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# Modeling and Dynamic Behavior of a Discontinuous Tourism-Based Social-Ecological Dynamical System

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**Abstract.** In this paper we study a four-dimensional tourism-based social-ecological system including two different types of tourists, i.e., eco-tourists and mass-tourists. First we develop the mathematical model of this system, in order to make it more realistic. In fact, the negative effects of eco-tourists on the environment can not be always ignored. Therefore, we consider a discontinuous function for describing the damage induced by tourist activities to ecosystem quality. By introducing a discontinuous model in this way, we can provide a tool for better investigating the behavior of tourism phenomenon. We analyse the dynamic of the obtained discontinuous system, by using the theory of discontinuous dynamical systems. Moreover, in order to discuss about profitability, compatibility and sustainability of the obtained system, stability regions for that will be found. Furthermore by verifying tangent points of the discontinuous system, more dynamic features of eco and mas-tourists will be shown. In this regard, we determine some regions for existence of the collision of two tangent points for the system. Some numerical simulations are carried out to demonstrate our theoretical results. Here our numerical and theoretical achievements can provide useful information for analysing the tourism industry.

## 1. Introduction

Tourism industry is one of the most important sources of economic growth for all touristic countries. Moreover it can promote social and cultural developments, and also distribute nations' history, civilization, and traditions. Therefore, recently many countries rely on it as one of the fastest growing economic development. In addition dynamical analysis of tourism phenomenon, based on investigating mathematical models of that, has been one of the most important and interesting topics of study and research in recent decades. The first minimal descriptive model for tourism phenomenon, considering interaction with ecosystems explicitly, was established by Casagrandi and Rinaldi in 2002 as follows; for more details see [4].

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$$\begin{cases} \frac{dT}{dt} = T\Big(\frac{\mu_E E}{E+\varphi_E} + \mu_C \frac{\frac{C}{(T+1)}}{\frac{C}{(T+1)}+\varphi_C} - \alpha T - a\Big) \\ \frac{dE}{dt} = rE\Big(1-\frac{E}{k}\Big) - E\Big(\beta C + \gamma T) \\ \frac{dC}{dt} = -\delta C + \epsilon T \end{cases} . \tag{1}$$

Their model only had three state variables, i.e., $T(t)$ presenting the tourists in the area at time $t$, the natural environment $E(t)$ and the capital $C(t)$, as tourist convenience. Although the model (1) provided theoretical results which were in good agreement with conventional wisdom and observations, it needed to be modified. Since the components of their suggested model were not sufficient enough to describe all the social, cultural and political features of tourism system. In fact, there are different tourist typologies due to explanatory variables applied and reflect differences among situations, purposes and interests of the study, [6]. Thus different kinds of tourists should be considered in (1), in order to make the model more realistic. Therefore Lacitignola et al. in 2007, to preserve the low-dimensionality of the system, took into account just two main tourist typologies, i.e., mass and eco-tourists; see [9]. Then by defining a degradation coefficient of tourist site as a bifurcation parameter, they could develop the tourism model

$$\begin{cases} \frac{dT_1}{dt} = T_1\Big(\frac{\mu_{1E}E^2}{E^2+\varphi_{1E}^2} - \beta_1 \frac{C}{T_1+T_2+1} - \alpha_1 T_1 - \alpha_1^* T2 - \gamma_1^* T_1 T_2 - \omega\Big) \\ \frac{dT_2}{dt} = T_2\Big(\frac{\mu_{2E}E}{E+\varphi_{2E}} + \mu_{2C} \frac{\frac{C}{T_1+T_2+1}}{\frac{C}{T_1+T_2+1}+\varphi_{2C}} - \alpha_2 T_2 - \alpha_2^* T1 - \gamma_2^* T_1 T_2 - \omega\Big) \\ \frac{dE}{dt} = \frac{rE^2}{k} - rE - E(\beta C + \gamma_2 T_2) + \gamma_1 T_1 E \\ \frac{dC}{dt} = -\delta C + \epsilon_1 T_1 + \epsilon_2 T_2 \end{cases} , \tag{2}$$

where $t, T_1(t), T_2(t), E(t), C(t)$ and all parameters are non-negative. In addition, eco-tourists $T_1$ and mass-tourists $T_2$ are two state variables of the system (2) with environment $E$ and capital $C$. Details of the construction of the model (2) can be found in [9, 10].

On the other hand, in the field of resource management, studying the important concept sustainability has become a crucial scientific issue. Sustainability refers to the possibility of keeping alive forever all meaning full social and natural compartments of an evolving system, [4]. For a tourism system, sustainability means a chance of maintaining the tourism industry indefinitely without putting the environment at risk. Moreover, profitability and compatibility are two components of sustainability which take into account the social and environmental aspects of the system respectively. In [4, 5] three significant features profitability, compatibility and sustainability are based on asymptotic properties and interpreted in terms of structural properties of the attractors of system; for more information see [4, 5]. In this regard, only those attractors of the system which are positive with respect to the social or to the environmental variables are involved, [5]. Furthermore, sustainability analysis of the system (2) and similar systems are investigated by some researchers, for instance see [1–3, 14].

## 2. Preliminaries

In this section first we shortly describe some definitions of discontinuous dynamical systems which are needed for next sections. Then, we will define profitability, compatibility and sustainability concepts for system (2), in terms of structural properties of its attractors.

### 2.1. discontinuous dynamical systems

Let us consider the system

$$\dot{x} = f(t,x) = \begin{cases} f^-(t,x); & x \in S^- \ , \ t \in J \\ f^+(t,x); & x \in S^+ \ , \ t \in J \end{cases} \tag{3}$$

$$x(t_0) = x_0,$$

such that $f : U \longrightarrow \mathbb{R}^n$, and $U = J \times U' \subseteq \mathbb{R} \times \mathbb{R}^n$ is a domain. Moreover, suppose that

(1) $U'$ is divided into two open and disjoint sets $S^-$ and $S^+$ by a hyper surface $\Sigma$. The discontinuity boundary $\Sigma$ and sets $S^+$ and $S^-$ can be defined by a scalar function $h : U' \longrightarrow \mathbb{R}, h \in C^r(U', \mathbb{R}), r \geq 1$ as

$$S^- = \{x \in U' \mid h(x) < 0\}, \tag{4}$$

$$S^+ = \{x \in U' \mid h(x) > 0\}, \tag{5}$$

and

$$\Sigma = \{x \in U' \mid h(x) = 0\}. \tag{6}$$

(2) The normal of the hyper surface $\Sigma$, given by $n(x) = \left[\nabla h(x)\right]^T$ is chosen such that it is always nonzero, i.e., $n(x) \neq 0$ for each $x \in \Sigma$.

