arguments, we can slso obtain the other optimal control function

$$u_2^*(t) = min\{1, max\{0, \frac{1}{c_2}(\lambda_2 - \lambda_3)I + (\lambda_3 - \lambda_4)\gamma I\}\}.$$

Then the proof is completed. ■

We point out that the optimal system consists of the state system with the initial conditions, the adjoint system with the terminal conditions and the optimal condition. Any optimal control pairs must satisfy this optimal system as follow:

$$
\begin{aligned}
S' &= \mu(N-S) - S\frac{\alpha I+\beta P}{N} \\
I' &= (1 - min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\})S\frac{\alpha I+\beta P}{N} \\
&\quad -(min\{1, max\{0, \frac{1}{c_2}(\lambda_2-\lambda_3)I + (\lambda_3-\lambda_4)\gamma I\}\} + \mu)I \\
P' &= min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\}S\frac{\alpha I+\beta P}{N} \\
&\quad +min\{1, max\{0, \frac{1}{c_2}(\lambda_2-\lambda_3)I + (\lambda_3-\lambda_4)\gamma I\}\}(1-\gamma)I - (\delta+\mu)P \\
Q' &= min\{1, max\{0, \frac{1}{c_2}(\lambda_2-\lambda_3)I + (\lambda_3-\lambda_4)\gamma I\}\}\gamma I + \delta P - \mu Q \\
\lambda_1' &= \lambda_1\mu + \lambda_1\frac{\alpha I+\beta P}{N} - \lambda_2(1 - min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\}) \\
&\quad \frac{\alpha I+\beta P}{N} - \lambda_3 min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\}\frac{\alpha I+\beta P}{N} \\
\lambda_2' &= -1 + \lambda_1\frac{S\alpha}{N} - \lambda_2(1 - min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\})\frac{S\alpha}{N} \\
&\quad +\lambda_2(min\{1, max\{0, \frac{1}{c_2}(\lambda_2-\lambda_3)I + (\lambda_3-\lambda_4)\gamma I\}\} + \mu) \\
&\quad -\lambda_3 min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\}\frac{S\alpha}{N} \\
&\quad -\lambda_3(1-\gamma)min\{1, max\{0, \frac{1}{c_2}(\lambda_2-\lambda_3)I + (\lambda_3-\lambda_4)\gamma I\}\} \\
&\quad -\lambda_4\gamma min\{1, max\{0, \frac{1}{c_2}(\lambda_2-\lambda_3)I + (\lambda_3-\lambda_4)\gamma I\}\} \\
\lambda_3' &= \lambda_1\frac{S\beta}{N} - \lambda_2(1 - min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\})\frac{S\beta}{N} \\
&\quad -\lambda_3 min\{1, max\{0, \frac{1}{c_1}(\lambda_2-\lambda_3)S\frac{\alpha I+\beta P}{N}\}\}\frac{S\beta}{N} \\
&\quad +\lambda_3(\delta+\mu) - \lambda_4\delta \qquad (38) \\
\lambda_4' &= \mu\lambda_4 \\
S(0) &= S_0, \qquad I(0) = I_0, \qquad P(0) = P_0, \qquad Q(0) = Q_0. \\
\lambda_i(t_f) &= 0, \qquad i = 1,2,3,4. \qquad (39)
\end{aligned}
$$

5.3. The Uniqueness of Optimal Control

Lemma 7 (see [12]) The function $u^*(s) = min(b, max(s, a))$ is Lipschitz continuous at s, where $a < b$ are some fixed positive constants.

