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On Some Sufficient Conditions for Periodicity of Meromorphic Function Under New Shared Sets

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Abstract. This paper deals with the two set sharing problem related to the uniqueness of a function and its shift operator. With the help of two new range sets we shall significantly improve a number of results in the literature. At the last section we shall exhibit certain examples to show that some conditions used in our results are the best possible.

1. Introduction, Definitions and Results

Throughout the paper we shall assume that all considered meromorphic functions are defined on $\mathbb{C}$ and that they are non-constant.

For such a function f and $a \in \overline{\mathbb{C}} =: \mathbb{C} \cup \{\infty\}$, each z with $f(z) = a$ will be called a-point of f. We also denote $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Next we need the following definition of set sharing.

Definition 1.1. *For a non-constant meromorphic function f and any set $\mathcal{S} \subset \mathbb{C} \cup \{\infty\}$, we define*

$$E_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,p) \in \mathbb{C} \times \mathbb{N} : f(z) = a, \text{ with multiplicity } p\Big\},$$

$$\overline{E}_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,1) \in \mathbb{C} \times \{1\} : f(z) = a\Big\}.$$

If $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$) then we simply say f and g share S Counting Multiplicities(CM) (Ignoring Multiplicities(IM)).

In 2001, *Lahiri* [13, 14] further refined the definition of sharing and introduced a scaling between CM and IM known as weighted sharing of values and sets. Gradually in terms of relaxation of sharing, this notion renders an useful tool to find new directions of research in the uniqueness theory. Below we are recalling the well known definition.

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Definition 1.2. *[13, 14] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_f(a,k)$, the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_f(a,k) = E_g(a,k)$, we see that f and g share the value a with weight k.*

We write f and g share (a,k) to mean that f and g share the value a with weight k.

Definition 1.3. *[13, 14] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S,k)$ the set $\bigcup_{a\in S} E_f(a,k)$. If $\bigcup_{a\in S} E_f(a,k) = \bigcup_{a\in S} E_g(a,k)$, then we say that f and g share the set S with k. Thus we see that f and g share the set $\mathcal{S}$ CM or IM if $\bigcup_{a\in S} E_f(a,\infty) = \bigcup_{a\in S} E_g(a,\infty)$ or if $\bigcup_{a\in S} E_f(a,0) = \bigcup_{a\in S} E_g(a,0)$ respectively.*

For a non-constant meromorphic function, we define its shift and difference operator respectively by $f(z+\omega)$ and $\Delta_\omega f = f(z+\omega) - f(z)$, where ω is a non-zero constant.

In connection with the question of *Gross* [8], a handful number of results have been obtained by many mathematicians [1, 2, 5, 6, 9, 10, 17, 19] concerning the uniqueness of meromorphic functions sharing two sets. But in most of the earlier results, in the direction, one set has always been kept fixed as the set of poles of a meromorphic function.

Recently set sharing corresponding to a function and its shift or difference operator have been given priority by the researchers than that of the original one.

In this respect, we would first like to mention here a result of *Zhang* [18].

Theorem 1.4. *[18] Let $m \geq 2$, $n \geq 2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero constant such that the equation $w^n + aw^{n-m} + b = 0$ has no multiple roots. Let $\mathcal{S} = \{w : w^n + aw^{n-m} + b = 0\}$. Suppose that $f(z)$ is a non-constant meromorphic function of finite order. Then $E_{f(z)}(\mathcal{S},\infty) = E_{f(z+\omega)}(\mathcal{S},\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ imply that $f(z) \equiv f(z+\omega)$.*

With the help of some extra supposition *Qi-Dou-Yang* [16] studied the above Theorem for $m = 1$ and reduced the lower bound of the range set as follows.

Theorem 1.5. *[16] Let $n \geq 6$ be an integer and S be given same as in Theorem 1.4. Suppose f is a non-constant meromorphic function of finite order. Then $E_{f(z)}(\mathcal{S},\infty) = E_{f(z+\omega)}(\mathcal{S},\infty)$, $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ and $\overline{N}(r,f) \leq \frac{n-3}{n-1}T(r,f) + S(r,f)$ implies that $f(z) \equiv f(z+\omega)$.*

As in *Theorem 1.4*, $gcd(n,m) = 1$, so we see that the lower bound of cardinality of the range set considered in *Theorem 1.4*, is 9, and that in *Theorem 1.5*, is 6. However, in 2013, *Bhoosnurmath-Kabbur* [7] improved *Theorem 1.4* by reducing the lower bound of the cardinality of range set and obtained the following result.