(3) The functions $f^{\mp} : J \times S^{\mp} \longrightarrow \mathbb{R}^n$ are $C^r, r \geq 1$.

The system described by (3) does not define $f(t, x(t))$ for all $x(t)$ on the discontinuity boundary $\Sigma$. For this purpose, we define the set-valued extension of system (3) for each $t \in J$ as follows

$$x' \in F(t,x) = \begin{cases} f^-(t,x); & x \in S^-, \\ \bar{co}\{f^-(t,x), f^+(t,x)\} = \{(1-\lambda) f^-(t,x) + \lambda f^+(t,x), \forall \lambda \in [0,1]\}; & x \in \Sigma, \\ f^+(t,x); & x \in S^+. \end{cases} \tag{7}$$

where $0 \leq \lambda \leq 1$ is a parameter which defines the convex combination and has no physical meaning. Moreover, the extension of a discontinuous system (3) into a convex differential inclusion (7) is known as Filippov's convex method.

**Definition 2.1** (Fillippov solution). *Function $x : I \longrightarrow \mathbb{R}^n$, at which $I \subseteq \mathbb{R}$ is an interval, is called a solution of differential inclusion* (7) *if $x$ is almost everywhere continuous and $x'(t) \in F(t, x(t))$, for almost all $t \in I$,*

The assumptions (1) – (3) assure that differential equations (7) and (3) have a solution in the sense of Filippov solution; for more details see [8, 13].

**Definition 2.2** (Transversality condition). *Suppose that $x : I \longrightarrow \mathbb{R}^n$ is a solution of* (3) *which reaches to the discontinuity boundary $\Sigma$ at the point $x_\Sigma \in \Sigma$ and time $t_\Sigma \in I$, i.e., $x(t_\Sigma) = x_\Sigma$. We say that solution $x(t)$ crosses the hyper surface $\Sigma$ transversally at $x_\Sigma$ if*

$$\sigma(x_\Sigma) = \left[n^T(x_\Sigma)\, f^-(t_\Sigma, x_\Sigma)\right] . \left[n^T(x_\Sigma)\, f^+(t_\Sigma, x_\Sigma)\right] > 0, \tag{8}$$

*where $n(x) = \left[Dh(x)\right]^T$, and $n^T(x_\Sigma)\, f^-(t_\Sigma\ x_\Sigma)$ and $n^T(x_\Sigma)\, f^+(t_\Sigma\ x_\Sigma)$ are the projections of $f^-(t_\Sigma\ x_\Sigma)$ and $f^+(t_\Sigma\ x_\Sigma)$ onto the normal to the hyper surface $\Sigma$.*

*In addition, when $n^T(x_\Sigma)\, f^-(t_\Sigma\ x_\Sigma) < 0$ we will enter $S^-$, and when $n^T(x_\Sigma)\, f^+(t_\Sigma\ x_\Sigma) > 0$ we will enter $S^+$; see [8, 11, 13]. Any solution of* (7) *with initial condition not in $\Sigma$, which reaches to $\Sigma$ at the time $t_\Sigma \in I$ and has a transversal intersection, there exists and is unique.*

**Definition 2.3** (Direct crossing set). *The direct crossing set is*

$$\Sigma_c = \left\{x_\Sigma \in \Sigma;\ \sigma(x_\Sigma) > 0\right\}, \tag{9}$$

*where $\sigma(x_\Sigma)$ is defined as in relation* (8). *In fact direct crossing implies that all trajectories of* (3) *approaching the hyperspace $\Sigma$ cross it immediately, see [7, 13].*

**Definition 2.4** (Sliding mode set)**.** *The sliding mode set is the complement of $\Sigma_c$ in $\Sigma$, i.e.,*

$$\Sigma_s = \Big\{x_\Sigma \in \Sigma;\ \sigma(x_\Sigma) \leq 0\Big\}, \tag{10}$$

*where the vector fields are both pointing towards or away from $\Sigma$.*

**Definition 2.5** (Tangent point)**.** *A point $T \in \Sigma_s$ is a tangent point of* (3)*, if both vectors $f^{\mp}(t,T)$ are nonzero, but one of them is tangent to $\Sigma$, i.e.,*

$$\langle \nabla h(T), f^-(t,T)\rangle = 0, \ \text{or} \ \langle \nabla h(T), f^+(t,T)\rangle = 0. \tag{11}$$

*Note that if for $T \in \Sigma_s$ both nonzero vectors $f^{\mp}(t,T)$ are tangent to $\Sigma$, i.e.,*

$$\langle \nabla h(T), f^-(t,T)\rangle = 0, \ \text{and} \ \langle \nabla h(T), f^+(t,T)\rangle = 0, \tag{12}$$

*then the collision of two tangent points (of vector fields $f^{\mp}(t,T)$) will occur.*

**Definition 2.6** (Generalised Jacobian matrix)**.** *Consider the following parametric discontinuous system*

$$\dot{x}(t) = f(t,x(t),\mu) = \begin{cases} f^-(t,x(t),\mu) & x(t) \in S^- \\ f^+(t,x(t),\mu) & x(t) \in S^+ \end{cases}, \tag{13}$$

*which has a discontinuity boundary $\Sigma$. Let $x$ be a fixed point of* (13) *at some value for $\mu$.*

(I) *If $x$ is not on $\Sigma$, then we can find a single-valued Jacobian matrix $J(x,\mu)$*

(II) *If $x$ is on $\Sigma$, then there exist two Jacobian matrices $J^-(x,\mu)$ and $J^+(x,\mu)$ on either side of $\Sigma$ associated with the vector fields $f^-$ and $f^+$ in $S^-$ and $S^+$. Assume that we vary $\mu$ such that the fixed point $x$ moves from $S^-$ to $S^+$ via $\Sigma$. Let $x_\Sigma$ denotes the unique fixed point on $\Sigma$ for $\mu = \mu_\Sigma$. The Jacobian matrix $J(x,\mu)$ varies as $\mu$ is varied and is discontinuous at $\mu = \mu_\Sigma$ for which $x = x_\Sigma$. Then, the generalized Jacobian matrix of system* (13) *at $(x_\Sigma, \mu_\Sigma)$ can be expressed as the set-valued matrix*

$$J^\Sigma(x,\mu) = J(x_\Sigma,\mu_\Sigma) := \Big\{J_\lambda(x_\Sigma,\mu_\Sigma) = (1-\lambda)J^-(x_\Sigma,\mu_\Sigma) + \lambda J^+(x_\Sigma,\mu_\Sigma)\,;\, 0 \leq \lambda \leq 1\Big\}. \tag{14}$$

