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On Submanifolds of an Almost Contact Metric Manifold Admitting a Quarter-Symmetric Non-Metric Connection

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Abstract. We study submanifolds of an almost contact metric manifold admitting a quarter-symmetric non-metric connection. We prove the induced connection on a submanifold is also quarter-symmetric non-metric connection. We consider the total geodesicness and minimality of a submanifold with respect to the quarter-symmetric non-metric connection. We obtain the Gauss, Cadazzi and Ricci equations for submanifolds with respect to the quarter-symmetric non-metric connection and show some applications of these equations. Finally, we give two examples verifying the results.

1. Introduction

In modern geometry and analysis, the study of the geometry of submanifolds has been an active field over the seven decades, and the geometry of submanifolds has become a subject of growing interest for its significant applications in applied mathematics and theoretical physics. It is well known that Gauss-Codazzi-Ricci equations are very important instruments for describing a submanifold in a Riemannian space. By nature, these equations appear in the Cauchy problem of general relativity [20]. For a submanifold M of a Riemannian manifold $(\overline{M}, g)$, if the Riemannian curvature tensors are denoted by R and $\overline{R}$, respectively, then the usual Gauss, Codazzi and Ricci equations are given by

$$g(\overline{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)), \tag{1}$$

$$(\overline{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z), \tag{2}$$

$$g(\overline{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V) + g([A_V, A_U]X,Y), \tag{3}$$

for all X, Y, Z tangent to M and U, V normal to M, where h is the second fundamental form, A is the shape operator, and $R^{\perp}$ is the curvature tensor of the normal bundle.

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In 1975, Golab [7] introduced a notion of the quarter-symmetric connection on a differential manifold. A linear connection $\overline{\nabla}^*$ on a Riemannian manifold $(\overline{M}, g)$ is called a quarter-symmetric connection if its torsion tensor $\overline{T}^*$ defined by $\overline{\nabla}^*_{\overline{X}}\overline{Y} - \overline{\nabla}^*_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}]$, is of the form

$$\overline{T}^*(\overline{X}, \overline{Y}) = u(\overline{Y})\psi(\overline{X}) - u(\overline{X})\psi(\overline{Y}), \tag{4}$$

where u is a 1-form and ψ is a $(1, 1)$-tensor field.

When $\overline{T}^*$ vanishes, the connection $\overline{\nabla}$ is said to be symmetric. Otherwise, it is non-symmetric. If in (4), $\psi(\overline{X}) = \overline{X}$, then the quarter-symmetric connection reduces to the semi-symmetric connection [6]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. A quarter-symmetric connection $\overline{\nabla}^*$ is said to be a quarter-symmetric metric connection if

$$\overline{\nabla}^* g = 0. \tag{5}$$

Many authors studied the quarter-symmetric metric connection, you can see [3, 4, 8, 17–19] for details. If moreover, a quarter-symmetric connection $\overline{\nabla}^*$ satisfies

$$\overline{\nabla}^* g \neq 0, \tag{6}$$

then $\overline{\nabla}^*$ is said to be a quarter-symmetric non-metric connection.

The quarter-symmetric non-metric connection was further developed by Sengupta and Biswas in [15], Jun, De and Pathak [9], Prakash and Narain [13], Barman [2], Mondal and De [11], Prakash and Pandey [14], Singh and Srivastava [16], and investigated by many other geometers. In [1, 10, 12], the authors obtained several results including the equations of Gauss, Codazzi and Ricci for submanifolds of a Riemannian manifold with a particular type of semi-symmetric non-metric connection. De and Mondal [5] studied hypersurfaces of Kenmotsu manifolds endowed with a quarter-symmetric non-metric connection. In this paper, we generalize the results of submanifolds in following these papers above. We consider submanifolds of any codimension of an almost contact metric manifold admitting a type of quarter-symmetric non-metric connection as introduced by Barman [2].

The present paper is organized as follows:
In section 2, we give a type of quarter-symmetric non-metric connection on an almost contact metric manifold. In section 3, we consider submanifolds of an almost contact metric manifold endowed with the quarter-symmetric non-metric connection and show that the induced connection on the submanifold is also a quarter-symmetric non-metric connection. We also consider the total geodesicness and the minimality of a submanifold of an almost contact metric manifold with respect to the quarter-symmetric non-metric connection; In section 4, we deduce the Gauss, Codazzi and Ricci equations with respect to the quarter-symmetric non-metric connection and obtain some results applying these equations. In section 5, we provide two examples verifying some results.

2. Quarter-symmetric non-metric connection on an almost contact metric manifold

Let $\overline{M}$ be an $(n + p)$-dimensional (where $n + p$ is odd) differential manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields on $\overline{M}$ of type $(1, 1), (1, 0), (0, 1)$, respectively, and g is a compatible metric with the almost structure, such that,

$$\begin{aligned}
&\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \\
&g(\phi\overline{X}, \phi\overline{Y}) = g(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \\
&g(\phi\overline{X}, \overline{Y}) = -g(\phi\overline{Y}, \overline{X}),
\end{aligned} \tag{7}$$

for all vector fields $\overline{X}, \overline{Y}$ on $\overline{M}$.