Theorem 1.6. *[7] Let $n \geq 8$ be an integer and $c(\neq 0,\ 1)$ is a constant such that the equation $P(w) = \frac{(n-1)(n-2)}{2}w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c$. Let us suppose that $\mathcal{S} = \{w : P(w) = 0\}$ and f is a non-constant meromorphic functions of finite order, then $E_{f(z)}(\mathcal{S},\infty) = E_{f(z+\omega)}(\mathcal{S},\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ imply that $f(z) \equiv f(z+\omega)$.*

By considering "entire" function, *Bhoosnurmath-Kabbur* [7] obtained the following result.

Theorem 1.7. *[7] Let $n \geq 7$ be an integer and $c(\neq 0,\ 1)$ is a constant such that the equation $P(w) = \frac{(n-1)(n-2)}{2}w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c$. Let us suppose that $\mathcal{S} = \{w : P(w) = 0\}$ and f is a non-constant entire functions of finite order, then $E_{f(z)}(\mathcal{S},\infty) = E_{f(z+\omega)}(\mathcal{S},\infty)$ and $E_{f(z)}(\{\infty\},\infty) = E_{f(z+\omega)}(\{\infty\},\infty)$ imply that $f(z) \equiv f(z+\omega)$.*

It is worth noting that no attempts have so far been made by any researcher to deal with a range set from $\overline{\mathbb{C}}$. That is to say, to associate elements of $\mathbb{C}$ with ∞ in the range set.

In this paper, we would like to investigate in this direction. In fact, we shall show that under this new approach of construction of the range sets, the cardinality of the main range set can significantly be reduced. We have also paid attention to relax the nature of sharing of the range sets by the help of weighted sharing method. Thus the purpose of the paper is to improve all the above theorems in two directions at the expense of suitable choice of the sets.

To this end, we next suppose that $Q(z)$ is defined by

$$Q(z) = az^n + bz^{2m} + cz^m + d,$$

where $n, m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{C}^*$ be such that $n > 2m$, $\gcd(n, m) = 1$, $\frac{c^2}{4bd} = \frac{n(n-2m)}{(n-m)^2} \neq 1$ and $a \notin \left\{\gamma_i, \frac{\gamma_i}{2}\right\}$; $\gamma_i = -\frac{(2mbe_i^{2m} + cme_i^m)}{ne_i^n}$ with e_i be the roots of the equation

$$z^m = -\frac{2nd}{(n-m)c}.$$

One can easily check that the polynomial $Q(z)$ has distinct zeros and let them be $\theta_1, \theta_2, \ldots, \theta_n$. Clearly, for any zero 's' of $Q'(z)$ we have $nas^{n-1} + 2mbs^{2m-1} + cms^{m-1} = 0$. i.e., $as^n = -\frac{(2mbs^{2m} + cms^m)}{n}$.

So for $s = 0$,

$$Q(0) = d \neq 0$$

and hence for $s \neq 0$

$$\begin{aligned} Q(s) &= -\frac{(2mbs^{2m} + cms^m)}{n} + bs^{2m} + cs^m + d \\ &= \frac{(n-2m)bs^{2m} + (n-m)cs^m + nd}{n} \\ &= \frac{c^2(n-m)^2 s^{2m} + 4cdn(n-m)s^m + 4d^2n^2}{4nd^2} \\ &= \frac{(c(n-m)s^m + 2dn)^2}{4dn^2}. \end{aligned}$$

So, 's' is a zero of $Q(z)$, if $s^m = \frac{-2nd}{(n-m)c}$ i.e., if $s \in \{e_1, e_2, \ldots, e_m\}$. But then we would have $ae_i^n = -\frac{(2mbe_i^{2m} + cme_i^m)}{n}$ for $i \in \{1, 2, \ldots, m\}$, which is a contradiction as $a \neq \gamma_i = -\frac{(2mbe_i^{2m} + cme_i^m)}{ne_i^n}$. Hence, $Q(z)$ has only simple zeros.