Theorem 8 For all $t \in [0, t_f]$, the solution to the optimal system (39) is unique.
Proof. Suppose $(S, I, P, Q, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\bar{S}, \bar{I}, \bar{P}, \bar{Q}, \bar{\lambda_1}, \bar{\lambda_2}, \bar{\lambda_3}, \bar{\lambda_4})$ are two different solutions of our

optimal system (39). Let

$$\begin{array}{llll} S = e^{\lambda t}m & I = e^{\lambda t}n & P = e^{\lambda t}p & Q = e^{\lambda t}q \\ \lambda_1 = e^{-\lambda t}r & \lambda_2 = e^{-\lambda t}s & \lambda_3 = e^{-\lambda t}w & \lambda_4 = e^{-\lambda t}v \\ \bar{S} = e^{\lambda t}\bar{m} & \bar{I} = e^{\lambda t}\bar{n} & \bar{P} = e^{\lambda t}\bar{p} & \bar{Q} = e^{\lambda t}\bar{q} \\ \bar{\lambda_1} = e^{-\lambda t}\bar{r} & \bar{\lambda_2} = e^{-\lambda t}\bar{s} & \bar{\lambda_3} = e^{-\lambda t}\bar{w} & \bar{\lambda_4} = e^{-\lambda t}\bar{v} \end{array}$$

where $\lambda > 0$ is to be chosen. Accordingly, we have

$$\begin{aligned} u_1^*(t) &= min\{1, max\{0, \frac{m(\alpha n + \beta p)(s - w)e^{\lambda t}}{c_1 N}\}\} \\ u_2^*(t) &= min\{1, max\{0, \frac{n(s - w + w\gamma - v\gamma)}{c_2}\}\} \\ \bar{u_1^*}(t) &= min\{1, max\{0, \frac{\bar{m}(\alpha\bar{n} + \beta\bar{p})(\bar{s} - \bar{w})e^{\lambda t}}{c_1 N}\}\} \\ \bar{u_2^*}(t) &= min\{1, max\{0, \frac{\bar{n}(\bar{s} - \bar{w} + \bar{w}\gamma - \bar{v}\gamma)}{c_2}\}\} \end{aligned}$$

Now we substitute above equations into the first ODE of (39), we can obtain

$$\begin{aligned} m' + \lambda m &= \mu N e^{-\lambda t} - \mu m - \frac{m e^{\lambda t}(\alpha n + \beta p)}{N} \\ \bar{m}' + \lambda \bar{m} &= \mu N e^{-\lambda t} - \mu \bar{m} - \frac{\bar{m} e^{\lambda t}(\alpha\bar{n} + \beta\bar{p})}{N} \end{aligned}$$

Similarly, we can derive the rest of the formula of $n' + \lambda n, p' + \lambda p, q' + \lambda q, r' + \lambda r, s' + \lambda s, w' + \lambda w, v' + \lambda v$. By Lemma 7, we can get

$$\begin{aligned} |u_1^*(t) - \bar{u_1^*}(t)| &\le \frac{m e^{\lambda t}}{c_1 N}|(\alpha n + \beta p)(s - w) - (\alpha\bar{n} + \beta\bar{p})(\bar{s} - \bar{w})| \\ |u_2^*(t) - \bar{u_2^*}(t)| &\le \frac{1}{c_2}|n(s - w + w\gamma - v\gamma) - \bar{n}(\bar{s} - \bar{w} + \bar{w}\gamma - \bar{v}\gamma)| \end{aligned}$$

As the following calculation is similar, we only take m and $\bar{m}$ for an example:

$$m' - \bar{m}' + (\mu + \lambda)(m - \bar{m}) = \frac{e^{\lambda t}}{N}[m(\alpha n + \beta p) - \bar{m}(\alpha\bar{n} + \beta\bar{p})]$$

Multiplying both sides of the above equation by $(m - \bar{m})$ and integrating from 0 to t_f gives