*Indeed,* (14) *defines how the Jacobian jumps at $\Sigma$; for more details see [12, 13].*

### *2.2. Profitability, compatibility and sustainability of tourism systems*

Here by using [4, 5], we state three significant features of tourism system(2), i.e., profitability, compatibility and sustainability of that, in the following definition:

**Definition 2.7.** *For system* (2) *the economic impact of a policy applied to a given site is profitable if at least one of the associated attractors of model is characterized by $T_1(t), T_2(t) > 0$ for all $t$. In this case if all attractors of the system are characterized by $T_1(t), T_2(t) > 0$ for all $t$ , the policy is called profitable and safe and if they characterized by the absence of the tourism industry (i.e., $T_1(t), T_2(t) = 0$) we say that the policy is profitable but risky. Moreover, the policy is called compatible if at least one of the associated attractors has $E(t) > 0$ for all $t$. Finally a policy is sustainable when one of its associated attractors is characterized by$E(t) > 0$ and $T_1(t), T_2(t) > 0$ for all $t$.*
*Notice that since system* (2) *is a third order model, its attractors can be stable equilibria and limit cycles.*

## 3. A discontinuous model for tourism-based social-ecological systems

Consider the system (2). A detailed analysis of this model has been performed, in [9, 10]. Here to better describe the damages induced by tourist activities on ecosystem quality, we extend the model (2). Due to [9], the dynamics of the ecosystem quality $E(t)$ has been considered as:

$$\frac{dE}{dt} = F(E) - D(T_1, T_2, E, C) + G(T_1, T_2, E, C), \tag{15}$$

where $F(E)$ defines the dynamics of ecosystem quality $E(t)$, when no interaction exists with two types of tourists and capital $C(t)$ and

$$F(E) = \frac{rE^2}{k} - rE. \tag{16}$$

Moreover, $G(T_1, T_2, E, C)$ describes the positive effects made by tourists on ecosystem quality; whereas $D(T_1, T_2, E, C)$ displays the damage induced by tourist activities to ecosystem quality. These two functions are defined as

$$G(T_1, T_2, E, C) = G(T_1, E) = +\gamma_1 T_1 E, \tag{17}$$

and

$$D(T_1, T_2, E, C) = D(T_2, E, C) = E\Big(\beta C + \gamma_2 T_2\Big). \tag{18}$$

In [9, 10], only mass-tourists $T_2$ are supposed to be responsible for the reduction in ecosystem quality together with accommodation and entertainment facilities. It is also assumed that eco-tourists $T_1$ just have positive effects on the environmental quality $E(t)$, even indirectly, they cause to maintain or develop the environmental quality.

But in the real world, although eco-tourists are often considered useful for the environment, they can have negative effects on that. For example, eco-tourists can have harmful effects on vegetation, and a series of vegetation may be lost due to carelessness or fire. Also, they may cause to prevail some kinds of contagious and infectious diseases. In addition, they can generate some kinds of environmental pollution such as water pollution in the rivers or lakes, and air pollution. However depending on the environment, these negative effects sometimes do not have observable results on the ecosystem quality until the number of eco-tourists is smaller than a positive number $N$. But when their number is greater than or equal to $N$, they can produce visible negative effects on the environment. In this case, the influence of the damages induced by eco-tourists on the environment can not be ignored. It should be mention that the value of the positive number $N$ depends on the ecosystem and it may be different for each environment.

According to the above description, the model (2) should be improved to include some more realistic features of tourism; specially to better describe the ecosystem quality dynamics. In this regard, to develop the tourism system (2), it is assumed that eco-tourists, in addition to positive effects on the quality of the ecosystem, can also have negative effects on that. Moreover, we suppose that when the number of eco-tourists is smaller than a positive number $N$, they only have positive environmental effects. But, if their number is greater than or equal to $N$, they have positive and also negative effects on the environment. So in this paper we define the function D as follows:

$$D(T_1, T_2, E, C) = \begin{cases} E\Big(\beta C + \gamma_2 T_2\Big); & T_1 < N \\ E\Big(\beta C + \gamma_2 T_2 + \bar{\gamma}_1 T_1\Big); & T_1 \geq N \end{cases}. \tag{19}$$

Therefore, we can introduce the following extended model of (2) for tourism which is a discontinuous system:

$$\begin{pmatrix} \frac{dT_1}{dt} \\ \frac{dT_2}{dt} \\ \frac{dE}{dt} \\ \frac{dC}{dt} \end{pmatrix} = \begin{cases} \begin{pmatrix} T_1\Big(\frac{\mu_{1E}E^2}{E^2+\varphi_{1E}^2} - \beta_1 \frac{C}{T_1+T_2+1} - \alpha_1 T_1 - \alpha_1^* T2 - \gamma_1^* T_1 T_2 - \omega\Big) \\ T_2\Big(\frac{\mu_{2E}E}{E+\varphi_{2E}} + \mu_{2C}\frac{C}{C+\varphi_{2C}(T_1+T_2+1)} - \alpha_2 T_2 - \alpha_2^* T1 - \gamma_2^* T_1 T_2 - \omega\Big) \\ \frac{rE^2}{k} - rE - E(\beta C + \gamma_2 T_2) + \gamma_1 T_1 E \\ -\delta C + \epsilon_1 T_1 + \epsilon_2 T_2 \end{pmatrix}; & T_1 < N \\ \begin{pmatrix} T_1\Big(\frac{\mu_{1E}E^2}{E^2+\varphi_{1E}^2} - \beta_1 \frac{C}{T_1+T_2+1} - \alpha_1 T_1 - \alpha_1^* T2 - \gamma_1^* T_1 T_2 - \omega\Big) \\ T_2\Big(\frac{\mu_{2E}E}{E+\varphi_{2E}} + \mu_{2C}\frac{C}{C+\varphi_{2C}(T_1+T_2+1)} - \alpha_2 T_2 - \alpha_2^* T1 - \gamma_2^* T_1 T_2 - \omega\Big) \\ \frac{rE^2}{k} - rE - E(\beta C + \gamma_2 T_2) + (\gamma_1 - \bar{\gamma_1}) T_1 E \\ -\delta C + \epsilon_1 T_1 + \epsilon_2 T_2 \end{pmatrix}; & T_1 \geq N \end{cases}, \tag{20}$$

in which $N > 0$.
Considering the scalar indicator functions $h(T_1, T_2, E, C) = T_1 - N$, the switching discontinuity boundary $\Sigma$ can be defined as

$$\Sigma = \{(T_1, T_2, E, C) \in \mathbb{R}^4;\ T_1 = N\}. \tag{21}$$

The hyperspace $\Sigma$ splits two adjacent regions $S^-$ and $S^+$, where $S^-$, $S^+$ take the forms