Let $\mathcal{T}(z) = bz^{2m} + cz^m + d$. We claim that all the roots $\alpha_1, \alpha_2, \ldots, \alpha_{2m}$ (say) of the polynomial $\mathcal{T}(z)$ are simple. Now $\mathcal{T}'(z) = 0$ implies

$$mz^{m-1}(2bz^m + c) = 0. \tag{1.1}$$

We see that 0 and the roots $\delta_j (j = 1, 2, \ldots, m)$ of the polynomial of $2bz^m + c$ are the only roots of the polynomial $\mathcal{T}'(z)$. Again we see that $\mathcal{T}(0) = d \neq 0$ and since $c^2 \neq 4bd$

$$\mathcal{T}(\delta_j) = b\delta_j^{2m} + c\delta_j^m + d = b\left(-\frac{c}{2b}\right)^2 + c\left(-\frac{c}{2b}\right) + d = \frac{4bd - c^2}{4b} \neq 0.$$

Thus we conclude that all the zeros of $\mathcal{T}(z) = 0$ are simple.

Next, we see that

$$\begin{aligned}\mathcal{R}'(z) &= -\frac{(bz^{2m}+cz^m+d)naz^{n-1}-az^n(2mbz^{2m-1}+mcz^{m-1})}{(bz^{2m}+cz^m+d)^2}\\ &= -\frac{az^{n-1}b(n-2m)\left(z^m+\dfrac{c(n-m)}{2b(n-2m)}\right)^2}{(bz^{2m}+cz^m+d)^2}\\ &= -\frac{az^{n-1}b(n-2m)\left(z^m+\dfrac{2nd}{c(n-m)}\right)^2}{(bz^{2m}+cz^m+d)^2}\\ &= -\frac{az^{n-1}b(n-2m)\prod_{i=1}^{m}(z-e_i)^2}{(bz^{2m}+cz^m+d)^2}\end{aligned}$$

Also we see that

$$\begin{aligned}\mathcal{R}(z)-\frac{a}{\gamma_i} &= -a\frac{\gamma_i z^n+bz^{2m}+cz^m+d}{\gamma_i(bz^{2m}+cz^m+d)}\\ &= -a\frac{\Gamma(z)}{\gamma_i(bz^{2m}+cz^m+d)},\end{aligned}$$

where $\Gamma(z)=\gamma_i z^n+bz^{2m}+cz^m+d$. One can easily check that $\Gamma(e_i)=0$, $\Gamma'(e_i)=0$ and $\Gamma''(e_i)=0$ but $\Gamma^{(k)}(e_i)\neq 0$ for $3\leq k<n$. Therefore we get

$$\mathcal{R}(z)-\frac{a}{\gamma_i} = -a\frac{\prod_{i=1}^{m}(z-e_i)^3\,\Delta_{n-3m}(z)}{b\prod_{i=1}^{2m}(z-\alpha_i)},$$

where $\Delta_{n-3m}(z)$ is a polynomial of degree $n-3m$.

Theorem 1.8. *Let* $\mathcal{S}_1=\left\{z:\mathcal{Q}(z)=0\right\}$, $\mathcal{S}_2=\left\{e_1,e_2,\ldots,e_m\right\}\cup\{\infty\}$, *where* $n\geq 2m+3$, $a,b,c,d\in\mathbb{C}^*$, $\dfrac{c^2}{4bd}=\dfrac{n(n-2m)}{(n-m)^2}\neq 1$, $gcd(n.m)=1$, $a\notin\left\{\gamma_i,\dfrac{\gamma_i}{2}\right\}$. *Let $f(z)$ be a finite order meromorphic function satisfying*

(i) $E_{f(z)}(\mathcal{S}_1,3)=E_{f(z+\omega)}(\mathcal{S}_1,3)$ *and* $E_{f(z)}(S_2,1)=E_{f(z+\omega)}(S_2,1)$, *or*

(ii) $E_{f(z)}(\mathcal{S}_1,2)=E_{f(z+\omega)}(\mathcal{S}_1,2)$ *and* $E_{f(z)}(S_2,2)=E_{f(z+\omega)}(S_2,2)$,

then $f(z)\equiv f(z+\omega)$.

Next for the sake of convenience for $n\geq 3$, $c,d\in\mathbb{C}^*$, we define

$$\delta^n_{c,d}=\frac{1}{n-2}\left(\frac{n-1}{2d}\right)^{n-1}\left(\frac{c}{n}\right)^n,$$

for $n\geq 3$, $c,d\in\mathbb{C}^*$.

Putting $m=1$ in *Theorem 1.8*, we can easily deduce the following corollary.