A linear connection $\overline{\nabla}^*$ on $\overline{M}$ is defined by

$$\overline{\nabla}^*_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} - \eta(\overline{X})\phi\overline{Y} + g(\overline{X},\overline{Y})\xi - \eta(\overline{Y})\overline{X} - \eta(\overline{X})\overline{Y}, \tag{8}$$

where $\overline{\nabla}$ is the Levi-Civita connection associated with g. Then the torsion tensor $\overline{T}^*$ of $\overline{\nabla}^*$ is given by

$$\overline{T}^*(\overline{X},\overline{Y}) = \eta(\overline{Y})\phi\overline{X} - \eta(\overline{X})\phi\overline{Y}. \tag{9}$$

Therefore, the connection $\overline{\nabla}^*$ is a quarter-symmetric connection. Also,

$$(\overline{\nabla}^*_{\overline{X}}g)(\overline{Y},\overline{Z}) = 2\eta(\overline{X})g(\overline{Y},\overline{Z}) \neq 0. \tag{10}$$

Hence the linear connection $\overline{\nabla}^*$ given by (8) is a quarter-symmetric non-metric connection.

Conversely, we show that a linear connection $\overline{\nabla}^*$ defined on $\overline{M}$ satisfying (9) and (10) is given by (8). For any smooth vector field $\overline{X},\overline{Y},\overline{Z}$ on $\overline{M}$, we have

$$\begin{aligned}&\overline{X}g(\overline{Y},\overline{Z}) + \overline{Y}g(\overline{Z},\overline{X}) - \overline{Z}g(\overline{X},\overline{Y})\\ &= 2g(\overline{\nabla}^*_{\overline{X}}\overline{Y},\overline{Z}) - g(\overline{T}^*(\overline{X},\overline{Y}),\overline{Z}) + g(\overline{T}^*(\overline{Y},\overline{Z}),\overline{X}) + g(\overline{T}^*(\overline{X},\overline{Z}),\overline{Y}) - g([\overline{X},\overline{Y}],\overline{Z})\\ &\quad + g([\overline{X},\overline{Z}],\overline{Y}) + g([\overline{Y},\overline{Z}],\overline{X}) + (\overline{\nabla}^*_{\overline{X}}g)(\overline{Y},\overline{Z}) + (\overline{\nabla}^*_{\overline{Y}}g)(\overline{Z},\overline{X}) - (\overline{\nabla}^*_{\overline{Z}}g)(\overline{X},\overline{Y}).\end{aligned}$$

By (9), (10) and Kosul's formula, the above formula becomes

$$\begin{aligned}2g(\overline{\nabla}^*_{\overline{X}}\overline{Y},\overline{Z}) = {}& 2g(\overline{\nabla}_{\overline{X}}\overline{Y},\overline{Z}) + g(\eta(\overline{Y})\phi\overline{X} - \eta(\overline{X})\phi\overline{Y},\overline{Z})\\ &+ g(\eta(\overline{Y})\phi\overline{Z} - \eta(\overline{Z})\phi\overline{Y},\overline{X}) + g(\eta(\overline{X})\phi\overline{Z} - \eta(\overline{Z})\phi\overline{X},\overline{Y})\\ &- 2\eta(\overline{X})g(\overline{Y},\overline{Z}) - 2\eta(\overline{Y})g(\overline{X},\overline{Z}) + 2\eta(\overline{Z})g(\overline{X},\overline{Y}).\end{aligned} \tag{11}$$

From (11), we can obtain

$$\overline{\nabla}^*_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} - \eta(\overline{X})\phi\overline{Y} + g(\overline{X},\overline{Y})\xi - \eta(\overline{Y})\overline{X} - \eta(\overline{X})\overline{Y}.$$

Analogous to the definition of the curvature tensor $\overline{R}$ of $\overline{M}$ with respect to the Levi-Civita connection $\overline{\nabla}$, we define the curvature tensor $\overline{R}^*$ of $\overline{M}$ with respect to the quarter-symmetric non-metric connection $\overline{\nabla}^*$ by

$$\overline{R}^*(\overline{X},\overline{Y})\overline{Z} = \overline{\nabla}^*_{\overline{X}}\overline{\nabla}^*_{\overline{Y}}\overline{Z} - \overline{\nabla}^*_{\overline{Y}}\overline{\nabla}^*_{\overline{X}}\overline{Z} - \overline{\nabla}^*_{[\overline{X},\overline{Y}]}\overline{Z}.$$

The Riemannian Christoffel tensors of the connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are defined by

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = g(\overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W})$$

and

$$\overline{R}^*(\overline{X},\overline{Y},\overline{Z},\overline{W}) = g(\overline{R}^*(\overline{X},\overline{Y})\overline{Z},\overline{W}),$$

respectively.

3. Submanifolds of an almost contact metric manifold admitting the quarter-symmetric non-metric connection

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional almost contact metric manifold $(\overline{M},g)$ admitting the quarter-symmetric non-metric connection $\overline{\nabla}^*$. Let $\overline{\nabla}$ be the Levi-Civita connection associated with g on $\overline{M}$. The usual Gauss and Weingarten formulae for the submanifold M are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X,Y \in TM; \tag{12}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad N \in T^{\perp}M \tag{13}$$

where ∇ is the induced Riemannian connection on M, h is the second fundamental form, A is the shape operator, and $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$, the normal bundle of M. From (12) and (13), one gets

$$g(h(X,Y),N) = g(A_N X, Y). \tag{14}$$

The submanifold M of an almost contact metric manifold $\overline{M}$ is said to be invariant (resp. anti-invariant) if for each point $p \in M$, $\phi T_pM \subset T_pM$(resp. $\phi T_pM \subset T_p^{\perp}M$). Let ∇^* be the induced connection on M from the quarter-symmetric non-metric connection $\overline{\nabla}^*$. Then we have

$$\overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y), \quad X, Y \in TM, \tag{15}$$

where h^* is a (1,2)-tensor field in $T^{\perp}M$. We call h^* the second fundamental form with respect to the quarter-symmetric non-metric connection.