$$\begin{aligned} &\frac{1}{2}(m - \bar{m})^2(t_f) + (\mu + \lambda)\int_0^{t_f}(m - \bar{m})^2 dt \\ &= \frac{e^{\lambda t}}{N}\int_0^{t_f}(m - \bar{m})[m(\alpha n + \beta p) - \bar{m}(\alpha\bar{n} + \beta\bar{p})]dt \\ &= \frac{e^{\lambda t}}{N}\int_0^{t_f}(m - \bar{m})[\alpha(mn - \bar{m}\bar{n}) + \beta(mp - \bar{m}\bar{p})]dt \\ &= \frac{e^{\lambda t}}{N}\int_0^{t_f}(m - \bar{m})[\alpha n(m - \bar{m})^2 + \alpha\bar{m}(m - \bar{m})(n - \bar{n}) + \beta p(m - \bar{m})^2 + \beta\bar{m}(m - \bar{m})(p - \bar{p})]dt \\ &\le \frac{e^{\lambda t}}{N}\int_0^{t_f}\{|\alpha n|(m - \bar{m})^2 + \frac{|\alpha\bar{m}|}{2}[(m - \bar{m})^2 + (n - \bar{n})^2] + |\beta p|(m - \bar{m})^2 + \frac{|\beta\bar{m}|}{2}[(m - \bar{m})^2 + (p - \bar{p})^2]\}dt \end{aligned}$$

Because $|\alpha n|$, $\frac{|\alpha \bar{m}|}{2}$, $|\beta p|$, $\frac{|\beta \bar{m}|}{2}$ are nonnegative and bounded, there exists a positive constant k_1 such that

$$\frac{1}{2}(m-\bar{m})^2(t_f) + (\mu+\lambda)\int_0^{t_f}(m-\bar{m})^2 dt \le k_1 \frac{e^{\lambda t}}{N}\int_0^{t_f}[(m-\bar{m})^2+(n-\bar{n})^2+(p-\bar{p})^2]dt$$

Use the similar way on the other seven equations, then

$$\begin{aligned}&\frac{1}{2}[(m-\bar{m})^2(t_f)+(n-\bar{n})^2(t_f)+(p-\bar{p})^2(t_f)+(q-\bar{q})^2(t_f)+(r-\bar{r})^2(0)+(s-\bar{s})^2(0)+(w-\bar{w})^2(0)+(v-\bar{v})^2(0)]\\&+(\mu+\lambda)\int_0^{t_f}[(m-\bar{m})^2+(n-\bar{n})^2+(p-\bar{p})^2+(q-\bar{q})^2+(r-\bar{r})^2+(s-\bar{s})^2+(w-\bar{w})^2+(v-\bar{v})^2]dt\\&\le k_2\int_0^{t_f}[(m-\bar{m})^2+(n-\bar{n})^2+(p-\bar{p})^2+(q-\bar{q})^2+(r-\bar{r})^2+(s-\bar{s})^2+(w-\bar{w})^2+(v-\bar{v})^2]dt\end{aligned}$$

Thus, we can obtain that

$$(\mu+\lambda-k_2)\int_0^{t_f}[(m-\bar{m})^2+(n-\bar{n})^2+(p-\bar{p})^2+(q-\bar{q})^2+(r-\bar{r})^2+(s-\bar{s})^2+(w-\bar{w})^2+(v-\bar{v})^2]dt \le 0$$

where k_2 depends on the coefficients and the bounds depend on m, n, p, q, r, s, w, v. If we choose λ such that $\mu+\lambda \ge k_2$, $m=\bar{m}, n=\bar{n}, p=\bar{p}, q=\bar{q}, r=\bar{r}, s=\bar{s}, w=\bar{w}$ and $v=\bar{v}$, then the proof is completed. ∎

6. Numerical Simulation

6.1. The Simulation of State System without Control

In this part, numerical simulation will be used to study the stability of the solution of the model without human intervention. We give some simulations using the parameter values in Table 1. Considering that the control measures can not be 100% effective[16], the values of u_1 and u_2 are selected in the range of $[0, 0.8]$[17]. If both u_1 and u_2 are 0, this is in contradiction with the fact that the susceptible people become professional directly through education, and some of the infective people choose to quit. In order to compare with the optimal control, in the case of natural development we choose that the value of u_1 is 0.1 and that of u_2 is 0.1.