Corollary 1.9. *Let* $\mathcal{S}_1=\left\{z:az^n+bz^2+cz+d=0\right\}$, $\mathcal{S}_2=\left\{-\dfrac{2nd}{(n-1)c}\right\}\cup\{\infty\}$, *where* $n\geq 5$, $a,b,c,d\in\mathbb{C}^*$, $\dfrac{c^2}{4bd}=\dfrac{n(n-2)}{(n-1)^2}\neq 1$, $a\notin\left\{\delta^n_{c,d},\dfrac{\delta^n_{c,d}}{2}\right\}$. *Let $f(z)$ be a finite order meromorphic function satisfying*

(i) $E_{f(z)}(\mathcal{S}_1,3)=E_{f(z+\omega)}(\mathcal{S}_1,3)$ *and* $E_{f(z)}(\mathcal{S}_2,1)=E_{f(z+\omega)}(\mathcal{S}_2,1)$, *or*

(ii) $E_{f(z)}(S_1,2)=E_{f(z+\omega)}(S_1,2)$ *and* $E_{f(z)}(S_2,2)=E_{f(z+\omega)}(S_2,2)$,

then $f(z)\equiv f(z+\omega)$.

2. Auxiliary definitions and lemmas

Though the standard definitions and notations for the value distribution are available in [11], we now explain here notations which are used throughout the paper.

Definition 2.1. *[12] Let $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f. For a positive integer k, we denote by $N(r, a; f| \geq k)(N(r, a; f| \leq k))$ the counting function of those a-points of f whose multiplicities are not less (greater) than k, where each a-point is counted according to its multiplicity.*

$\overline{N}(r, a; f| \geq k)(\overline{N}(r, a; f| \leq k))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r, a; f| > k)$, $N(r, a; f| < k)$, $\overline{N}(r, a; f| > k)$ and $\overline{N}(r, a; f| < k)$ are defined analogously.

Theorem 2.2. *Let f and g be two non-constant meromorphic functions such that f and g share (a, k) where $a \in \mathbb{C}\cup\{\infty\}$. Let z_0 be a a-point of f with multiplicity p, a a-point of g of multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g where $p > q$, by $\overline{N}_E^{(k+1}(r, a; f)$ the counting function of those a-points of f and g where $p = q \geq k + 1$; each point in these counting function is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$, $\overline{N}_E^{(k+1}(r, a; g)$. It is clear that $\overline{N}_E^{(k+1}(r, a; f) = \overline{N}_E^{(k+1}(r, a; g)$.*

Definition 2.3. *[13, 14] Let f and g share a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Next, we are going to discuss the lemmas which will be needed in sequel. Given meromorphic functions $f(z)$ and $f(z + \omega)$, we define $\mathcal{F}, \mathcal{G}$

$$\mathcal{F} = \mathcal{R}(f), \quad \mathcal{G} = \mathcal{R}(f(z + \omega)), \text{ where } \mathcal{R}(z) = -\frac{az^n}{bz^{2m} + cz^m + d}. \tag{2.1}$$

and to $\mathcal{F}, \mathcal{G}$ we associate $\mathcal{H}$ and Ψ by the following formulas

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F} - 1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G} - 1}\right), \tag{2.2}$$

$$\Psi = \left(\frac{\mathcal{F}'}{\mathcal{F} - 1} - \frac{\mathcal{F}'}{\mathcal{F}}\right) - \left(\frac{\mathcal{G}'}{\mathcal{G} - 1} - \frac{\mathcal{G}'}{\mathcal{G}}\right) = \frac{\mathcal{F}'}{\mathcal{F}(\mathcal{F} - 1)} - \frac{\mathcal{G}'}{\mathcal{G}(\mathcal{G} - 1)}. \tag{2.3}$$

Lemma 2.4. *[15] Let f be a non-constant meromorphic function and let*

$$\mathcal{R}^{\#}(f) = \frac{\sum_{i=1}^{n} a_i f^i}{\sum_{j=1}^{m} b_j f^j},$$

be an irreducible rational function in f with constant coefficients $\{a_i\}$, $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, \mathcal{R}^{\#}(f)) = \max\{n, m\}\, T(r, f) + S(r, f).$$

