



## Further Nonlinear Version of Inequalities and Their Applications

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**Abstract.** In this article, some new generalized nonlinear versions are established for integral and discrete analogues of inequalities, with advanced arguments that provide explicit bounds on unknown functions. The estimation given here can be used as a handy and powerful tool in the study of some classes of sum difference and integral equations. Some applications are also discussed here in order to illustrate the usefulness of our results.

### 1. Introduction

Linear and nonlinear integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the study of qualitative properties of solutions of differential, integral and integro-differential equations. During the past few years, such type of inequalities and their applications have been undertaken by many scholars. For details, we refer to literature [2–5, 7, 8, 10, 13, 16, 18, 19, 21] and references therein.

In 1973, Pachpatte [9] established the following useful integral inequality: If  $v, z, j$  are non-negative continuous functions on  $R_+ = [0, \infty)$ ,  $v_0 \geq 0$  is a constant and

$$v(l) \leq v_0 + \int_0^l z(\vartheta) \left[ v(\vartheta) + \int_0^{\vartheta} j(\zeta) v(\zeta) d\zeta \right] d\vartheta, \quad (1)$$

then

$$v(l) \leq v_0 \left[ 1 + \int_0^l z(\vartheta) \exp \left( \int_0^{\vartheta} [z(\zeta) + j(\zeta)] d\zeta \right) d\vartheta \right], \quad l \in R_+.$$

In 2010, he [14] further studied the integral inequality of two variables of the type

$$v(l, u) \leq p(l, u) + q(l, u) \int_0^l \int_B [f(y, t)v(y, t) + g(y, t)] dt dy,$$

such that  $l, u \in \Delta$ ,  $B$  be a bounded domain in  $R^n$ ,  $n$ -dimensional Euclidean space,  $\Delta = B \times R_+$  and  $B = \sum_{i=1}^n [a_i, b_i] (a_i < b_i)$ . Later, Tian et al. [17] discovered the integral inequality as

$$\varphi(v(l, u)) \leq c + \int_0^{\alpha(l)} \int_0^{\beta(u)} [f(y, t)\psi(v(y, t))] dt dy$$

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$c$  is a constant,  $\varphi$  be increasing function with  $\varphi(\infty) = \infty$ ,  $\alpha, \beta, \psi$  be nondecreasing and  $\alpha(l) \leq l, \beta(u) \leq u$  on  $R_+$ . In this paper based on the work above, we establish some variations of (1) involving functions of two independent variables, that can be beneficial to demonstrate the classifications of integral inequalities and integral equations. Lastly, we give examples to prove the validity of our results.

**2. Preliminaries**

For the reader convenience, let  $R$  be the set of real numbers,  $R_+ = [0, \infty)$  and  $N_0 = \{0, 1, 2, \dots\}$ .  $I_1 = [l_0, L)$ ,  $I_2 = [u_0, U)$  are the subsets of  $R$  and  $\theta = I_1 \times I_2$ . Let  $C(X, W)$  denotes the class of continuous functions from  $X$  to  $W$  in which empty sums and products are taken to be 0 and 1 respectively. All the integrals, sums and products exist on the definitions of their respective domain and the functions be real-valued.  $D_1v(l, u)$ ,  $D_2v(l, u)$  and  $D_1D_2v(l, u) = D_2D_1v(l, u)$  are the partial derivatives of a function  $v(l, u)$  with respect to  $r, u$  and  $ru$  respectively for  $l, u \in R$ . Also  $\Delta w(b) = w(b + 1) - w(b)$ ,  $\Delta_1w(b, k) = z(b + 1, k) - w(b, k)$ ,  $\Delta_2w(b, k) = w(b, k + 1) - w(b, k)$  and  $\Delta_2\Delta_1w(b, k) = \Delta_2(\Delta_1w(b, k))$  are the operators  $\Delta, \Delta_1, \Delta_2$  for the function  $w(b), w(b, k)$ ,  $b, k \in N_0$ . In addition  $P_1 = \{(\vartheta, \zeta) \in R_+^2 : 0 \leq \vartheta \leq \zeta < \infty\}$ ,  $P_2 = \{(l, u, \vartheta, \zeta) \in R_+^4 : 0 \leq \vartheta \leq l < \infty, 0 \leq \zeta \leq u < \infty\}$ ,  $H_1 = \{(b, k) \in N_0^2 : 0 \leq k \leq b < \infty\}$  and  $H_2 = \{(l, u, b, k) \in N_0^4 : 0 \leq b \leq l < \infty, 0 \leq k \leq u < \infty\}$ .

**3. Results and discussion**

Our principle results depend on the accompanying hypotheses of fundamental integral inequalities with two variables that can be utilized in specific circumstances.