Table 1: Estimation of parameters

Parameter	Estimated value	Date source
μ	0.2 year^{-1}	[3]
α	0.2 year^{-1}	Estimate
β	0.3 year^{-1}	Estimate
δ	0.3 year^{-1}	[11]
γ	0.2 year^{-1}	[11]
u_1	Variable	
u_2	Variable	

We consider the population in mainland China. According to [1], the initial population as $N = 829$, unit (million). We assume that the initial population of infective individuals $I(0) = 138$; the initial population of professionals is given as $P(0) = 40$; the initial quitting population is $Q(0) = 71$. Hence, the initial susceptible population is given as $S(0) = N - I(0) - P(0) - Q(0) = 580$. It should be noted that the initial number of professionals we selected include e-sports athletes, game developers, e-sports media organization planning, e-sports hosts, game commentaries and so on.

Numerical results are displayed in the following figures.

First, we choose $\alpha = 0.2, \beta = 0.3$, numerical simulation gives $R_0 = 0.804 < 1$. Then the disease-free equilibrium $E_0 = \{829, 0, 0, 0\}$ is globally asymptotically stable (Figure 2).

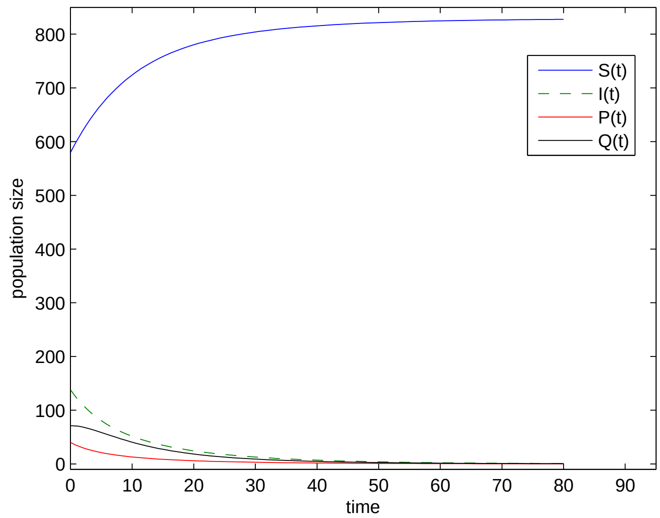

Fig.2. The disease-free equilibrium $E_0 = \{829, 0, 0, 0\}$ is globally asymptotically stable.

Next, we choose $\alpha = 0.3, \beta = 0.35$, numerical simulation gives $R_0 = 1.138 > 1$. Then the disease equilibrium $E^* = \{721, 64, 14, 29\}$ is globally asymptotically stable (Figure 3).

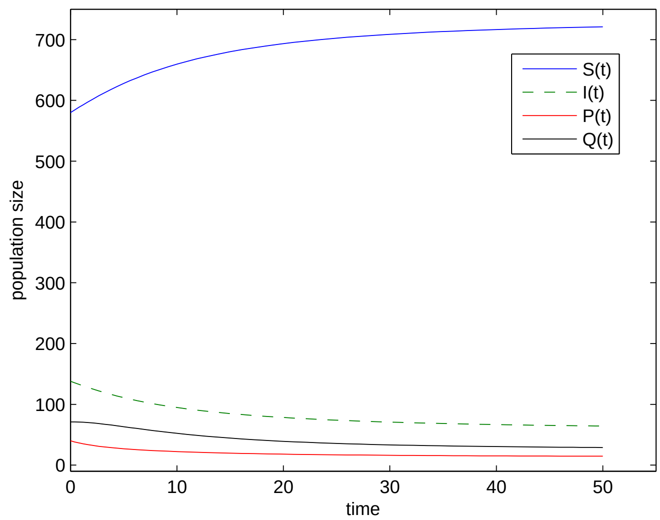

Fig.3. The disease equilibrium $E^* = \{721, 64, 14, 29\}$ is globally asymptotically stable.

6.2. The Sensitivity of R_0 about Two Control Parameters

We simulate the three-dimensional curves of R_0 about u_1 and u_2 with MATLAB, see Fig.4. It is easy to see that R_0 decreases with respect to these two variables from the graph, so some preventive measures and education measures are taken to control the problem of game addiction. This is also the significance of this paper's control strategy.