Lemma 2.5. *Let $\mathcal{F}$, $\mathcal{G}$ be given by (2.1) and $\mathcal{S}_1$, $\mathcal{S}_2$ be defined as in Theorem 1.8 with $\mathcal{H} \not\equiv 0$. If $E_{f(z)}(\mathcal{S}_1, q) = E_{f(z+\omega)}(\mathcal{S}_1, q)$ and $E_{f(z)}(\mathcal{S}_2, k) = E_{f(z+\omega)}(\mathcal{S}_2, k)$, where $1 \leq q < \infty$, $0 \leq k < \infty$, then*

$$\begin{aligned}
& (3k + 2) \left[\sum_{i=1}^{m} \overline{N}(r, e_i; f \,|\geq k + 1) + \overline{N}(r, \infty; f \,|\geq k + 1)\right] \\
\leq\ & \overline{N}(r, 0; f(z)) + \overline{N}(r, 0; f(z + \omega)) + \overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + S(r, f(z)) + S(r, f(z + \omega)).
\end{aligned}$$

Proof. We shall now discuss the following cases.
Case 1. Let if possible $\Psi \not\equiv 0$. Let z_0 be a pole or a 'e_i'-point ($i \in \{1, 2, \ldots, m\}$) of $f(z)$ of multiplicity p_i. Since $E_f(\mathcal{S}_2, k) = E_{f(z+\omega)}(\mathcal{S}_2, k)$, then that would be a zero of Ψ of multiplicity at least $min\{(n-2m)p_i - 1, 3p_i - 1\} = 3p_i - 1$ when $p_i \leq k$ and is of multiplicity at least $min\{(n-2m)(k+1) - 1, 3k+2\} = 3k+2$, when $p_i > k$. Therefore in view of the definition of Ψ, we get that

$$\begin{aligned}
& (3k+2)\left[\sum_{i=1}^{m} \overline{N}(r, e_i; f \mid\geq k+1) + \overline{N}(r, \infty; f \mid\geq k+1)\right] \\
\leq\ & \overline{N}(r, 0; \Psi) \\
\leq\ & \overline{N}(r, \infty; \Psi) + S(r, f(z)) + S(r, f(z+\omega)) \\
\leq\ & \overline{N}(r, 0; f(z)) + \overline{N}(r, 0; f(z+\omega)) + \overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + S(r, f(z)) + S(r, f(z+\omega)).
\end{aligned}$$

Case 2. Let Ψ=0. Then after integrating, we have

$$\frac{\mathcal{F}-1}{\mathcal{F}} \equiv \mathcal{A}\frac{\mathcal{G}-1}{\mathcal{G}}, \tag{2.4}$$

where $\mathcal{A}(\neq 0) \in \mathbb{C}$. Clearly in view of *Lemma 2.4*, from (2.4), we have

$$T(r, f(z)) = T(r, f(z+\omega)) + S(r, f(z+\omega)). \tag{2.5}$$

It is obvious that $\mathcal{A} \neq 1$, otherwise we would have $\mathcal{F} \equiv \mathcal{G}$, which implies $\mathcal{H} \equiv 0$.
From (2.4), we get

$$1 - \frac{1}{\mathcal{F}} \equiv \mathcal{A}\left(1 - \frac{1}{\mathcal{G}}\right). \tag{2.6}$$

After rewriting (2.6), we get

$$\frac{\mathcal{A}\mathcal{F}}{\mathcal{F}(\mathcal{A}-1)+1} = \mathcal{G} \tag{2.7}$$

and consider the following subcases.
Subcase 2.1. Let us consider $f(z)$ and $f(z+\omega)$ share $(\infty, 0)$. Then we discuss the following subcases.
Subcase 2.1.1. Let if possible '∞' is an e.v.P of both $f(z)$ and $f(z+\omega)$.
Subcase 2.1.1.1. Suppose $\dfrac{-1}{\mathcal{A}-1} \neq \dfrac{a}{\gamma_i}$. Then $\mathcal{F}(\mathcal{A}-1)+1$ has only simple zeros (say) $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ i.e., we get from (2.7) that

$$\frac{\mathcal{A}f^n}{(\mathcal{A}-1)\prod_{j=1}^{n}(f-\zeta_j)} = \frac{f^n(z+\omega)}{b\prod_{i=1}^{2m}(f(z+\omega)-\alpha_i)}. \tag{2.8}$$

Now by using *Second Fundamental Theorem* for finite order meromorphic functions and *Lemma 2.4*, (2.5), we get

$$\begin{aligned}
(n-1)T(r, f(z)) &\leq \sum_{i=1}^{n} \overline{N}(r, \zeta_i; f) + \overline{N}(r, \infty; f) + S(r, f) \\
&\leq \sum_{i=1}^{2m} \overline{N}(r, \alpha_i; f(z+\omega)) + S(r, f) \\
&\leq 2m\ T(r, f(z)) + S(r, f),
\end{aligned}$$

which contradics $n \geq 2m+3$.