We put $\xi = \xi^{\top} + \xi^{\perp}$ where $\xi^{\top} \in TM$, $\xi^{\perp} \in T^{\perp}M$. For $X \in TM$ and $N \in T^{\perp}M$, we let

$$\phi X = PX + QX, \quad PX \in TM, QX \in T^{\perp}M. \tag{16}$$

$$\phi N = tN + qN, \quad tN \in TM, qN \in T^{\perp}M. \tag{17}$$

Using (8), (12) and (15), we get

$$\nabla^*_X Y + h^*(X,Y) = \nabla_X Y + h(X,Y) - \eta(X)PY - \eta(X)QY + g(X,Y)\xi^{\top} + g(X,Y)\xi^{\perp} - \eta(Y)X - \eta(X)Y.$$

Then we have

$$\nabla^*_X Y = \nabla_X Y - \eta(X)PY + g(X,Y)\xi^{\top} - \eta(Y)X - \eta(X)Y, \tag{18}$$

and

$$h^*(X,Y) = h(X,Y) - \eta(X)QY + g(X,Y)\xi^{\perp}. \tag{19}$$

From (18), the torsion tensor of ∇^* is given by

$$T^*(X,Y) = \eta(Y)PX - \eta(X)PY \tag{20}$$

and

$$(\nabla^*_X g)(Y,Z) = 2\eta(X)g(Y,Z) \neq 0. \tag{21}$$

From (20) and (21) we know that the connection ∇^* is also a quarter-symmetric non-metric connection on M. So we obtain the following result.

Theorem 3.1. *On a submanifold of an almost contact metric manifold admitting the quarter-symmetric non-metric connection, the induced connection is also a quarter-symmetric non-metric connection.*

If a submanifold is anti-invariant, that is, $PX = 0$, then from (20) and (21), we have the following:

Theorem 3.2. *The induced connection on an anti-invariant submanifold of an almost contact metric manifold admitting the quarter-symmetric non-metric connection is a symmetric non-metric connection.*

If $h^*(X,Y) = 0$ for all $X, Y \in TM$, then M is said to be totally geodesic with respect to the quarter-symmetric non-metric connection. For an invariant submanifold tangent to ξ, from (19) we have

$$h^* = h. \tag{22}$$

Furthermore, we have the following:

Proposition 3.3. *Any invariant submanifold tangent to ξ of an almost metric manifold admitting the quarter-symmetric non-metric connection is totally geodesic with respect to the quarter-symmetric non-metric connection if and only if it is totally geodesic with respect to the Levi-Civita connection.*

The equation (15) is called the Gauss formula for the quarter-symmetric non-metric connection. Also from (8), (13) and (17), we have

$$\begin{aligned}\overline{\nabla}^*_X N &= \overline{\nabla}_X N - \eta(X)\phi N - \eta(N)X - \eta(X)N \\ &= -A_N X + \nabla^\perp_X N - \eta(X)tN - \eta(X)qN - \eta(N)X - \eta(X)N \\ &= -A_N X - \eta(X)tN - \eta(N)X + \nabla^\perp_X N - \eta(X)qN - \eta(X)N \\ &= -A^*_N X + \nabla^\perp_X N - \eta(X)qN - \eta(X)N,\end{aligned} \tag{23}$$

where $A^*_N X = A_N X + \eta(X)tN + \eta(N)X$ is called the shape operator corresponding to the quarter-symmetric non-metric connection. The equation (23) may be called the Weingarten formula with respect to the quarter-symmetric non-metric connection.

By simple calculations, we can obtain

$$g(h^*(X,Y),N) = g(A^*_N X, Y). \tag{24}$$

Remark 3.4. *Unlike the second fundamental form corresponding to the Levi-Civita connection, h^* is neither symmetric nor skew-symmetric, in general, which is evident from (19). Thus, the shape operator A^* corresponding to the quarter-symmetric non-metric connection is also not symmetric. However, for invariant submanifolds, both of them are symmetric.*

Let H be the mean curvature vector of the submanifold M. Thus we have

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of TM. We can define the mean curvature vector H^* of M with respect to the quarter-symmetric non-metric connection by

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i).$$

Theorem 3.5. *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ with the quarter-symmetric non-metric connection. Then we have*
(i). If $\xi \in TM$, then $H^* = H$;
(ii). If $\xi \in T^\perp M$, then $H^* = H + \xi$;
(iii). If M is invariant, then $H^* = H + \xi^\perp$.

Proof. Let $\{e_1, \cdots, e_n\}$ be an orthornormal basis of TM.
Case 1: $\xi \in TM$. Let $e_n = \xi$. Then from (19), we have

$$h^*(e_i, e_i) = h(e_i, e_i) - \eta(e_i)Qe_i.$$

Since $\eta(e_i) = g(\xi, e_i) = 0$ for $i = 1, \cdots, n-1$ and $\phi e_n = \phi\xi = 0$, i.e., $Qe_n = 0$, summing up for $i = 1, \cdots, n$ and dividing by n, we obtain

$$H^* = H.$$

Case 2: $\xi \in T^\perp M$. Then from (19), we have

$$h^*(e_i, e_i) = h(e_i, e_i) - \eta(e_i)Qe_i + g(e_i, e_i)\xi.$$

Since $\eta(e_i) = g(\xi, e_i) = 0$ for $i = 1, \cdots, n$, we obtain

$$h^*(e_i, e_i) = h(e_i, e_i) + g(e_i, e_i)\xi.$$

Summing up for $i = 1, \cdots, n$ and dividing by n, we have

$$H^* = H + \xi.$$

Case 3:M is an invariant submanifold, i.e. $QX = O$. Then from (19), we have

$$h^*(e_i, e_i) = h(e_i, e_i) + g(e_i, e_i)\xi^{\perp} = h(e_i, e_i) + g(e_i, e_i)\xi^{\perp}.$$

Summing up for $i = 1, \cdots, n$ and dividing by n, we have

$$H^* = H + \xi^{\perp}.$$

□

If $h^*(X, Y) = g(X, Y)H^*$ for all $X, Y \in TM$, then M is said to be totally umbilical with respect to the quarter-symmetric non-metric connection. If $H^* = 0$, then M is said to be minimal with respect to the quarter-symmetric non-metric connection.