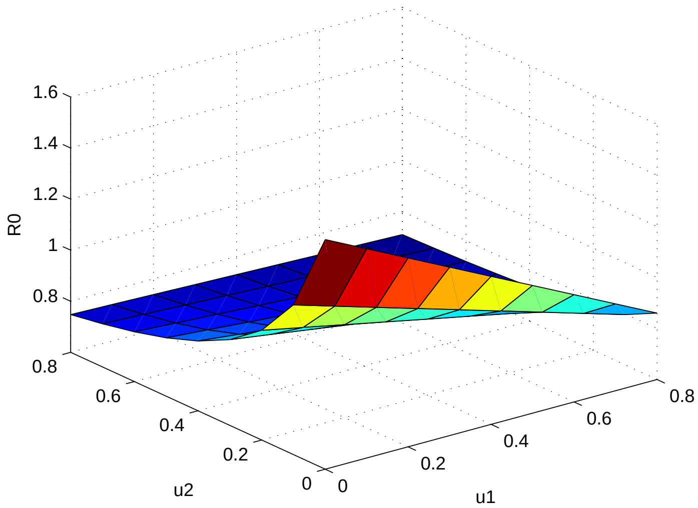

Fig.4. When the relationship between R_0 and two control parameters u_1, u_2.

6.3. The Simulation of Optimal System

In this part, we will use the forward and backward Runge-Kutta method to solve the optimal system in [16, 19]. First, the fourth order Runge-Kutta iteration method is used to solve the state system, and then the state values obtained are to solve the adjoint variables by backward Runge-Kutta method. The obtained state values are used to update the control solution. When the state value in the previous iteration is very close to that in the current iteration, the iteration terminates.

In application, it is difficult to obtain the ideal weight, which requires a lot of data mining, analysis and fitting work, and it still needs further study. The costs associated with I and u_1 are mainly caused by the dangerous behavior of addicted players and the cost of public education. The costs associated with u_2 are mental health treatment, etc. Considering these, after many experiments, the final weight coefficients are $C_1 = 15, C_2 = 30$.

In this simulation, the time level is chosen in weeks up to 52. The figures from simulating the model, given below, help to compare the susceptible individuals, the infective individuals, the professional individuals and the quitting individuals at different control levels, that is, (1)optimal control: $u_1 = u_1^*, u_2 = u_2^*$; (2)$u_1 = u_2 = 0$; (3)$u_1 = 0, u_2 = 0.4$; (4)$u_1 = 0.4, u_2 = 0.4$; (5)$u_1 = 0.4, u_2 = 0.8$. It should be pointed out that we do not consider the maximum control strategy, because maintaining the maximum control strategy from beginning to end will lead to excessive cost consumption.

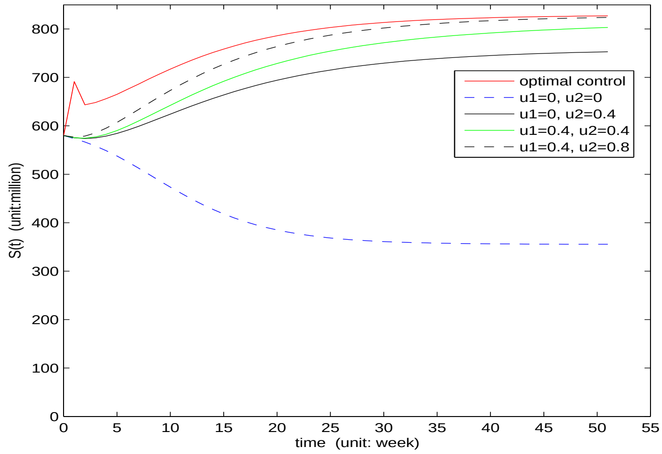

Fig.5. Number of the susceptible compartment under different control strategies.