$$S^- = \{(T_1, T_2, E, C) \in \mathbb{R}^4;\ T_1 < N\}, \tag{22}$$

$$S^+ = \{(T_1, T_2, E, C) \in \mathbb{R}^4;\ T_1 > N\}, \tag{23}$$

## 4. Sustainability analysis

Consider the discontinuous system (20) and let due to [9, 10], all the parameters of the system except $\omega$ and $\bar{\gamma_1}$, be chosen as follows:

$$\begin{aligned} &\mu_{1E} = 3.5,\ \varphi_{1E} = 1.5,\ \alpha_1 = 0.5,\ \alpha_1^* = 1.5,\ \gamma_1^* = 0.4,\ \gamma_2^* = 3, \alpha_2 = 0, \beta_1 = 1,\ k = 5, \\ &\beta = 1,\ \mu_{2E} = 2, \alpha_2^* = 0.5,\ \delta = 0.5, \epsilon_1 = 1.5,\ \epsilon_2 = 0.4,\ \gamma_2 = 1.5, \varphi_{2C} = 1, \mu_{2C} = 3, \\ &\varphi_{2E} = 0.5,\ \gamma_1 = 3.65,\ r = 0.005. \end{aligned} \tag{24}$$

These values of parameters are gathered from different research activities on tourism dynamics by using questionnaires, administrated through personal interviews in a region located in southern Italy; further details can be found in [9, 10]. Then we have the following system:

$$\begin{pmatrix} \frac{dT_1}{dt} \\ \frac{dT_2}{dt} \\ \frac{dE}{dt} \\ \frac{dC}{dt} \end{pmatrix} = \begin{cases} \begin{pmatrix} T_1\Big(\frac{3.5E^2}{E^2+2.25} - \frac{C}{T_1+T_2+1} - 0.5T_1 - 1.5T2 - 0.4T_1T_2 - \omega\Big) \\ T_2\Big(\frac{2E}{E+0.5} + 3\frac{C}{C+(T_1+T_2+1)} - 0.5T1 - 3T_1T_2 - \omega\Big) \\ 0.001E^2 - 0.005E - E(C+1.5T_2) + 3.65T_1E \\ -0.5C + 1.5T_1 + 0.4T_2 \end{pmatrix}; & T_1 < N \\ \begin{pmatrix} T_1\Big(\frac{3.5E^2}{E^2+2.25} - \frac{C}{T_1+T_2+1} - 0.5T_1 - 1.5T2 - 0.4T_1T_2 - \omega\Big) \\ T_2\Big(\frac{2E}{E+0.5} + 3\frac{C}{C+(T_1+T_2+1)} - 0.5T1 - 3T_1T_2 - \omega\Big) \\ 0.001E^2 - 0.005E - E(C+1.5T_2) + (3.65-\bar{\gamma}_1)T_1E \\ -0.5C + 1.5T_1 + 0.4T_2 \end{pmatrix}; & T_1 \geq N \end{cases}, \tag{25}$$

All the equilibria of system (25) can be divided into the following cases:

(1) The equilibria in the region $S^-$ which have the forms

$$(0,0,0,0),\ (0,0,5,0),\ (0,T_2^-,0,C^-),\ (T_1^- < N,0,E^-,C^-), \tag{26}$$
$$(0,T_2^-,E^-,C^-),\ (T_1^- < N,T_2^-,E^-,C^-).$$

(2) The equilibrium points $(N,0,E^0,3N)$ and $(N,T_2^0,E^0,C^0)$ which lie on $\Sigma$.

(3) The equilibria $(T_1^+ > N,0,E^+,C^+)$ and $(T_1^+ > N,T_2^+,E^+,C^+)$ that belong to the region $S^+$.

Now, let $p$ be an equilibrium of the system (25). Then by definition 2.6, the Jacobian matrix of (25) at $p$ can be written as:

$$J(p) := \begin{cases} J^-(p); & p \in S^- \\ J^0(p); & p \in \Sigma \\ J^+(p); & p \in S^+ \end{cases}. \tag{27}$$

Such that for $p \in \Sigma$, the set $J^0(p) = \Big\{J_\lambda(p) = (1-\lambda)J^-(p) + \lambda J^+(p),\ 0 \leq \lambda \leq 1\Big\}$ is the generalised Jacobian matrix of the system which takes the value $J^{\mp}$ at each side of the switching boundary $\Sigma$ and we have:

$$J^-(p) \tag{28}$$
$$= \begin{pmatrix} a_{11} & \frac{CT_1}{(T_1+T_2+1)^2} - 1.5T_1 - 0.4T_1^2 & \frac{15.75T_1E}{(E^2+2.25)^2} & \frac{-T_1}{T_1+T_2+1} \\ a_{21} & a_{22} & \frac{T_2}{(E+0.5)^2} & \frac{\frac{3T_2}{T_1+T_2+1}}{(\frac{C}{T_1+T_2+1}+1)^2} \\ 3.65E & -1.5E & 0.002E - 0.005 - C - 1.5T_2 + 3.65T_1 & -E \\ 1.5 & 0.4 & 0 & -0.5 \end{pmatrix}, \quad T_1 < N$$

$$J^+(p) \tag{29}$$
$$= \begin{pmatrix} a_{11} & \frac{CT_1}{(T_1+T_2+1)^2} - 1.5T_1 - 0.4T_1^2 & \frac{15.75T_1E}{(E^2+2.25)^2} & \frac{-T_1}{T_1+T_2+1} \\ a_{21} & a_{22} & \frac{T_2}{(E+0.5)^2} & \frac{\frac{3T_2}{T_1+T_2+1}}{(\frac{C}{T_1+T_2+1}+1)^2} \\ (3.65-\bar{\gamma}_1)E & -1.5E & 0.002E - 0.005 - C - 1.5T_2 + (3.65-\bar{\gamma}_1)T_1 & -E \\ 1.5 & 0.4 & 0 & -0.5 \end{pmatrix}, \quad T_1 > N$$

and

$$J_{\lambda}(p) \tag{30}$$
$$= \begin{pmatrix} a_{11} & \frac{CT_1}{(T_1+T_2+1)^2} - 1.5T_1 - 0.4T_1^2 & \frac{15.75T_1E}{(E^2+2.25)^2} & \frac{-T_1}{T_1+T_2+1} \\ a_{21} & a_{22} & \frac{T_2}{(E+0.5)^2} & \frac{\frac{3T_2}{T_1+T_2+1}}{(\frac{C}{T_1+T_2+1}+1)^2} \\ (3.65-\lambda\,\bar{\gamma_1})E & -1.5E & 0.002E - 0.005 - C - 1.5T_2 + (3.65-\lambda\,\bar{\gamma_1})T_1 & -E \\ 1.5 & 0.4 & 0 & -0.5 \end{pmatrix}, \; T_1 = N$$

where

$$a_{11} = \frac{3.5E^2}{E^2+2.25} - \frac{C(T_2+1)}{(T_1+T_2+1)^2} - T_1 - 1.5T_2 - 0.8T_1T_2 - \omega,$$

$$a_{21} = \frac{\frac{-3T_2C}{(T_1+T_2+1)^2}}{(\frac{C}{T_1+T_2+1}+1)^2} - 0.5T_2 - 3T_2^2,$$

$$a_{22} = \frac{3\frac{C(T_1+1)+C^2}{(T_1+T_2+1)^2}}{(\frac{C}{T_1+T_2+1}+1)^2} + \frac{2E}{E+0.5} - 0.5T_1 - 6T_1T_2 - \omega.$$