**Theorem 3.1.** Let  $v(l, u), z(l, u), m(l, u), a(l, u) \in C(R_+^2, R_+)$ . Moreover,  $j(l, u, \vartheta, \zeta), D_1j(l, u, \vartheta, \zeta), D_2j(l, u, \vartheta, \zeta), D_1D_2j(l, u, \vartheta, \zeta) \in C(P_2, R_+)$  and  $\eta > 1, \rho > 0$ . If

$$v^\eta(l, u) \leq a(l, u) + m(l, u) \int_0^l \int_0^u z(\vartheta, \zeta) \left[ v^\eta(\vartheta, \zeta) + \int_0^\vartheta \int_0^\zeta j(\vartheta, \zeta, b, k) v^\rho(b, k) dkdb \right] d\vartheta d\zeta, \tag{2}$$

then

$$v(l, u) \leq \left\{ a(l, u) + m(l, u) Q(l, u) \left[ 1 + \int_0^l \int_0^u F(\vartheta, \zeta) \exp\left( \int_0^\vartheta \int_0^\zeta [F(\sigma, \omega) + \frac{\rho}{\eta} B(\sigma, \omega)] d\omega d\sigma \right) d\zeta d\vartheta \right] \right\}^{\frac{1}{\eta}}, \tag{3}$$

provided with

$$Q(l, u) = \int_0^l \int_0^u z(\vartheta, \zeta) \left[ a(\vartheta, \zeta) + \int_0^\vartheta \int_0^\zeta j(\vartheta, \zeta, b, k) \left( \frac{\rho}{\eta} a(b, k) + \frac{\eta - \rho}{\eta} \right) dkdb \right] d\zeta d\vartheta, \tag{4}$$

$$B(l, u) = j(l, u, l, u) + \int_0^l D_1j(l, u, b, u) db + \int_0^u D_2j(l, u, l, k) dk + \int_0^l \int_0^u D_1D_2j(l, u, b, k) dkdb, \tag{5}$$

$$F(l, u) = m(l, u)z(l, u), \tag{6}$$

for  $l, u \in R_+$ .

*Proof.* Set a function  $n(l, u)$  by

$$n(l, u) = \int_0^l \int_0^u z(\vartheta, \zeta) \left[ v^\eta(\vartheta, \zeta) + \int_0^\vartheta \int_0^\zeta j(\vartheta, \zeta, b, k) v^\rho(b, k) dkdb \right] d\zeta d\vartheta. \tag{7}$$

Then (2) can be rewritten as

$$v^\eta(l, u) \leq a(l, u) + n(l, u) \implies v(l, u) \leq \left[ a(l, u) + m(l, u)n(l, u) \right]^{\frac{1}{\eta}}. \tag{8}$$

By applying elementary inequality (See [6, p 30]) and from (8), we deduce

$$v^{\frac{1}{\eta}}g^{\frac{1}{\varrho}} \leq \frac{v}{\eta} + \frac{g}{\varrho},$$

where  $v \geq 0, g \geq 0, \frac{1}{\eta} + \frac{1}{\varrho} = 1$  with  $\eta > 1$ , we notice that

$$v(l, u) \leq \frac{a(l, u)}{\eta} + \frac{\eta - 1}{\eta} + \frac{n(l, u)}{\eta}, \tag{9}$$

and

$$v^{\varrho}(l, u) \leq [a(l, u) + m(l, u)n(l, u)]^{\frac{\varrho}{\eta}} [1]^{\frac{\eta-\varrho}{\eta}} \leq \frac{\eta - \varrho}{\eta} + \frac{\varrho}{\eta}a(l, u) + \frac{\varrho}{\eta}m(l, u)n(l, u), \tag{10}$$

substitute (9) and (10) in (7), we get

$$n(l, u) \leq Q(l, u) + \int_0^l \int_0^u F(\vartheta, \zeta) \left[ n(\vartheta, \zeta) + \frac{\varrho}{\eta} \int_0^{\vartheta} \int_0^{\zeta} j(\vartheta, \zeta, b, k)n(b, k)dkdb \right] d\zeta d\vartheta, \tag{11}$$