From Theorem 3.4, we have the following:

Corollary 3.6. *Any submanifold tangent to ξ of an almost metric manifold with the quarter-symmetric non-metric connection is minimal with respect to the quarter-symmetric non-metric connection if and only if it is minimal with respect to the Levi-Civita connection.*

Corollary 3.7. *If a submanifold M tangent to ξ of an almost metric manifold with the quarter-symmetric non-metric connection is totally umbilical with respect to both connections, then M is invariant. Conversely, if M is invariant, then M is totally umbilical with respect to the quarter-symmetric non-metric connection if and only if it is totally umbilical with respect to the Levi-Civita connection.*

Proof. From (19), for all $X, Y \in TM$ we have

$$h^*(X, Y) = h(X, Y) - \eta(X)QY,$$

i.e.,

$$\eta(X)QY = h(X, Y) - h^*(X, Y). \tag{25}$$

If M is totally umbilical with respect to both quarter-symmetric non-metric connection and Levi-Civita connection, then from Theorem 3.2, we have

$$h^*(X, Y) = g(X, Y)H^* = g(X, Y)H = h(X, Y). \tag{26}$$

Using (25) and (26), we get

$$\eta(X)QY = 0 \tag{27}$$

for all $X, Y \in TM$. Putting $X = \xi$ in (27), we obtain

$$QY = 0$$

for all $Y \in TM$, which implies that M is an invariant submanifold.

Conversely, if M is invariant, then (19) turns into

$$h^*(X, Y) = h(X, Y) + g(X, Y)\xi^{\perp}. \tag{28}$$

From (28), we can obtain the result. □

4. Gauss, Codazzi and Ricci equations with respect to the quarter-symmetric non-metric connection

We denote the curvature tensor corresponding to the induced connections ∇ and ∇^* by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and

$$R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z,$$

respectively, where $X, Y, Z \in TM$.

From (15) and (23) we have

$$\begin{aligned}\overline{\nabla}^*_X\overline{\nabla}^*_Y Z = {}& \nabla^*_X\nabla^*_Y Z + h^*(X,\nabla^*_Y Z) - A^*_{h^*(Y,Z)}X \\ & + \nabla^\perp_X h^*(Y,Z) - \eta(X)qh^*(Y,Z) - \eta(X)h^*(Y,Z).\end{aligned} \tag{29}$$

$$\begin{aligned}\overline{\nabla}^*_Y\overline{\nabla}^*_X Z = {}& \nabla^*_Y\nabla^*_X Z + h^*(Y,\nabla^*_X Z) - A^*_{h^*(X,Z)}Y \\ & + \nabla^\perp_Y h^*(X,Z) - \eta(Y)qh^*(X,Z) - \eta(Y)h^*(X,Z).\end{aligned} \tag{30}$$

and

$$\overline{\nabla}^*_{[X,Y]}Z = \nabla^*_{[X,Y]}Z + h^*([X,Y],Z) \tag{31}$$

Using (29), (30) and (31), we obtain

$$\begin{aligned}\overline{R}^*(X,Y)Z = {}& R^*(X,Y)Z + h^*(X,\nabla^*_Y Z) - h^*(Y,\nabla^*_X Z) - h^*([X,Y],Z) \\ & - A^*_{h^*(Y,Z)}X + A^*_{h^*(X,Z)}Y + \nabla^\perp_X h^*(Y,Z) - \eta(X)qh^*(Y,Z) - \eta(X)h^*(Y,Z) \\ & - \nabla^\perp_Y h^*(X,Z) + \eta(Y)qh^*(X,Z) + \eta(Y)h^*(X,Z).\end{aligned} \tag{32}$$

Hence, the Gauss equation with respect to the quarter-symmetric non-metric connection is given by

$$\begin{aligned}\overline{R}^*(X,Y,Z,W) &= R^*(X,Y,Z,W) - g(A^*_{h^*(Y,Z)}X,W) + g(A^*_{h^*(X,Z)}Y,W) \\ &= R^*(X,Y,Z,W) - g(h^*(Y,Z),h^*(X,W)) + g(h^*(X,Z),h^*(Y,W)),\end{aligned} \tag{33}$$

where we used the formula (24).