The associated characteristic polynomial of system (25) can be given by

$$P(\omega, \bar{\gamma_1}, \lambda) = p_0(\lambda) + p_1(\lambda)\Lambda + p_2(\lambda)\Lambda^2 + p_3\Lambda^3 + \Lambda^4, \qquad 0 \le \lambda \le 1, \tag{31}$$

where

$$\begin{aligned} p_0(\lambda) = & -\frac{2.3T_2E.a_{11}}{2(E+0.5)^2} + \frac{36.225T_1E^2.a_{21}}{2(E^2+2.25)^2} + \frac{3T_2E.b}{2(E+0.5)^2} - \frac{47.25T_1E^2.a_{22}}{2(E^2+2.25)^2} \\ & + \frac{141.75T_1E^2.d}{4(E^2+2.25)^2} + \frac{9ET_1T_2}{4(E+0.5)^2(T_1+T_2+1)} - \frac{a_{11}.a_{22}.g}{2} + \frac{b.g.a_{21}}{2} \\ & - \frac{bT_2.(3.65-\lambda\,\bar{\gamma_1})E}{2(E+0.5)^2} + \frac{1.575T_1(5a_{22}+4d)(3.65-\lambda\,\bar{\gamma_1})E^2}{(E^2+2.25)^2} - \frac{2a_{11}.d.g}{5} \\ & + \frac{-2T_1.a_{21}.g}{5(T_1+T_2+1)} + \frac{2T_2T_1(3.65-\lambda\,\bar{\gamma_1})E}{5(E+0.5)^2(T_1+T_2+1)} + \frac{3b.d.g}{2} + \frac{3T_1.a_{22}.g}{2(T_1+T_2+1)}, \end{aligned}$$

$$\begin{aligned} p_1(\lambda) = & \frac{47.25T_1E^2(1+a_{21})}{2(E^2+2.25)^2} + \frac{2.3ET_2}{2(E+0.5)^2} + \frac{a_{11}.a_{22}}{2} - \frac{b.a_{21}}{2} - \frac{-2T_1.a_{21}}{5(T_1+T_2+1)} \\ & - \frac{3b.d}{2} + \frac{-3T_1.a_{22}}{2(T_1+T_2+1)} + \frac{a_{11}.g}{2} - \frac{15.75T_1E^2(3.65-\lambda\,\bar{\gamma_1})}{2(E^2+2.25)^2} + \frac{2a_{11}.d}{5} \\ & + \frac{-3T_1.g}{2(T_1+T_2+1)} + \frac{a_{22}.g}{2} - \frac{3T_2E.a_{11}}{2(E+0.5)^2} - a_{11}.a_{22}.g + \frac{2d.g}{5} + b.g.a_{21} \\ & - \frac{b.T_2(3.65-\lambda\,\bar{\gamma_1})E}{(E+0.5)^2} + \frac{15.75T_1.a_{22}.(3.65-\lambda\,\bar{\gamma_1})E^2}{(E^2+2.25)^2}, \end{aligned}$$

$$p_2(\lambda) = \frac{1.5ET_2}{(E+0.5)^2} - \frac{-1.5T_1}{T_1+T_2+1} - \frac{a_{22}+a_{11}+g}{2} + a_{11}.a_{22} - b.a_{21} + a_{11}.g$$
$$- \frac{15.75T_1(3.65-\lambda\bar{\gamma}_1)E^2}{(E^2+2.25)^2} + a_{22}.g - \frac{2d}{5},$$

$$p_3 = 0.5 - a_{22} - g - a_{11},$$

in which

$$b = \frac{CT_1}{(T_1+T_2+1)^2} - 1.5T_1 - 0.4T_1^2,$$

$$d = \frac{\frac{3T_2}{T_1+T_2+1}}{(\frac{C}{T_1+T_2+1}+1)^2},$$

$$g = 0.002E - 0.005 - C - 1.5T_2 + (3.65-\lambda\bar{\gamma}_1)T_1.$$

Moreover for $\lambda = 0$ and $T_1 < N$, $P(\omega, \bar{\gamma}_1, \lambda)$ is the characteristic polynomial related to the Jacobian matrix $J^-(p)$; for $\lambda = 1$ and $T_1 > N$, it is the characteristic polynomial of $J^+(p)$; also for $0 \le \lambda \le 1$ and $T_1 = N$, it is the characteristic polynomial related to $J_\lambda(p)$.
In addition, due to the Routh-Hurwitz criterion, the equilibrium $p$ is stable if the following conditions are satisfied

$$p_3 > 0, \quad A(\lambda) = p_3 p_2(\lambda) - p_1(\lambda) > 0, \quad p_0(\lambda) > 0, \tag{32}$$
$$B(\lambda) = p_1(\lambda)\big[p_3 p_2(\lambda) - p_1(\lambda)\big] - p_3^2 p_0(\lambda) > 0, \quad 0 \le \lambda \le 1.$$

Indeed, conditions (32) guarantees that the eigenvalues of $p$ have negative real part. It should be mentioned that when $p$ is a stable equilibrium, there is a coexistence between eco-tourists and mass-tourists. This type of coexistence is interesting since these two types of tourists have different effects on the environment quality.