$Q(l, u)$  and  $F(l, u)$  are given as in (4) and (6) respectively. First, we assume that  $Q(l, u) > 0$  for  $l, u \in R_+$ . From (11), it is easy to verify that

$$\frac{n(l, u)}{Q(l, u)} \leq g(l, u), \tag{12}$$

where

$$g(l, u) = 1 + \int_0^l \int_0^u F(\vartheta, \zeta) \left[ \frac{n(\vartheta, \zeta)}{Q(\vartheta, \zeta)} + \frac{\varrho}{\eta} \int_0^{\vartheta} \int_0^{\zeta} j(\vartheta, \zeta, b, k) \frac{n(b, k)}{Q(b, k)} dkdb \right] d\zeta d\vartheta, \tag{13}$$

and  $g(0, u) = g(l, 0) = v(0, 0) = 1$ . Differentiating (13) and from (12), we have

$$D_1 D_2 g(l, u) \leq F(l, u)M(l, u), \tag{14}$$

from which

$$M(l, u) = g(l, u) + \frac{\varrho}{\eta} \int_0^l \int_0^u j(l, u, b, k)g(b, k)dkdb, \tag{15}$$

$$M(0, u) = M(l, 0) = M(0, 0) = 1, \tag{16}$$

and

$$g(l, u) \leq M(l, u). \tag{17}$$

It is obvious that  $M(l, u)$  is nondecreasing and using (17), we get

$$D_2 \left[ \frac{D_1 M(l, u)}{M(l, u)} \right] \leq \left[ F(l, u) + \frac{\varrho}{\eta} B(l, u) \right], \tag{18}$$

where  $B(l, u)$  be defined as in (5). By keeping  $l$  fixed,  $u = \zeta$ , integrate first from 0 to  $u$  and then again keeping  $u$  fixed,  $l = \vartheta$  and integrate the resulting inequality from 0 to  $l$  for  $l, u \in R_+$  and using (16), we have

$$M(l, u) \leq \exp \left[ \int_0^l \int_0^u \left( F(\vartheta, \zeta) + \frac{\varrho}{\eta} B(\vartheta, \zeta) \right) d\zeta d\vartheta \right]. \tag{19}$$

Insert (19) in (14) and integrating the resultant inequality first from 0 to  $u$  and then from 0 to  $l$ , we obtain

$$g(l, u) \leq 1 + \int_0^l \int_0^u F(\vartheta, \zeta) \exp\left[\int_0^\vartheta \int_0^\zeta \left(F(\sigma, \omega) + \frac{\varrho}{\eta} B(\sigma, \omega)\right) d\sigma d\omega\right] d\zeta d\vartheta. \tag{20}$$

The desired inequality in (3) follows by combining (12) and (20) in (8).

The above procedure can be executed with  $Q(l, u) + \varepsilon$  instead of  $Q(l, u)$  if  $Q(l, u)$  is non-negative, where  $\varepsilon > 0$  is an arbitrary small constant so if  $\varepsilon \rightarrow 0$ , we get the required inequality (3).  $\square$

**Remark 3.2.** If we take  $a(l, u) = c$ ,  $m(l, u) = 1$ ,  $\varrho = 1$  and  $\eta = 1$ , then Theorem 3.1 can be reduced to Theorem 2.2( $b_1$ ) of [12].

**Remark 3.3.** It is interesting to note that if  $u$  fixed,  $m(l, u) = 1$ ,  $v(l, u) = u(x)$ ,  $a(l, u) = u_0$ ,  $z(\vartheta, \zeta) = f(s)$ ,  $j(\vartheta, \zeta, b, k)v(b, k) = h(m)u(m)$  and  $\varrho = 1$ , then Theorem 3.1 can be converted into Theorem 2.4 of [5].

**Remark 3.4.** The inequality established in Theorem 3.1 can be generalized into Theorem 2.2( $a_1$ ) of [11] with  $\eta = 1$ ,  $j(\vartheta, \zeta, b, k) = 0$  and  $\int_0^u z(\vartheta, \zeta) d\zeta = \int_y^\infty c(s, t) dt$ .

**Corollary 3.5.** Suppose that  $v, z, j, a, m, D_1 j(l, u, \vartheta, \zeta), D_2 j(l, u, \vartheta, \zeta), D_1 D_2 j(l, u, \vartheta, \zeta)$  and  $\eta$  be mentioned as in Theorem 3.1. Then

$$v^\eta(l, u) \leq a(l, u) + m(l, u) \int_0^l \int_0^u f(\vartheta, \zeta) \left[ v^\eta(\vartheta, \zeta) + \int_0^\vartheta \int_0^\zeta j(\vartheta, \zeta, b, k)v(b, k) dkdb \right] d\zeta d\vartheta,$$

implies

$$v(l, u) \leq \left\{ a(l, u) + m(l, u) Q^*(l, u) \left[ 1 + \frac{1}{\eta} \int_0^l \int_0^u F(\vartheta, \zeta) \exp\left(\int_0^\vartheta \int_0^\zeta [F(\sigma, \omega) + \frac{1}{\eta} B(\sigma, \omega)] d\omega d\sigma\right) d\zeta d\vartheta \right] \right\}^{\frac{1}{\eta}},$$

where  $B(l, u)$  and  $F(l, u)$  be given as in (3.4), (3.5) respectively and

$$Q^*(l, u) = \int_0^l \int_0^u z(\vartheta, \zeta) \left[ a(\vartheta, \zeta) + \int_0^\vartheta \int_0^\zeta j(\vartheta, \zeta, b, k) \left( \frac{a(b, k)}{\eta} + \frac{\eta - 1}{\eta} \right) dkdb \right] d\zeta d\vartheta,$$

for  $l, u \in R_+$ .