Let $\{N_\alpha\}$, $\alpha = 1,\cdots,p$, be a basis of $T^\perp M$. Then $h^*(X,Y) = \sum_{\alpha=1}^p h^*_\alpha(X,Y)N_\alpha$, where h^*_α is a (0,2) tensor. Hence the Gauss equation (33) can be written as

$$\overline{R}^*(X,Y,Z,W) = R^*(X,Y,Z,W) + \sum_{\alpha=1}^p [h^*_\alpha(Y,W)h^*_\alpha(X,Z) - h^*_\alpha(X,W)h^*_\alpha(Y,Z)]. \tag{34}$$

From (24), we can deduce

$$h^*_\alpha(X,Y) = g(A^*_{N_\alpha}X,Y). \tag{35}$$

Hence by (34) and (35), the Gauss equation can be also represented in terms of the shape operator as

$$\overline{R}^*(X,Y,Z,W) = R^*(X,Y,Z,W) + \sum_{\alpha=1}^p [g(A^*_{N_\alpha}Y,W)g(A^*_{N_\alpha}X,Z) - g(A^*_{N_\alpha}X,W)g(A^*_{N_\alpha}Y,Z)]. \tag{36}$$

From (36), we can get

$$\overline{R}^*(X,Y,X,Y) = R^*(X,Y,X,Y) + \sum_{\alpha=1}^p [g(A^*_{N_\alpha}Y,Y)g(A^*_{N_\alpha}X,X) - g(A^*_{N_\alpha}Y,X)g(A^*_{N_\alpha}X,Y)].$$

Combing with the Remark 3.4 in Section 3, we can state the following:

Theorem 4.1. *Let π be a 2-dimensional invariant subspace of T_pM, $p \in M$. Let $\overline{K}^*(\pi)$ and $K^*(\pi)$ be the sectional curvature of π in $\overline{M}$ and M, respectively, with respect to the quarter-symmetric non-metric connection. Let $\{X, Y\}$ be an orthonormal basis of π. Then*

$$\overline{K}^*(\pi) = K^*(\pi) + \sum_{\alpha=1}^{p}[g(A^*_{N_\alpha}X, X)g(A^*_{N_\alpha}Y, Y) - g(A^*_{N_\alpha}X, Y)^2].$$

As an application of Theorem 4.1, we can obtain the following Synger's inequality with respect to the quarter-symmetric non-metric connection.

Corollary 4.2. *Let M be an invariant submanifold with $\xi \in TM$ of an almost contact metric manifold $\overline{M}$ admitting the quarter-symmetric non-metric connection $\overline{\nabla}^*$ and γ be a geodesic in $\overline{M}$ which lies in M and T be a unit tangent vector field of γ. π is a subspace of the tangent space T_pM spanned by $\{X, T\}$. Then*
(i) $\overline{K}^*(\pi) \geq K^*(\pi)$ along γ.
(ii) if X is a unit tangent vector field on M which is parallel along γ, then the equality of (i) holds if and only if X is parallel along γ in $\overline{M}$.

Proof. (i) Let γ be a geodesic in $\overline{M}$ which lies in M and T be a unit tangent vector field of γ. Then we have

$$h(T, T) = 0. \tag{37}$$

Let π be a subspace of the tangent space T_pM spanned by $\{X, T\}$. Applying (22) and the Theorem 4.1, we obtain

$$\begin{aligned}\overline{K}^*(\pi) &= K^*(\pi) + g(h(X, T), h(X, T)) - g(h(X, X), h(T, T))\\ &= K^*(\pi) + g(h(X, T), h(X, T))\\ &\geq K^*(\pi).\end{aligned} \tag{38}$$

(ii) If X be parallel along γ, we have $\nabla_T X = 0$. Thus we have

$$\overline{\nabla}_T X = h(T, X).$$

Then the equality of (38) holds if and only if $h(X, T) = 0$, i.e. $\overline{\nabla}_T X = 0$. □

Now, if the shape operator A^* is symmetric, by contracting (36) we have the expression of Ricci tensor corresponding to the quarter-symmetric non-metric connection as

$$\begin{aligned}\overline{S}^*(Y, Z) &= S^*(Y, Z) + \sum_{\alpha=1}^{p}\overline{R}^*(N_\alpha, Y, Z, N_\alpha)\\ &\quad + \sum_{\alpha=1}^{p}[\sum_{i=1}^{n} g(A^*_{N_\alpha}Y, e_i)g(A^*_{N_\alpha}e_i, Z) - \sum_{i=1}^{n} h^*_\alpha(e_i, e_i)h^*_\alpha(Y, Z)]\\ &= S^*(Y, Z) + \sum_{\alpha=1}^{p}\overline{R}^*(N_\alpha, Y, Z, N_\alpha) + \sum_{\alpha=1}^{p}[g(A^*_{N_\alpha}Y, A^*_{N_\alpha}Z) - f_\alpha h^*_\alpha(Y, Z)\\ &= S^*(Y, Z) + \sum_{\alpha=1}^{p}\overline{R}^*(N_\alpha, Y, Z, N_\alpha) + \sum_{\alpha=1}^{p}[g(A^*_{N_\alpha}A^*_{N_\alpha}Y, Z) - f_\alpha h^*_\alpha(Y, Z)]\\ &= S^*(Y, Z) + \sum_{\alpha=1}^{p}\overline{R}^*(N_\alpha, Y, Z, N_\alpha) + \sum_{\alpha=1}^{p}[h^*_\alpha(A^*_{N_\alpha}Y, Z) - f_\alpha h^*_\alpha(Y, Z)],\end{aligned} \tag{39}$$

where f_α denotes the trace of $A^*_{N_\alpha}$.