Now we check the profitability, compatibility and sustainability of the tourism system (25) by means of definition 2.7. Suppose that the equilibrium point $p = (T_1, T_2, E, C)$ satisfies in conditions (32). Then, $p$ is a stable equilibrium of system (25). Hence according to definition 2.7, $p = (T_1, T_2, E, C)$ is called a profitable equilibrium if $T_1 > 0, T_2 > 0$. Since $p$ is an equilibrium, so in this case we have:

$$T_1T_2 = \frac{5}{13}\Big[E\Big(\frac{2}{E+0.5} - \frac{3.5E}{E^2+2.25}\Big) + C\Big(\frac{3}{C+(T_1+T_2+1)} + \frac{1}{T_1+T_2+1}\Big) + 1.5T_2\Big] \tag{33}$$
$$:= g_1(T_1, T_2, E, C) > 0,$$

and also

$$C = 3T_1 + 0.8T_2 := g_2(T_1, T_2) > 0. \tag{34}$$

Furthermore, $p = (T_1, T_2, E, C)$ is called a compatible equilibrium if $E > 0$; i.e.,

$$E = 5 + 1000(C + 1.5T_2) - 1000(3.65 - \lambda\bar{\gamma}_1)T_1 := g_3(T_1, T_2, C, \lambda) > 0, \tag{35}$$

Finally by definition 2.7, the point $p = (T_1, T_2, E, C)$ is called a sustainable equilibrium if $T_1, T_2 > 0$ (so $T_1T_2 = g_1(T_1, T_2, E, C) > 0$ and $C = g_2(T_1, T_2) > 0$), and also $E = g_3(T_1, T_2, C, \lambda) > 0$.

By the above explanations we can state the following theorem for investigating profitability, compatibility and sustainability of system (25):

**Theorem 4.1.** *Consider the system* (25). *This system can be profitable, compatible and sustainable in the regions* $S^-$, $S^+$ *and on the boundary* $\Sigma$, *under the following conditions:*

(i) *Let* $p = (T_1 < N, T_2, E, C) \in S^-$ *be an equilibrium point of system* (25). *If* $p_3 > 0, A(0) > 0, p_0(0) > 0, B(0) > 0$, *then p is a stable equilibrium in the region* $S^-$. *In this case, p is a profitable equilibrium, if* $T_1 > 0, T_2 > 0$ *(so* $T_1T_2 = g_1(T_1, T_2, E, C) > 0$ *and* $C = g_2(T_1, T_2) > 0$*); p is a compatible equilibrium, if* $E = g_3(T_1, T_2, C, 0) > 0$*; and it is a sustainable equilibrium, if* $T_1 > 0, T_2 > 0$, *and* $E = g_3(T_1, T_2, C, 0) > 0$.

(ii) *Suppose that for the equilibrium* $p = (N, T_2, E, C) \in \Sigma$, *the conditions* $p_3 > 0, A(\lambda) > 0, p_0(\lambda) > 0, B(\lambda) > 0$ *hold, for some* $0 \le \lambda \le 1$. *Then p is a stable equilibrium which lies on* $\Sigma$. *Moreover, p is a profitable equilibrium, if* $T_2 > 0$ *(so* $NT_2 = g_1(N, T_2, E, C) > 0$ *and* $C = g_2(N, T_2) > 0$*); p is a compatible equilibrium if* $E = g_3(N, T_2, C, \lambda) > 0$, *for some* $0 \le \lambda \le 1$*; and it is a sustainable equilibrium, if* $T_2 > 0$, *and* $E = g_3(N, T_2, C, \lambda) > 0$, *where* $0 \le \lambda \le 1$.

(iii) *Let the point* $p = (T_1 > N, T_2, E, C) \in S^+$ *satisfy in relations* $p_3 > 0, A(1) > 0, p_0(1) > 0, B(1) > 0$. *This means that p is a stable equilibrium of* (25), *in the region* $S^+$. *Then p is a profitable equilibrium, if* $T_2 > 0$ *(so* $T_1T_2 = g_1(T_1, T_2, E, C) > 0$ *and* $C = g_2(T_1 > N, T_2) > 0$*); p is a compatible equilibrium if* $E = g_3(T_1 > N, T_2, C, 1) > 0$*; and it is a sustainable equilibrium, if* $T_2 > 0$, *and* $E = g_3(T_1 > N, T_2, C, 1) > 0$.

*Proof.* $p$ is a stable equilibrium, provided that conditions (32) hold for that. Furthermore by definition 2.7 and previous explanations, the stable equilibria of (25) are profitable, if they satisfy in relation (33) and so in (34); and they are compatible, if they satisfy in relation (35). Also, they are sustainable equilibria if relations (33), (34) and (35) are true for them. Thus by definition 2.7, conditions (32) and relations 33-35 the proof is clear. □

Indeed, by the above theorem we can find the *profitability, compatibility and sustainability regions* for system (25) in the spaces $S^-$ and $S^+$ and also on the boundary $\Sigma$.

**Example 4.2.** *For system* (25) *suppose that* $\omega = 2.797$. *Then for every* $N \ge 1$, *the point* $p^- = (0.176, 0.0534, 13.42, 0.57072)$ *is an equilibrium in the space* $S^-$, *at which eco-tourists are higher in number than mass-tourists* ($T_1 > T_2$). *Moreover at* $p^-$ *and for* $\lambda = 0$, *conditions* (32) *are:*

$$p_3 \simeq 0.5784619482 > 0, \quad A(0) \simeq 0.0749076406 > 0,$$

$$p_0(0) \simeq 0.00369297206 > 0, \quad B(0) = 0.0003455985 > 0.$$

*Then by Routh-Hurwitz criterion,* $p^-$ *is a stable equilibrium in the region* $S^-$. *Therefore, a coexistence of mass- and eco-tourists is possible in the form of a stable equilibrium with a relatively high ecosystem quality. Furthermore, we have:*

$$T_1 \simeq 0.176 > 0, \quad T_2 \simeq 0.0534 > 0,$$

$$T_1T_2 = g_1(T_1, T_2, E, C) \simeq 0.0093984 > 0,$$

$$C = g_2(T_1, T_2) = 0.57072 > 0,$$

$$E = g_3(T_1, T_2, C, 0) \simeq 13.42 > 0.$$

*Hence by theorem 4.1,* $p^-$ *is a profitable and compatible equilibrium; thus it is a sustainable equilibrium. The motions of a trajectory starting near* $p^-$ *in the phase spaces* $T_1T_2, EC, T_1T_2E$ *and also time histories of this trajectory are illustrated in Figure 1.*