**Remark 3.6.** By letting  $u$  fixed,  $m(l, u) = 1$ ,  $v(l, u) = u(x)$ ,  $a(l, u) = u_0$ ,  $j(\vartheta, \zeta, b, k)v(b, k) = h(m)u(m)$  and  $\int_0^l \int_0^u z(\vartheta, \zeta) d\zeta = \int_0^{\alpha(t)} f(s) ds$  where  $\alpha(t) \leq t$ , Corollary 3.5 becomes Theorem 2.1 of [1].

**Remark 3.7.** Corollary 3.5 is the generalization of Theorem 2.2 ( $a_2$ ) [11] by putting  $\eta = 1$ ,  $m(l, u) = 1$ ,  $j(\vartheta, \zeta, b, k) = 0$  and  $\int_0^l \int_0^u z(\vartheta, \zeta) d\zeta d\vartheta = \int_x^\infty \int_y^\infty c(s, t) dt ds$ .

**Theorem 3.8.** Let  $v(l, u), z(l, u), j(l, u), a(l, u) \in C(\theta, R_+)$ . Further,  $\alpha \in C^1(I_1, I_1)$ ,  $\beta \in C^1(I_2, I_2)$  be nondecreasing with  $\alpha(l) \leq l$  on  $I_1$ ,  $\beta(u) \leq u$  on  $I_2$  and  $\eta > 1$ . If

$$v^\eta(l, u) \leq a(l, u) + \int_{l_0}^l \int_{u_0}^u z(\vartheta, \zeta) \left[ v(\vartheta, \zeta) + \int_{\alpha(l_0)}^{\alpha(l)} \int_{\beta(u_0)}^{\beta(u)} j(b, k)v(b, k) dkdb \right] d\zeta d\vartheta. \tag{21}$$

Then

$$v(l, u) \leq \left\{ a(l, u) + H(l, u) \left[ 1 + \frac{1}{\eta} \int_{l_0}^l \int_{u_0}^u z(\vartheta, \zeta) \exp\left[\int_{\vartheta_0}^\vartheta \int_{\zeta_0}^\zeta z(\sigma, \omega) + \int_{\alpha(\vartheta_0)}^{\alpha(\vartheta)} \int_{\beta(\zeta_0)}^{\beta(\zeta)} j(\sigma, \omega) d\omega d\sigma\right] d\zeta d\vartheta \right] \right\}^{\frac{1}{\eta}}, \tag{22}$$

where

$$H(l, u) = \int_{l_0}^l \int_{u_0}^u z(\vartheta, \zeta) \left[ \frac{a(\vartheta, \zeta)}{\eta} + \frac{\eta - 1}{\eta} + \int_{\alpha(l_0)}^{\alpha(l)} \int_{\beta(u_0)}^{\beta(u)} j(b, k) \left( \frac{a(b, k)}{\eta} + \frac{\eta - 1}{\eta} \right) dkdb \right] d\zeta d\vartheta, \tag{23}$$

for  $l, u \in \theta$ .

*Proof.* Denote

$$n_1(l, u) = \int_{l_0}^l \int_{u_0}^u z(\vartheta, \zeta) \left[ v(\vartheta, \zeta) + \int_{\alpha(l_0)}^{\alpha(l)} \int_{\beta(u_0)}^{\beta(u)} j(b, k) v(b, k) dkdb \right] d\zeta d\vartheta, \tag{24}$$

(21) can be restated as

$$v(l, u) \leq \left[ a(l, u) + n_1(l, u) \right]^{\frac{1}{\eta}}, \tag{25}$$

from (25) and (9) in (24), we attain

$$n_1(l, u) \leq H(l, u) + \int_{l_0}^l \int_{u_0}^u z(\vartheta, \zeta) \left[ \frac{n(\vartheta, \zeta)}{\eta} + \int_{\alpha(l_0)}^{\alpha(l)} \int_{\beta(u_0)}^{\beta(u)} j(b, k) \frac{n(b, k)}{\eta} dkdb \right] d\zeta d\vartheta, \tag{26}$$