Subcase 2.1.1.2. Suppose $\dfrac{-1}{\mathcal{A}-1} = \dfrac{a}{\gamma_i}$. Then we can rewrite (2.6) as

$$F = \frac{\mathcal{G}}{\mathcal{G}(1-\mathcal{A})+\mathcal{A}}. \tag{2.9}$$

It is obvious that $\dfrac{\mathcal{A}}{\mathcal{A}-1} \neq \dfrac{a}{\gamma_i}$. If so i.e., $\dfrac{\mathcal{A}}{\mathcal{A}-1} = \dfrac{a}{\gamma_i}$, then $\mathcal{A} = \dfrac{\frac{a}{\gamma_i}}{\frac{a}{\gamma_i}-1}$. Also we have $\dfrac{-1}{\mathcal{A}-1} = \dfrac{a}{\gamma_i}$ i.e., $\mathcal{A} = \dfrac{\frac{a}{\gamma_i}-1}{\frac{a}{\gamma_i}}$. So we get that $\dfrac{\frac{a}{\gamma_i}}{\frac{a}{\gamma_i}-1} = \dfrac{\frac{a}{\gamma_i}-1}{\frac{a}{\gamma_i}}$ and hence $\dfrac{a}{\gamma_i} = \dfrac{1}{2}$. i.e., $a = \dfrac{\gamma_i}{2}$ which is a contradiction.

Therefore $\mathcal{G}(1-\mathcal{A})+\mathcal{A}$ must have n distinct zeros say $\delta_1, \delta_2, \ldots, \delta_n$. Again from (2.9), we get $\overline{N}\left(r, \dfrac{\mathcal{A}}{\mathcal{A}-1}; \mathcal{G}\right) = \overline{N}(r,\infty;\mathcal{F}) = \displaystyle\sum_{i=1}^{2m} \overline{N}(r,\alpha_i; f(z))$.

By the *Second Fundamental Theorem* for the finite order meromorphic function $f(z+\omega)$ and using *Lemma 2.4* and (2.5), we have

$$\begin{aligned}
&(n-1)T(r, f(z+\omega)) \\
\leq\ & \sum_{i=1}^{n} \overline{N}(r,\delta_i; f(z+\omega)) + \overline{N}(r,\infty; f(z+\omega)) + S(r, f(z+\omega)) \\
\leq\ & \sum_{i=1}^{2m} \overline{N}(r,\alpha_1; f(z)) + S(r, f(z+\omega)) \\
\leq\ & 2m\ T(r, f(z+\omega)) + S(r, f(z+\omega)),
\end{aligned}$$

which is a contradiction for $n \geq 2m+3$.

Subcase 2.1.2. Let if possible '∞' is not an e.v.P of both $f(z)$ and $f(z+\omega)$.

Then proceeding exactly same way as done in *Subcases 2.1.1.1*, we get a contradiction for $n \geq 2m+3$.

Subcase 2.2. Let if possible $f(z)$ and $f(z+\omega)$ do not share $(\infty, 0)$.

So, there must exits at least one point z_0 such that $f(z_0) = e_i$, (i=1,2,...,n), $f(z_0+\omega) = \infty$, since otherwise '∞' will be an e.v.P of both $f(z)$ and $f(z+\omega)$ which would contradict the assumption of this subcase.

We omit the rest of the proof as the same can be done as in *Subcases 2.1.1.1*, to get a contradiction for $n \geq 2m+3$. □

Lemma 2.6. *[3] Let f, g be two meromorphic functions sharing $(1,q)$, where $1 \leq q < \infty$. Then*

$$\begin{aligned}
&\overline{N}(r,1;f) + \overline{N}(r,1;g) - N(r,1;f| = 1) + \left(q - \frac{1}{2}\right)\overline{N}_*(r,1;f,g) \\
\leq\ & \frac{1}{2}[N(r,1;f) + N(r,1;g)].
\end{aligned}$$

Next we define $\chi_n = 0$, for $n = 5$ and $\chi_n = 1$, for $n \neq 5$.