Suppose that the quarter-symmetric non-metric connection $\overline{\nabla}^*$ is of constant sectional curvature. Then

$$\overline{R}^*(X, Y, Z, W) = \lambda[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \tag{40}$$

Therefore, from (39) and (40), we have

$$\begin{aligned}(n-1)\lambda g(Y, Z) = S^*(Y, Z) &+ \sum_{\alpha=1}^{p} \lambda[g(Y, Z)g(N_\alpha, N_\alpha) - g(N_\alpha, Z)g(N_\alpha, Y)] \\ &+ \sum_{\alpha=1}^{p}[h^*_\alpha(A^*_{N_\alpha} Y, Z) - f_\alpha h^*_\alpha(Y, Z)].\end{aligned}$$

It follows that

$$S^*(Y, Z) = (n-1-p)\lambda g(Y, Z) + \sum_{\alpha=1}^{p}[f_\alpha h^*_\alpha(Y, Z) - h^*_\alpha(A^*_{N_\alpha} Y, Z)]. \tag{41}$$

From (41) we have the following two theorems:

Theorem 4.3. *Let M be an invariant submanifold of an almost contact metric manifold $\overline{M}$ of constant sectional curvature with the quarter-symmetric non-metric connection. Then the Ricci tensor of M with respect to the quarter-symmetric non-metric connection is symmetric.*

Theorem 4.4. *Let M be a totally umbilical and invariant submanifold of an almost contact metric manifold $\overline{M}$ of constant sectional curvature admitting the quarter-symmetric non-metric connection. Then the submanifold M is Einstein manifold with respect to the quarter-symmetric non-metric connection.*

From (32), the normal component of $\overline{R}^*(X, Y)Z$ is given by

$$\begin{aligned}(\overline{R}^*(X, Y)Z)^\perp &= h^*(X, \nabla^*_Y Z) - h^*(Y, \nabla^*_X Z) - h^*([X, Y], Z) \\ &\quad + \nabla^\perp_X h^*(Y, Z) - \eta(X)qh^*(Y, Z) - \eta(X)h^*(Y, Z) \\ &\quad - \nabla^\perp_Y h^*(X, Z) + \eta(Y)qh^*(X, Z) + \eta(Y)h^*(X, Z). \\ &= (\widetilde{\nabla}^*_X h^*)(Y, Z) - (\widetilde{\nabla}^*_Y h^*)(X, Z) \\ &\quad - \eta(X)[h^*(Y, Z) + h^*(PY, Z) + qh^*(Y, Z)] \\ &\quad + \eta(Y)[h^*(X, Z) + h^*(PX, Z) + qh^*(X, Z)],\end{aligned} \tag{42}$$

where $(\widetilde{\nabla}^*_X h^*)(Y, Z) = \nabla^\perp_X h^*(Y, Z) - h^*(\nabla^*_X Y, Z) - h^*(Y, \nabla^*_X Z)$. The equation (42) is called the Codazzi equation with respect to the quarter-symmetric non-metric connection.

Remark 4.5. *$\widetilde{\nabla}^*$ is the connection in $TM \oplus T^\perp M$ built with ∇^* and $\nabla^\perp$. It can be called the van der Waerden-Bortolotti connection with respect to the quarter-symmetric non-metric connection.*

Let $\xi_1, \xi_2 \in T^\perp M$. From (15) and (23), we get

$$\begin{aligned}\overline{\nabla}^*_X \overline{\nabla}^*_Y \xi_1 &= -\nabla^*_X A^*_{\xi_1} Y - h^*(X, A^*_{\xi_1} Y) - A^*_{\nabla^\perp_Y \xi_1} X + \nabla^\perp_X \nabla^\perp_Y \xi_1 \\ &\quad - \eta(X)(\nabla^\perp_Y \xi_1 + q\nabla^\perp_Y \xi_1) - X(\eta(Y))(\xi_1 + q\xi_1) \\ &\quad - \eta(Y)[\overline{\nabla}_X(\xi_1 + q\xi_1) - \eta(X)\phi(\xi_1 + q\xi_1) \\ &\quad - \eta(\xi_1 + q\xi_1)X - \eta(X)(\xi_1 + q\xi_1)]\end{aligned} \tag{43}$$

$$
\begin{aligned}
\overline{\nabla}^*_Y \overline{\nabla}^*_X \xi_1 = & -\nabla^*_Y A^*_{\xi_1} X - h^*(Y, A^*_{\xi_1} X) - A^*_{\nabla^\perp_X \xi_1} Y + \nabla^\perp_Y \nabla^\perp_X \xi_1 \\
& - \eta(Y)(\nabla^\perp_X \xi_1 + q\nabla^\perp_X \xi_1) - Y(\eta(X))(\xi_1 + q\xi_1) \\
& - \eta(X)[\overline{\nabla}_Y(\xi_1 + q\xi_1) - \eta(Y)\phi(\xi_1 + q\xi_1) \\
& - \eta(\xi_1 + q\xi_1)Y - \eta(Y)(\xi_1 + q\xi_1)]
\end{aligned} \tag{44}
$$

$$
\overline{\nabla}^*_{[X,Y]} \xi_1 = -A^*_{\xi_1}[X, Y] + \nabla^\perp_{[X,Y]} \xi_1 - \eta([X, Y])(\xi_1 + q\xi_1). \tag{45}
$$

Using (43), (44) and (45), we can obtain

$$
\begin{aligned}
\overline{R}^*(X, Y, \xi_1, \xi_2) = & R^\perp(X, Y, \xi_1, \xi_2) + g(A^*_{\xi_1} X, A^*_{\xi_2} Y) - g(A^*_{\xi_1} Y, A^*_{\xi_2} X) \\
& - \eta(X)g(q\nabla^\perp_Y \xi_1 - \nabla^\perp_Y q\xi_1, \xi_2) - g(Y, \nabla_X \xi^\top)g(\xi_1 + q\xi_1, \xi_2) \\
& + \eta(Y)g(q\nabla^\perp_X \xi_1 - \nabla^\perp_X q\xi_1, \xi_2) - g(X, \nabla_Y \xi^\top)g(\xi_1 + q\xi_1, \xi_2).
\end{aligned} \tag{46}
$$

The equation (46) is called the Ricci equation corresponding to the quarter-symmetric non-metric connection.