$H(x, y)$  be shown as in (23). By the nondecreasing nature of  $H(l, u)$  and from (26), we have

$$\frac{n_1(l, u)}{H(l, u)} \leq g_1(l, u), \tag{27}$$

where

$$g_1(l, u) = 1 + \int_{l_0}^l \int_{u_0}^u z(\vartheta, \zeta) \left[ \frac{n_1(\vartheta, \zeta)}{\eta H(\vartheta, \zeta)} + \int_{\alpha(l_0)}^{\alpha(l)} \int_{\beta(u_0)}^{\beta(u)} j(b, k) \frac{n_1(b, k)}{\eta H(b, k)} dkdb \right] d\zeta d\vartheta. \tag{28}$$

and  $g_1(l_0, u) = g_1(l, u_0) = g_1(l_0, u_0) = 1$ . Obviously  $g_1(l, u) > 0$  and using (27), we obtain

$$D_1 g_1(l, u) \leq \int_{u_0}^u z(l, \zeta) M_1(l, \zeta) d\zeta,$$

$$M_1(l, u) = \frac{g_1(l, u)}{\eta} + \int_{\alpha(l_0)}^{\alpha(l)} \int_{\beta(u_0)}^{\beta(u)} j(b, k) \frac{g_1(b, k)}{\eta} dkdb; \quad M_1(l_0, u) = M_1(l, u_0) = M_1(l_0, u_0) = \frac{1}{\eta},$$

and  $g_1(l, u) \leq M_1(l, u)$ . The remaining proof can be completed by following a suitable modifications at the proof of Theorem 3.1. Here we omit the details.  $\square$

**Remark 3.9.** Take  $\eta = 1, v(l, u) = \Phi(u(x, y)), a(l, u) = a(x) + b(y), j(l, u) = 1, \int_{l_0}^l \int_{u_0}^u z(\vartheta, \zeta) d\zeta d\vartheta = \int_0^{\alpha(x)} \int_0^{\beta(y)} f(t, s) dsdt$  with  $\alpha(x) \leq x, \beta(y) \leq y$  and  $\int_{\alpha(l_0)}^{\alpha(l)} \int_{\beta(u_0)}^{\beta(u)} z(\vartheta, \zeta) d\zeta d\vartheta = g(t, s)$ , Theorem 3.8 converts to Theorem 1 of [17].

**Remark 3.10.** The inequality established in Theorem 3.8 generalizes Theorem 2.5 of [20] (with  $j = 0$  and  $v(l, u) = u(\tau_1(s), \tau_2(t))$ ) on time scales where  $(s, t) \in T_0 \times T_0^+$ ,  $\tau_1 \in (T_0, T), \tau_1(x) \leq x, -\infty < \alpha = \inf \{ \tau_1(x), x \in T_0 \} \leq x_0$  and  $\tau_2 \in (T_0^+, T), \tau_2(y) \leq y, -\infty < \beta = \inf \{ \tau_2(y), y \in T_0^+ \} \leq y_0$ .

**Theorem 3.11.** Assume that  $v(l, u), z(l, u), j(l, u), a(l, u)$  be non-negative functions defined on  $N_0$  and  $\eta > 1, \rho > 0$ . If

$$v^\eta(l, u) \leq a(xl, u) + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ v^\rho(\vartheta, \zeta) + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(b, k) v^\eta(b, k) \right], \tag{29}$$

then

$$v(l, u) \leq \left\{ a(l, u) + E(l, u) \left[ 1 + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \prod_{b=0}^{\vartheta-1} \left( 1 + \sum_{k=0}^{\zeta-1} \left[ \frac{\rho}{\eta} z(b, k) + j(b, k) \right] \right) \right] \right\}^{\frac{1}{\eta}}, \tag{30}$$

such that

$$E(l, u) = \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ \frac{\varrho}{\eta} a(\vartheta, \zeta) + \frac{\eta - \varrho}{\eta} + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} j(b, k) a(b, k) \right], \tag{31}$$

for  $l, u \in N_0$ .

*Proof.* Consider

$$n_2(l, u) = \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ v^\varrho(\vartheta, \zeta) + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(b, k) v^\eta(b, k) \right] \tag{32}$$

(29) takes the form

$$v^\eta(l, u) \leq a(l, u) + n_2(l, u), \tag{33}$$

utilizing (33) and (10) in (32), we easily obtain

$$n_2(l, u) \leq E(l, u) + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ \frac{\varrho}{\eta} n_2(\vartheta, \zeta) + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(b, k) n_2(b, k) \right], \tag{34}$$

where  $E(x, y)$  be mentioned as in (31). Clearly  $E(l, u)$  is non-negative, continuous and nondecreasing. Hence from (34)

$$\frac{n_2(l, u)}{E(l, u)} \leq g_2(l, u), \tag{35}$$

so that

$$g_2(l, u) = 1 + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ \frac{\varrho}{\eta} \frac{n_2(\vartheta, \zeta)}{E(\vartheta, \zeta)} + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(b, k) \frac{n_2(b, k)}{E(b, k)} \right]. \tag{36}$$