Lemma 2.7. *Let $\mathcal{F}$, $\mathcal{G}$ be given by (2.1) and $\mathcal{S}_1$, $\mathcal{S}_2$ be defined as in Theorem 1.8 with $\mathcal{H} \not\equiv 0$. If $E_{f(z)}(\mathcal{S}_1, q) = E_{f(z+\omega)}(\mathcal{S}_1, q)$, and $E_{f(z)}(\mathcal{S}_2, k) = E_{f(z+\omega)}(\mathcal{S}_2, k)$, where $0 \le q < \infty$, $0 \le k < \infty$, then*

$$\begin{aligned}
& N(r,1;\mathcal{F}| = 1) = N(r,1;\mathcal{G}| = 1)\\
\le\ & N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G})\\
\le\ & \overline{N}(r,0;f(z)) + \overline{N}(r,0;f(z+\omega)) + \sum_{i=1}^{m}\overline{N}(r,e_i;f(z)| \ge k+1) + \overline{N}(r,\infty;f(z)| \ge k+1)\\
& +\chi_n\left[\sum_{i=1}^{m}\overline{N}(r,e_i;f(z)| \le k) + \overline{N}(r,\infty;f(z)| \le k)\right] + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}_0(r,0;f'(z))\\
& +\overline{N}_0(r,0;f'(z+\omega)) + S(r,f(z)) + S(r,f(z+\omega)),
\end{aligned}$$

where $\overline{N}_0(r,0;f'(z))$ denotes the reduced counting function corresponding to the zeros of $f'(z)$ which are not the zeros of $f(z)\prod_{i=1}^{m}(f(z)-e_i)$ and $\mathcal{F}-1$, $\overline{N}_0(r,0;f'(z+\omega))$ is defined similarly.

Proof. Since $f(z)$, $f(z+\omega)$ share $(\mathcal{S}_1, q)$, hence $\mathcal{F}$ and $\mathcal{G}$ share $(1,q)$. Clearly any simple 1-point of $\mathcal{F}$ and $\mathcal{G}$ is a zero of $\mathcal{H}$. From the construction of $\mathcal{H}$, we know that $m(r,\mathcal{H}) = S(r,\mathcal{F}) + S(r,\mathcal{G})$. Therefore by the *First Fundamental Theorem*, we get

$$N_E^{1)}(r,1;\mathcal{F}) = N_E^{1)}(r,1;\mathcal{G}) \le N(r,0;\mathcal{H}) \le N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G}).$$

From (2.1) and (2.2), we see that

$$\frac{\mathcal{F}''}{\mathcal{F}'} = (n-1)\frac{f'(z)}{f(z)} + 2\sum_{i=1}^{m}\frac{f'(z)}{f(z)-e_i} + \frac{f''(z)}{f'(z)} - 2\sum_{i=1}^{2m}\frac{f'(z)}{f(z)-\alpha_i},$$

$$\frac{\mathcal{F}'}{\mathcal{F}-1} = \sum_{i=1}^{n}\frac{f'(z)}{f(z)-\theta_i} - \sum_{i=1}^{2m}\frac{f'(z)}{f(z)-\alpha_i}.$$

Similarly we obtain

$$\frac{\mathcal{G}''}{\mathcal{G}'} = (n-1)\frac{f'(z+\omega)}{f(z+\omega)} + 2\sum_{i=1}^{m}\frac{f'(z+\omega)}{f(z+\omega)-e_i} + \frac{f''(z+\omega)}{f'(z+\omega)} - 2\sum_{i=1}^{2m}\frac{f'(z+\omega)}{f(z+\omega)-\alpha_i},$$

$$\frac{\mathcal{G}'}{\mathcal{G}-1} = \sum_{i=1}^{n}\frac{f'(z+\omega)}{f(z+\omega)-\theta_i} - \sum_{i=1}^{2m}\frac{f'(z+\omega)}{f(z)-\alpha_i}.$$

Hence we get that

$$\begin{aligned}
\mathcal{H} =\ & \left((n-1)\frac{f'(z)}{f(z)} + 2\sum_{i=1}^{m}\frac{f'(z)}{f(z)-e_i} + \frac{f''(z)}{f'(z)} - 2\sum_{i=1}^{n}\frac{f'(z)}{f(z)-\theta_i}\right)\\
& -\left((n-1)\frac{f'(z+\omega)}{f(z+\omega)} + 2\sum_{i=1}^{m}\frac{f'(z+\omega)}{f(z+\omega)-e_i} + \frac{f''(z+\omega)}{f'(z+\omega)} - 2\sum_{i=1}^{n}\frac{f'(z+\omega)}{f(z+\omega)-\theta_i}\right).
\end{aligned}$$

Observe that

(i) if any e_i-point (for some i) of $f(z)$ is a e_i-point of $f(z+\omega)$ of multiplicity $p \le k$, then for the pole of $\mathcal{H}$, the contribution of this e_i-point of $f(z)$ and those of $f(z+\omega)$ will nullify each other as the construction of $\mathcal{H}$ is symmetrical in terms of $f(z)$ and $f(z+\omega)$.