Remark 4.6. *If M is an invariant submanifold, then the shape operator is symmetric. Thus we can express the Ricci equation in the following form:*

$$
\begin{aligned}
\overline{R}^*(X, Y, \xi_1, \xi_2) = & R^\perp(X, Y, \xi_1, \xi_2) + g([A^*_{\xi_2}, A^*_{\xi_1}]X, Y) \\
& - \eta(X)g(q\nabla^\perp_Y \xi_1 - \nabla^\perp_Y q\xi_1, \xi_2) - g(Y, \nabla_X \xi^\top)g(\xi_1 + q\xi_1, \xi_2) \\
& + \eta(Y)g(q\nabla^\perp_X \xi_1 - \nabla^\perp_X q\xi_1, \xi_2) - g(X, \nabla_Y \xi^\top)g(\xi_1 + q\xi_1, \xi_2).
\end{aligned}
$$

5. Examples

Example 5.1. *Consider 5-Euclidean space $\mathbb{R}^5$ with the cartesian coordinates (x_1, x_2, y_1, y_2, t) and the almost contact structure*

$$
\phi(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i}, \quad \phi(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial x_j}, \quad \phi(\frac{\partial}{\partial t}) = 0, \quad 1 \leq i, j \leq 2.
$$

It is easy to show that (ϕ, ξ, η, g) is an almost metric structure on $\mathbb{R}^5$ with $\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and g, the Euclidean metric of $\mathbb{R}^5$. Let M be a submanifold of $\mathbb{R}^5$ defined by the immersion f as follows:

$$
f(u, v, t) = (u + v, 0, u - v, 0, u + v + t).
$$

Then the tangent bundle TM of M is spanned by the following unit vector fields

$$
e_1 = \frac{1}{\sqrt{3}}(1, 0, 1, 0, 1), \quad e_2 = \frac{1}{\sqrt{3}}(1, 0, -1, 0, 1), \quad e_3 = (0, 0, 0, 0, 1) \tag{47}
$$

and the normal bundle $T^\perp M$ is spanned by the following unit vector fields

$$
e_4 = (0, 1, 0, 0, 0), \quad e_5 = (0, 0, 0, 1, 0).
$$

Clearly,

$$
\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.
$$

So M is an invariant submanifold of $\mathbb{R}^5$ with $\xi \in TM$.

Differentiating (47), we get

$$
\begin{aligned}
& \overline{\nabla}_{e_1} e_1 = 0, \overline{\nabla}_{e_1} e_2 = 0, \overline{\nabla}_{e_1} e_3 = 0, \\
& \overline{\nabla}_{e_2} e_1 = 0, \overline{\nabla}_{e_2} e_2 = 0, \overline{\nabla}_{e_2} e_3 = 0, \\
& \overline{\nabla}_{e_3} e_1 = 0, \overline{\nabla}_{e_3} e_2 = 0, \overline{\nabla}_{e_3} e_3 = 0.
\end{aligned} \tag{48}
$$

So by Gauss formula (13), we have

$$\nabla_{e_i} e_j = 0, \quad i, j = 1, 2, 3, \tag{49}$$

and

$$h(e_i, e_j) = 0, \quad i, j = 1, 2, 3. \tag{50}$$

By (8), the quarter-symmetric non-metric connection $\overline{\nabla}^*$ *on* $\mathbb{R}^5$ *is given by*

$$\begin{aligned} &\overline{\nabla}^*_{e_1} e_1 = e_3, \overline{\nabla}^*_{e_1} e_2 = 0, \overline{\nabla}^*_{e_1} e_3 = -e_1, \\ &\overline{\nabla}^*_{e_2} e_1 = 0, \overline{\nabla}^*_{e_2} e_2 = e_3, \overline{\nabla}^*_{e_2} e_3 = -e_2, \\ &\overline{\nabla}^*_{e_3} e_1 = -e_1 - e_2, \overline{\nabla}^*_{e_3} e_2 = e_1 - e_2, \overline{\nabla}^*_{e_3} e_3 = -e_3. \end{aligned} \tag{51}$$

So by Gauss formula (15) with respect to the quarter-symmetric connection, we have

$$\begin{aligned} &\nabla^*_{e_1} e_1 = e_3, \nabla^*_{e_1} e_2 = 0, \nabla^*_{e_1} e_3 = -e_1, \\ &\nabla^*_{e_2} e_1 = 0, \nabla^*_{e_2} e_2 = e_3, \nabla^*_{e_2} e_3 = -e_2, \\ &\nabla^*_{e_3} e_1 = -e_1 - e_2, \nabla^*_{e_3} e_2 = e_1 - e_2, \nabla^*_{e_3} e_3 = -e_3. \end{aligned} \tag{52}$$

Then

$$h^*(e_i, e_j) = 0, \quad i, j = 1, 2, 3. \tag{53}$$

From (52), we can obtain

$$T^*(e_1, e_2) = 0, T^*(e_1, e_3) = e_2, T^*(e_2, e_3) = -e_1, T^*(e_i, e_i) = 0, i = 1, 2, 3.$$

and

$$\begin{aligned} &(\nabla^*_{e_1} g)(e_i, e_j) = 0, \quad (\nabla^*_{e_2} g)(e_i, e_j) = 0 \quad (i, j = 1, 2, 3), \\ &(\nabla^*_{e_3} g)(e_i, e_i) = 0 \quad (i = 1, 2, 3), \quad (\nabla^*_{e_3} g)(e_i, e_j) = 0 \quad (i \neq j = 1, 2, 3). \end{aligned}$$

So ∇^* *is a quarter-symmetric non-metric connection on* M.