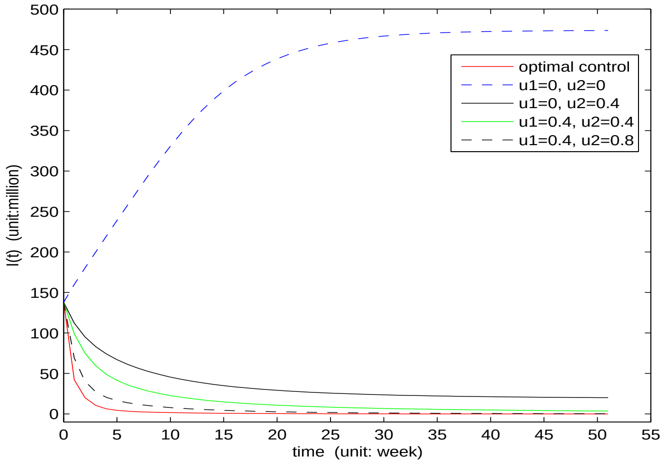

Fig.6. Number of the infective compartment under different control strategies.

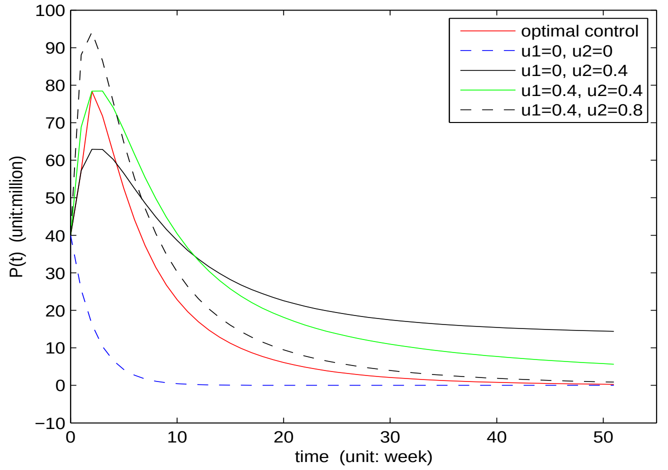

Fig.7. Number of the professional compartment under different control strategies.

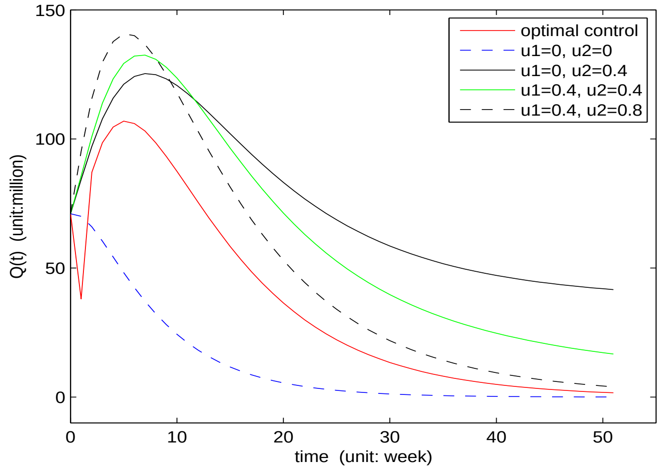

Fig.8. Number of the quitting compartment under different control strategies.

The number of susceptible people at different control levels is given in Fig.5; The number of people who indulge in games at different control levels is given in Fig.6. Fig.7 and Fig.8 show that the number of the professional and quitting individuals at different control levels.

The result that we want is that the fewer the addicts, the better, and the more susceptible people, the better. From the Figure 5 and 6, we can see that the optimal control strategy outperforms other control strategies in terms of the number of susceptible and infective compartment. There is a significant difference between the optimal control case and other control case, as observed in above figures. The difference suggests a very positive impact on reducing the number of infective population if we adopt the optimal control strategy. The variation of the optimal control pairs is shown in Fig.9 and Fig.10.