The inequality (36) implies the estimate

$$g_2(l, u) \leq 1 + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \prod_{b=0}^{\vartheta-1} \left[ 1 + \sum_{k=0}^{\zeta-1} \left( \frac{\varrho}{\eta} z(b, k) + j(b, k) \right) \right], \tag{37}$$

from (37) and (35), we get

$$n_2(l, u) \leq E(l, u) \left[ 1 + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \prod_{b=0}^{\vartheta-1} \left[ 1 + \sum_{k=0}^{\zeta-1} \left( \frac{\varrho}{\eta} z(b, k) + j(b, k) \right) \right] \right]. \tag{38}$$

Using (38) in (33) to get the acquired inequality in (30).  $\square$

**Remark 3.12.** When  $a(l, u) = c$ ,  $j(b, k) = k(s, t, m, n)$  and  $\eta = \varrho = 1$ , the inequality given in Theorem 3.11 can be changed into Theorem 2.4 ( $d_1$ ) of [12].

**Remark 3.13.** If we put  $\eta = 1$ ,  $j = 0$ ,  $\varrho = 1$  and  $z(l, u) = b(x, y)c(s, t)$ , then the inequality established in Theorem 3.11 reduces to Theorem 2.6 ( $p_2$ ) of [11].

**Remark 3.14.** Setting  $\varrho = 1$ ,  $j = 0$  and  $z(\vartheta, \zeta)v(\vartheta, \zeta) = b(x, y)[c(s, t)u(s, t) + e(s, t)]$ , the inequality given in Theorem 3.11 changes to Theorem 1 of [6].

**Theorem 3.15.** Let  $v(l, u)$ ,  $z(l, u)$ ,  $m(l, u)$ ,  $a(l, u)$ ,  $j(l, u, \vartheta, \zeta)$ ,  $\Delta_1 j(l, u, \vartheta, \zeta)$ ,  $\Delta_2 j(l, u, \vartheta, \zeta)$  and  $\Delta_1 \Delta_2 j(l, u, \vartheta, \zeta)$  be nonnegative functions for  $0 \leq \vartheta \leq l$ ,  $0 \leq \zeta \leq u$  for  $\vartheta, l, \zeta, u$  in  $N_0$  and  $\eta, \rho$  be same as in Theorem 3.11. The inequality

$$v^\eta(l, u) \leq a(l, u) + m(l, u) \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ v^\rho(\vartheta, \zeta) + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(\vartheta, \zeta, b, k) v(b, k) \right], \tag{39}$$

satisfies

$$v(l, u) \leq \left\{ a(l, u) + m(l, u) A(l, u) \left[ 1 + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} F(\vartheta, \zeta) \prod_{b=0}^{\vartheta-1} \left( \frac{1}{\eta} \sum_{k=0}^{\zeta-1} [\rho F(b, k) + N(b, k)] \right) \right] \right\}^{\frac{1}{\eta}}, \tag{40}$$

where

$$A(l, u) = \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ \frac{\rho}{\eta} a(\vartheta, \zeta) + \frac{\eta - \rho}{\eta} + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(\vartheta, \zeta, b, k) \left( \frac{a(b, k)}{\eta} + \frac{\eta - 1}{\eta} \right) \right], \tag{41}$$

$$N(l, u) = j(l + 1, u + 1, l, u) + \sum_{b=0}^{l-1} \Delta_1 j(l, u + 1, b, u) + \sum_{k=0}^{u-1} \Delta_2 j(l + 1, u, l, k) + \sum_{b=0}^{l-1} \sum_{k=0}^{u-1} \Delta_2 \Delta_1 j(l, u, b, k), \tag{42}$$

and  $F(l, k)$  be given as in (22) for  $l, u \in N_0$ .

*Proof.* Define a function  $n_3(l, u)$  by

$$n_3(l, u) = \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} z(\vartheta, \zeta) \left[ v^\rho(\vartheta, \zeta) + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(\vartheta, \zeta, b, k) v(b, k) \right] \tag{43}$$