From (50) and (53), we know that M *is totally geodesic with respect to both Levi-Civita connection and quarter-symmetric non-metric connection. Thus this result verifies the Proposition 3.3.*

Example 5.2. *Consider a 3-dimensional manifold* $\overline{M} = \{(x, y, z) \in \mathbb{R}^3) : x \neq 0\}$, *where* (x, y, z) *are the standard coordinates in* $\mathbb{R}^3$. *Let* $\{e_1, e_2, e_3\}$ *be a linearly independent global field field on* $\overline{M}$ *given by*

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = x\frac{\partial}{\partial y}, \quad e_3 = x(y\frac{\partial}{\partial x} + \frac{\partial}{\partial z}).$$

Let g *be the Riemannian metric defined by*

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

The (ϕ, ξ, η) *is given by*

$$\phi e_1 = 0, \phi e_2 = -e_3, \phi e_3 = e_2, \eta = dx - ydy, \xi = e_1 = \frac{\partial}{\partial x}.$$

It is easy to show that (ϕ, ξ, η, g) *is an almost metric metric structure on* $\overline{M}$. *By the definition of Lie bracket, we have*

$$[e_1, e_2] = \frac{1}{x}e_2, \quad [e_2, e_3] = x^2 e_1 - ye_2, \quad [e_3, e_1] = -\frac{1}{x}e_3.$$

Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the above metric g. Then using the Koszul's formula, we have

$$\begin{aligned}&\overline{\nabla}_{e_1}e_1 = 0, \overline{\nabla}_{e_1}e_2 = -\frac{x^2}{2}e_3, \overline{\nabla}_{e_1}e_3 = \frac{x^2}{2}e_2,\\&\overline{\nabla}_{e_2}e_1 = -\frac{x^2}{2}e_3 - \frac{1}{x}e_2, \overline{\nabla}_{e_2}e_2 = \frac{1}{x}e_1 + ye_3, \overline{\nabla}_{e_2}e_3 = \frac{x^2}{2}e_1 - ye_2,\\&\overline{\nabla}_{e_3}e_1 = \frac{x^2}{2}e_2 + \frac{1}{x}e_3, \overline{\nabla}_{e_3}e_2 = -\frac{x^2}{2}e_1, \overline{\nabla}_{e_3}e_3 = \frac{1}{x}e_1.\end{aligned} \tag{54}$$

By (8), the quarter-symmetric non-metric connection $\overline{\nabla}^*$ on $\overline{M}$ is given by

$$\begin{aligned}&\overline{\nabla}^*_{e_1}e_1 = -e_1, \overline{\nabla}^*_{e_1}e_2 = -e_2 + (1-\frac{x^2}{2})e_3, \overline{\nabla}^*_{e_1}e_3 = (\frac{x^2}{2}-1)e_2 - e_3,\\&\overline{\nabla}^*_{e_2}e_1 = -(\frac{1}{x}+1)e_2 - \frac{x^2}{2}e_3, \overline{\nabla}^*_{e_2}e_2 = (\frac{1}{x}+1)e_1 + ye_3, \overline{\nabla}^*_{e_2}e_3 = \frac{x^2}{2}e_1 - ye_2,\\&\overline{\nabla}^*_{e_3}e_1 = \frac{x^2}{2}e_2 + (\frac{1}{x}-1)e_3, \overline{\nabla}^*_{e_3}e_2 = -\frac{x^2}{2}e_1, \overline{\nabla}^*_{e_3}e_3 = (\frac{1}{x}+1)e_1.\end{aligned} \tag{55}$$

Now, let M be a subset of $\overline{M}$ and consider an isometric immersion $f: M \to \overline{M}$ by

$$f(x, y, z) = (x, y, 0).$$

It can be easily seen that M is a 2-dimensional anti-invariant submanifold of the 3-dimensional almost contact metric manifold $\overline{M}$. The tangent bundle TM of M is spanned by $\{e_1, e_2\}$ and e_3 is a normal vector of M.

From (55) we have

$$\nabla^*_{e_1}e_1 = -e_1, \nabla^*_{e_1}e_2 = -e_2, \nabla^*_{e_2}e_1 = -(\frac{1}{x}+1)e_2, \nabla^*_{e_2}e_2 = (\frac{1}{x}+1)e_1, \tag{56}$$

where ∇^* is the induced connection on M by $\overline{\nabla}^*$.

From (56), we can obtain

$$T^*(e_i, e_j) = 0, \quad i, j = 1, 2.$$

and

$$\begin{aligned}&(\nabla^*_{e_1}g)(e_1, e_1) = 2, \quad (\nabla^*_{e_1}g)(e_i, e_j) = 0 \quad i \neq j = 1, 2,\\&(\nabla^*_{e_1}g)(e_2, e_2) = 2, \quad (\nabla^*_{e_2}g)(e_i, e_j) = 0, \quad i, j = 1, 2.\end{aligned}$$

So ∇^* is a symmetric non-metric connection on M. This result verifies Theorem 3.2.

From (54) and (55), we have

$$h(e_1, e_1) = 0, h(e_1, e_2) = h(e_2, e_1) = -\frac{x^2}{2}e_3, h(e_2, e_2) = ye_3.$$

$$h^*(e_1, e_1) = 0, h^*(e_1, e_2) = (1-\frac{x^2}{2})e_3, h(e_2, e_1) = -\frac{x^2}{2}e_3, h(e_2, e_2) = ye_3.$$

Thus the mean curvature vectors of M with respect to the Levi-Civita connection and the semi-symmetric non-metric connection are

$$H = \frac{1}{2}ye_3 = H^*.$$

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