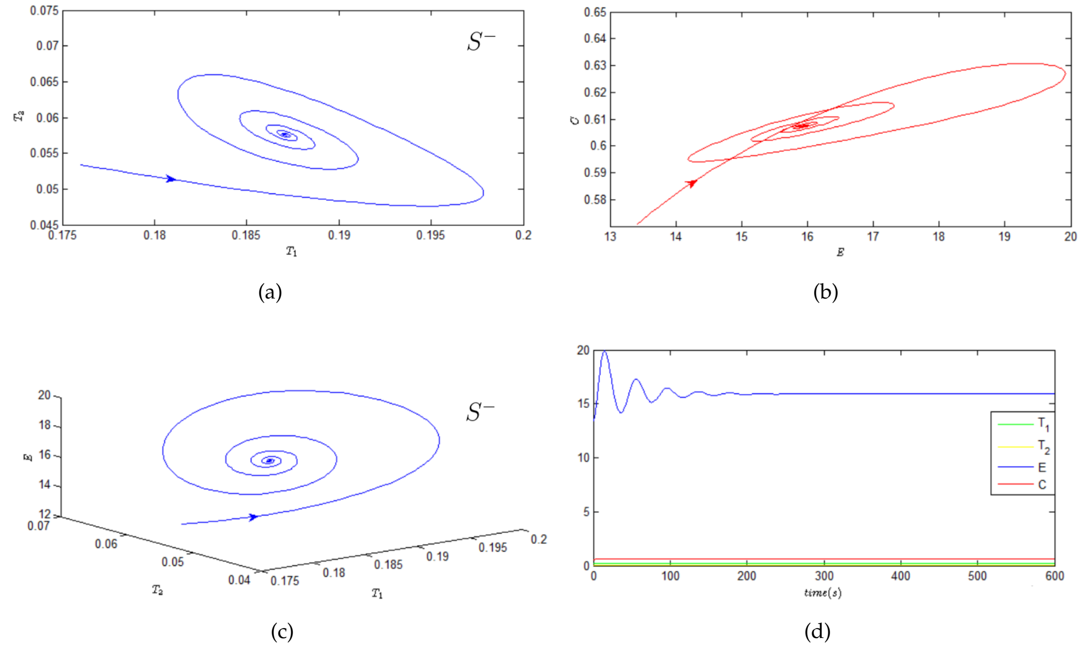



Figure 1: Phase portrait of a trajectory in $S^-$, for $\omega = 2.797$ and every $N \geq 1$; the initial conditions are near $p^-$. (a) The related projections in the phase space $T_1T_2$. (b) The related projections in the phase space $EC$. (c) The related projections in the $3D$ phase space $T_1T_2E$. (d) Time dependent behavior of the model taking initial conditions near $p^-$.

## 5. Tangent points

Here in order to analyse more dynamic features of eco and mas-tourists, we will investigate tangent points of the discontinuous system (25). These points belong to the discontinuity boundary $\Sigma$ and can be computed according to definition 2.5. In fact due to definition 2.5, $T = (N, T_2, E, C)$ is a tangent point of (25) if

$$\langle \nabla h(T), f^-(t, T) \rangle = 0, \text{ or } \langle \nabla h(T), f^+(t, T) \rangle = 0. \tag{36}$$

On the other hand we have

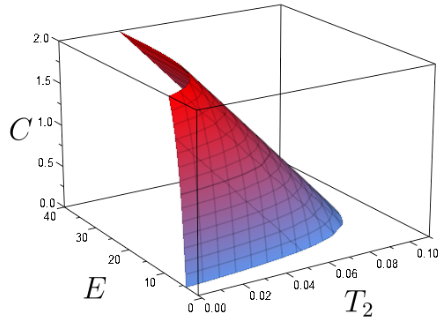
$$\langle \nabla h(T), f^-(t, T) \rangle = \langle \nabla h(T), f^+(t, T) \rangle \tag{37}$$

$$= \frac{3.5E^2}{E^2 + 2.25} - \frac{C}{N + T_2 + 1} - 0.5N - 1.5T2 - 0.4NT_2 - \omega$$

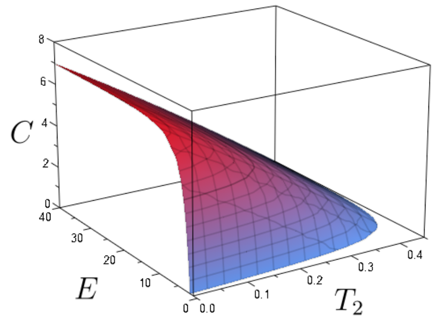
Therefore by definition 2.5, the collision of two tangent points (of vector fields $f^{\mp}(t, T)$) will occur at $T = (N, T_2, E, C)$ provided that

$$\frac{3.5E^2}{E^2 + 2.25} - \frac{C}{N + T_2 + 1} - 0.5N - 1.5T2 - 0.4NT_2 = \omega \tag{38}$$

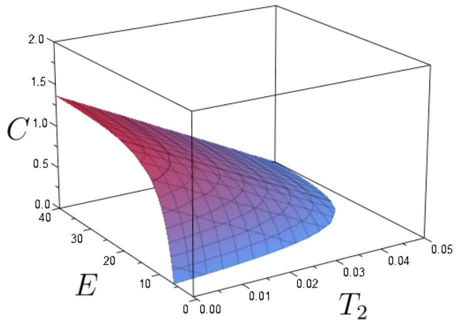
*Indeed, a region for existence of tangent points (the collision of two tangent points) of* (25) *can be obtained as all the points* $(N, T_2, E, C, \omega)$ *that satisfy in relation* (38). Some of these regions for different values of $\omega$ and $N$ are shown in Figure 2.



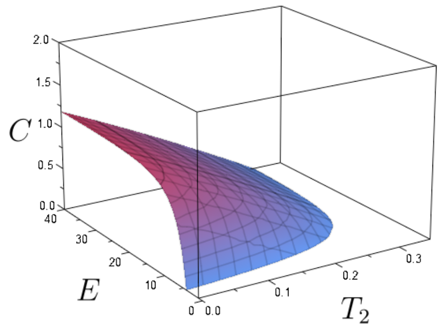

(a)




(b)




(c)




(d)

Figure 2: Plotted surfaces as regions for existence of tangent points of system (25) in the space $T_2EC$ for (a) $N = 6$, $\omega = 0.1$; (b) $N = 4$, $\omega = 0.1$; (c) $N = 6$, $\omega = 0.3$; (d) and $N = 2$, $\omega = 0.1$.