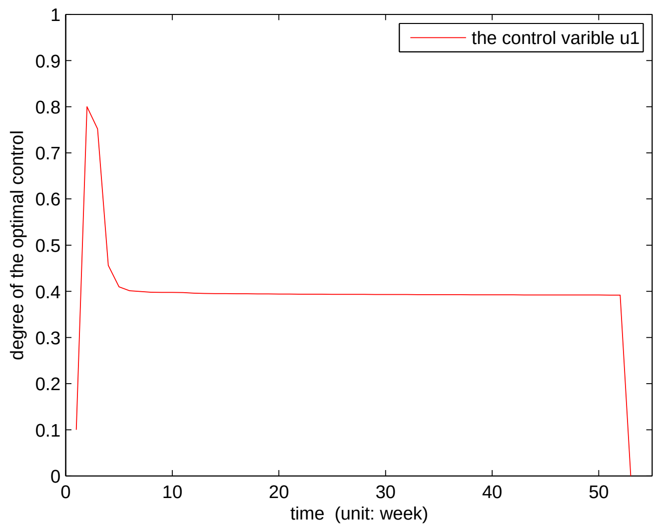

Fig.9. u1 of the optimal control strategies.

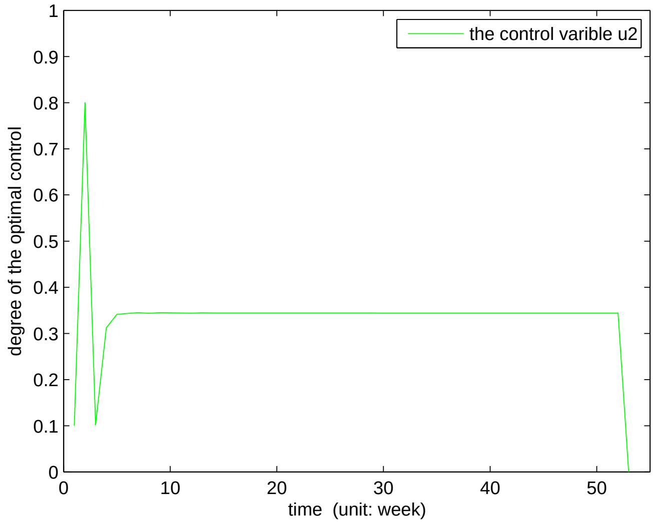

Fig.10. u2 of the optimal control strategies.

Fig.9 describes the time-varying curves of the optimal control for u_1. At the beginning of the simulation, u_1 gradually changed from 0.1 to 0.8 in the first two weeks. This suggests that in the early stages of control efforts should be made to prevent the growth of infective groups and to maximize the proportion of professionals. It can be achieved by increasing media publicity and policy control of game operators by national regulatory department. By the fifth week, it was gradually reduced to 0.4, then keep to the end. It shows that addiction has been mitigated. There is no need to maintain maximum control, only 0.4 control level can be maintained.

In Fig.10, as for the control u_2, it changed from 0.1 to 0.8 in the first two weeks too. It indicates that the intensity of treatment should be maximized at the early stage of control and let the infected population drop rapidly. With the effective treatment measures, only 0.1 of the intensity of u_2 can be controlled in the third week. In order to consolidate the effect of treatment and prevent possible rebound, the control of u_2 should gradually increase to 0.34 before the 5th week and keep to the end.

It should also be pointed out: when replacing different weight coefficients, the results are not sensitive to the weight coefficients. The reason is that the two control measures here are almost equally important, so the weight coefficients basically do not distinguish.

7. Conclusions

In this paper, we constructed an online game addiction model(including susceptible, infective, professional and quitting compartments). We also considered that the direct transfer from the susceptible individuals to the professional individuals. The basic reproduction number of the model was derived through the method of next generation regeneration matrix. The global stability of the two equilibria is given by constructing a suitable Lyapunov function. After completing the mathematical analysis of the model, we used the Pontriagin's maximum principle to solve the optimal control strategy that included two control measures(i.e., prevention and treatment). Using the forward and backward Runge-Kutta method, we solved the optimal control model with control and without control in the numerical simulation. From the results, we can see that in the early stage of indulgence outbreak, the two control forces should be rapidly increased to the maximum. When the situation has been alleviated, it is not necessary to maintain the maximum control intensity, which can be reduced to a certain extent. The optimal control strategy can not only save the cost of treatment, but also effectively inhibit the development of addiction.

Disclosure statement

There are no conflicts of interest by the authors.

Acknowledgements

The authors are grateful to all of the anonymous reviewers for their valuable suggestions.

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