Then (43) leads to

$$v(l, u) \leq \left[ a(l, u) + m(l, u) n_3(l, u) \right]^{\frac{1}{\eta}}, \tag{44}$$

using (9) and (10) in (43), we achieve

$$n_3(l, u) \leq A(l, u) + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} F(\vartheta, \zeta) \left[ \frac{\rho}{\eta} n_3(\vartheta, \zeta) + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(\vartheta, \zeta, b, k) \frac{n_3(b, k)}{\eta} \right], \tag{45}$$

where  $A(l, u)$  and  $F(l, u)$  are defined by (41) and (6) simultaneously. Now by the definition of  $A(x, y)$  and (45), we observe that

$$\frac{n_3(l, u)}{A(l, u)} \leq g_3(l, u), \tag{46}$$

such that

$$g_3(l, u) = 1 + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} F(\vartheta, \zeta) \left[ \frac{\rho}{\eta} \frac{n_3(\vartheta, \zeta)}{A(\vartheta, \zeta)} + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(l, u, b, k) \frac{n_3(b, k)}{\eta A(b, k)} \right], \tag{47}$$

and  $g_3(l, u)$  is nondecreasing function, we have

$$\Delta_1 \Delta_2 g_3(l, u) \leq F(l, u) M_2(l, u), \tag{48}$$

from (46) and the function  $M_2(l, u)$  is defined by

$$M_2(l, u) = \frac{\rho}{\eta} g_3(l, u) + \sum_{b=0}^{\vartheta-1} \sum_{k=0}^{\zeta-1} j(l, u, b, k) \frac{g_3(b, k)}{\eta}, \tag{49}$$

and

$$g_3(l, u) \leq M_2(l, u), \tag{50}$$

also  $M_2(l, u) > 0, M_2(l, u + 1) \leq M_2(l, u)$  and from (48) and (50) in (49), we attain

$$\Delta_1 \Delta_2 M_2(l, u) \leq \frac{1}{\eta} [\rho F(l, u) + N(l, u)] M_2(l, u). \tag{51}$$

or, equivalently

$$\left[ \frac{M_2(l + 1, u + 1) - M_2(l, u + 1)}{M_2(l, u)} \right] - \left[ \frac{M_2(l + 1, u) - M_2(l, u)}{M_2(l, u)} \right] \leq \frac{1}{\eta} [\rho F(l, u) + N(l, u)], \tag{52}$$

take  $l$  fixed with  $u = \zeta$  and summing over  $\zeta = 0, 1, 2, \dots, u - 1$  in (3.51) first and then again  $u$  fixed in the resulting inequality,  $l = \vartheta$  and summing over  $\vartheta = 0, 1, 2, \dots, l - 1$ , where  $\vartheta$  and  $\zeta$  are an arbitrary in  $N_0$ , we get

$$M_2(l, u) \leq \prod_{\vartheta=0}^{l-1} \left[ 1 + \frac{1}{\eta} \sum_{\zeta=0}^{u-1} [\rho F(\vartheta, \zeta) + N(\vartheta, \zeta)] \right]. \tag{53}$$

(53) and (48) give

$$\Delta_1 \Delta_2 g_3(l, u) \leq F(l, u) \prod_{\vartheta=0}^{l-1} \left[ 1 + \frac{1}{\eta} \sum_{\zeta=0}^{u-1} [\rho F(\vartheta, \zeta) + N(\vartheta, \zeta)] \right], \tag{54}$$

which implies the estimate

$$g_3(l, u) \leq 1 + \sum_{\vartheta=0}^{l-1} \sum_{\zeta=0}^{u-1} F(\vartheta, \zeta) \prod_{b=0}^{\vartheta-1} \left[ 1 + \frac{1}{\eta} \sum_{k=0}^{\zeta-1} [\rho F(b, k) + N(b, k)] \right], \tag{55}$$

the required inequality (40) can be obtained by putting (55) in (46) and the resulting inequality in (44).  $\square$

**Remark 3.16.** Theorem 3.15 converts to Theorem 2.4(d<sub>2</sub>) of [12] if  $m(l, u) = 1$  and  $\eta = \rho = 1$ .

**Remark 3.17.** When  $j = 0, m(l, u) = 1$  and  $\eta = \rho = 1$ , the inequality given in Theorem 3.15 becomes to Theorem 2.1 of [15].

**Remark 3.18.** Put  $\rho = 1, j = 0$  and  $z(\vartheta, \zeta)v(\vartheta, \zeta) = c(s, t)u(s, t) + e(s, t)$ , then the inequality established in Theorem 3.15 reduces to Theorem 2 of [6].

**Remark 3.19.** Setting  $\eta = 1, j = 0, \rho = 1$  and  $z(\vartheta, \zeta) = c(s, t)$ , the inequality given in Theorem 3.15 changes to Theorem 2.6 (p<sub>1</sub>) of [11].



**4. Application**

In this segment, we are presenting some theorem 3.1 implementations. Consider the following nonlinear hyperbolic partial integro-differential equation

$$v_{lu}^\eta(l, u) = X(l, u, v(l, u), \int_0^l \int_0^u h(l, u, \sigma, \omega, v(\sigma, \omega))d\omega d\sigma), \tag{56}$$

with the boundary conditions

$$v^\eta(l, 0) = a_1(l), v^\eta(0, u) = a_2(u), v^\eta(0, 0) = 0, \tag{57}$$

where  $v \in C(R_+^2, R)$ ,  $h \in C(M_2 \times R, R)$ ,  $X \in C(R_+^2 \times R^2, R)$  and  $\eta > 1$ .