**Example 5.1.** *Consider system* (25) *with* $N = 6$ , $\omega = 0.1$ *and* $\bar{\gamma}_1 = 1$*. Then, for* $(T_2, E, C) = (0.10105, 40, 0.0071288)$ *belonging to the region* (*a*) *in Figure 2, the point* $T = (N, T_2, E, C) = (6, 0.10105, 40, 0.0071288)$ *is a tangent point of system* (25)*. In fact, at this point the collision of two tangent points occurs for* (25)*. Related trajectories of the system near* $T = (6, 0.10105, 40, 0.0071288)$ *are illustrated in Figure 3.*

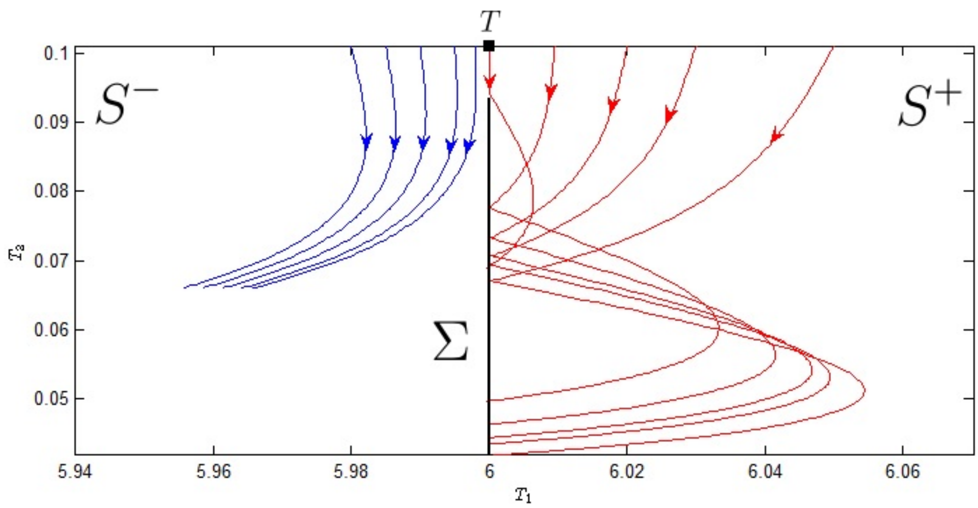



Figure 3: Trajectories of system (25) in the spaces $S^{\mp}$ and on the switching boundary $\Sigma$, for $N = 6$, $\omega = 0.1$ and $\bar{\gamma}_1 = 1$.

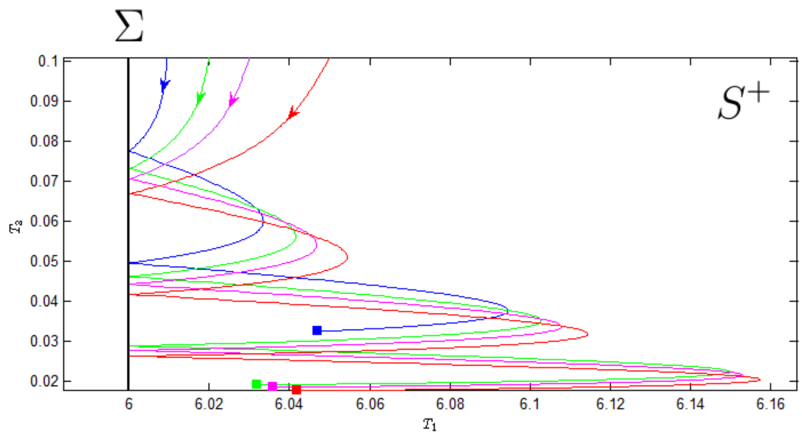



Figure 4: Trajectories of system (25) in the space $S^+$, for more passing the time and $N = 6$, $\omega = 0.1$, $\bar{\gamma}_1 = 1$.

*In Figure 3, there are three types of trajectories, i.e., trajectories in the space $S^-$; a trajectory on the switching boundary $\Sigma$ which starts from the tangent point T and slides on the boundary; and trajectories in $S^+$. As it is shown, for all trajectories in $S^-$, the number of eco-tourists at first is next to $N = 6$. Then their number will decrease in $S^-$ by passing the time. Moreover, the trajectory on the boundary $\Sigma$ implies that the number of eco-tourists will become fixed for a short period of time. Finally, the trajectories in $S^+$ show that there are some oscillations in the number of eco-tourists in the space $S^+$. To find more information about this bahavior, the trajectories in $S^+$ for a larger time period are plotted in Figure 4. This figure demonstrates that the trajectories in $S^+$ hit to the discontinuity boundary several times and eventually they lead to some equilibrium points. This means that in $S^+$, the number of eco-tourists which at the beginning is more that $N = 6$, reduces until it becomes equal to 6. Then, it first increases and then decreases to $N = 6$ again. Finally after some vibrations, the number of eco-tourists will become constant by leading to a stable condition. In this case, there exists a coexistence between eco-tourists and mass-tourists.*
*It should be mention that for some values of $\omega > 0.1$, the dynamic behavior of system (25) is completely different. Phase portrait of system (25), for $\omega = 0.2$ is presented in Figure 5. One can see that for $\omega = 0.2$, all the trajectories in $S^+$ tend to cross the switching boundary $\Sigma$ into $S^-$ transversally. Also for all related trajectories, the number of eco-tourists in this case will reduce by passing the time.*

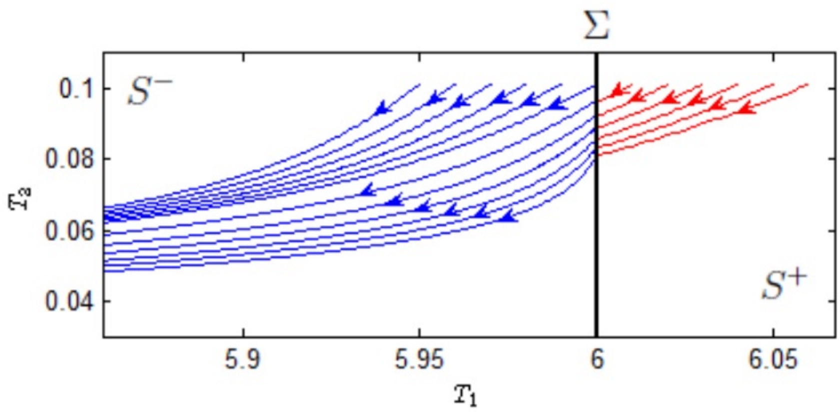



Figure 5: Related trajectories of system (25), for $N = 6$, $\omega = 0.2$ and $\bar{\gamma}_1 = 1$.

## 6. Conclusion

In this paper we introduced an extended discontinuous model (20) for the system (2) to include some more realistic features of tourism. Indeed, the developed system (20) contains the observable negative effects of eco-tourists on the environment which can not be ignored for all tourist sites. Moreover, by the aid of the theory of discontinuous dynamical systems and also some structural properties of the stable equilibria of the system (20), we could analyse profitability, compatibility and sustainability of that. In fact, we obtained some profitability, compatibility and sustainability regions for (25). It was shown that in such regions there is a kind of coexistence between mass-tourists and eco-tourists. Finally we studied more dynamic behaviors of eco and mas-tourists by investigating tangent points of (25) and finding some regions for existence of the collision of two tangent points for (25). Some numerical simulations as two examples were performed to display our theoretical results.

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