Example 4.1: Now, we deal with the assumptions as follows:

$$|X(l, u, t, v) \leq z(l, u)[|t| + |v|], \tag{58}$$

$$|a_1(l) + a_2(u)| \leq a(l, u), \tag{59}$$

$$|h(l, u, \vartheta, \zeta, v) \leq j(l, u, \vartheta, \zeta)|v|, \tag{60}$$

Every solution  $v(l, u)$  of (56) satisfying (57) implies

$$|v(l, u)| \leq \left\{ |a(l, u)| + |Q(l, u)| \left[ 1 + \frac{1}{\eta} \int_0^l \int_0^u |F(\vartheta, \zeta)| \exp\left(\frac{1}{\eta} \int_0^\vartheta \int_0^\zeta [|F(\sigma, \omega) + B(\sigma, \omega)]d\omega d\sigma\right) dt ds \right] \right\}^{\frac{1}{\eta}}, \tag{61}$$

for  $l, u \in R_+$ , where  $Q(l, u)$ ,  $B(l, u)$ ,  $F(l, u)$ ,  $j$ ,  $z$  and  $a(l, u)$  with  $m(l, u) = 1$  be defined as in Theorem 3.1.

Certainly, the solution  $v(l, u)$  of (56) satisfies the following equivalent equation

$$v^\eta(l, u) = a_1(l) + a_2(u) + \int_0^l \int_0^u F(\vartheta, \zeta, v, \int_0^\vartheta \int_0^\zeta h(\vartheta, \zeta, \sigma, \omega, v(\sigma, \omega))d\omega d\sigma) d\zeta d\vartheta, \tag{62}$$

it follows from (58)-(60) that

$$|v(l, u)|^\eta \leq a(l, u) + \int_0^l \int_0^u z(\vartheta, \zeta) [|u(s, t)| + \int_0^\vartheta \int_0^\zeta j(\vartheta, \zeta, \sigma, \omega) |u(\sigma, \omega)| d\omega d\sigma] d\zeta d\vartheta. \tag{63}$$

An appropriate application of the Theorem 3.1 in (63) yields the preferred estimate in (61).

Our next result manages with the uniqueness of the solutions (56) and (57).

Example 4.2: The hypotheses

$$|X(l, u, v_1^\eta, v_2^\eta) - X(l, u, p_1^\eta, p_2^\eta)| \leq z(l, u) [|v_1^\eta - v_2^\eta| + |p_1^\eta - p_2^\eta|], \tag{64}$$

$$|h(l, u, \vartheta, \zeta, v_1^\eta) - h(l, u, \vartheta, \zeta, v_2^\eta)| \leq j(l, u, \vartheta, \zeta) |v_1^\eta - v_2^\eta|, \tag{65}$$

Then the problem (56) and (57) has at most one solution on  $R_+^2$ . Indeed, let  $v_1(l, u)$  and  $v_2(l, u)$  be two solutions of (56)-(57). It follows from (62) and using (64)- (65) that

$$|v_1^\eta(l, u) - v_2^\eta(l, u)| \leq \int_0^l \int_0^u z(\vartheta, \zeta) [|v_1^\eta(\vartheta, \zeta) - v_2^\eta(\vartheta, \zeta)| + \int_0^\vartheta \int_0^\zeta j(\vartheta, \zeta, \sigma, \omega) |v_1^\eta(\sigma, \omega) - v_2^\eta(\sigma, \omega)| d\omega d\sigma] d\zeta d\vartheta, \tag{66}$$

by Theorem 3.8 and from (66), we have

$$|v_1^\eta(l, u) - v_2^\eta(l, u)| \leq 0 \implies v_1^\eta(l, u) = v_2^\eta(l, u),$$

which shows that the problem (56) and (57) has at most one solution on  $R_+^2$ . This completes the proof of example 4.2.

The following partial sum-difference equation can be discussed in order to get the boundedness and uniqueness of Theorem 3.15

$$\Delta_2 \Delta_1 v^\eta(l, u) = B(l, u, v(l, u), \sum_{b=0}^{l-1} \sum_{k=0}^{u-1} g(l, u, b, k)v(b, k)), \quad (67)$$

with the conditions

$$v^\eta(l, 0) = \gamma_1(l), v^\eta(0, u) = \gamma_2(u), v^\eta(0, 0) = 0, \quad (68)$$

under some suitable conditions on the functions involved in (67) and (68). The proof can be completed by closely looking at the proof of Theorem 3.15 given above. Here we omit the details.

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