(ii) if any pole of $f(z)$ is a pole of $f(z+\omega)$ of multiplicity $p \le k$, then also the above things happen again due to the symmetrical structure of $\mathcal{H}$.

(iii) if any e_i-point of $f(z)$(or $f(z+\omega)$) is a pole of $f(z+\omega)$(or $f(z)$) of multiplicity $p \le k$, then it contributes to the poles of $\mathcal{H}$ with co-efficient of the pole as $[5-n]p$. i.e., it contributes to the pole of $\mathcal{H}$ for all values of n except $n = 5$.

Since $\mathcal{H}$ has only simple poles, so the result is obvious by some simple calculation. □

Lemma 2.8. *Let $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1) and S_1, S_2 be defined as in Theorem 1.8, with $m = 1$ and $\mathcal{H} \neq 0$. If $\mathcal{F}$ and $\mathcal{G}$ share $(1, q)$ for $2 \le q < \infty$. Then*

$$\begin{aligned} & \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \\ \le\ & \frac{1}{2q-1}\Big[\overline{N}(r,0;f(z)) + \overline{N}(r,0;f(z+\omega))\Big] + S(r,f(z)) + S(r,f(z+\omega)). \end{aligned}$$

Proof. Using *Lemma 2.5* with $k = 0$, we get

$$\begin{aligned} & \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \\ \le\ & \frac{1}{q}\Big[\overline{N}(r,\infty;f(z)) + \overline{N}(r,e_1;f(z))\Big] + S(r,f(z)) \\ \le\ & \frac{1}{2q}\Big[\overline{N}(r,0;f(z)) + \overline{N}(r,0;f(z+\omega)) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G})\Big] + S(r,f(z)) + S(r,f(z+\omega)). \end{aligned}$$

i.e.,

$$\begin{aligned} & N_*(r,1;\mathcal{F},\mathcal{G}) \\ \le\ & \frac{1}{2q-1}\Big[\overline{N}(r,0;f(z)) + \overline{N}(r,0;f(z+\omega))\Big] + S(r,f(z)) + S(r,f(z+\omega)). \end{aligned}$$

□

Lemma 2.9. *Let $\mathcal{F}$, $\mathcal{G}$ be given by (2.1) and S_1, S_2 be defined as in Theorem 1.8 with $\mathcal{H} \not\equiv 0$. If $E_{f(z)}(\mathcal{S}_1, q) = E_{f(z+\omega)}(\mathcal{S}_1, q)$ and $E_{f(z)}(\mathcal{S}_2, k) = E_{f(z+\omega)}(\mathcal{S}_2, k)$, where $1 \le q < \infty$, $0 \le k < \infty$, then*

$$\begin{aligned} & \left(\frac{n}{2} + m - 1\right)\Big\{T(r,f(z)) + T(r,f(z+\omega))\Big\} \\ \le\ & \overline{N}(r,0;f(z)) + \overline{N}(r,0;f(z+\omega)) + 2\left(\overline{N}(r,\infty;f(z)) + \sum_{i=1}^{m}\overline{N}(r,e_i;f(z))\right) \\ & + \left[\sum_{i=1}^{m}\overline{N}(r,e_i;f(z)| \ge k+1) + \overline{N}(r,\infty;f(z)| \ge k+1)\right] \\ & + \chi_n\left[\sum_{i=1}^{m}\overline{N}(r,e_i;f(z)| \le k) + \overline{N}(r,\infty;f(z)| \le k)\right] - \left(q - \frac{3}{2}\right)\overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \\ & + S(r,f(z)) + S(r,f(z+\omega)). \end{aligned}$$

Proof. By the *Second Fundamental Theorem* for the finite order meromorphic functions $f(z)$, $f(z+\omega)$ and using

Lemma 2.4, we get

$$\begin{aligned}
& (n+m-1)\Big\{T(r,f(z))+T(r,f(z+\omega))\Big\} \qquad (2.10)\\
\leq\ & \overline{N}(r,1;\mathcal{F})+\sum_{i=1}^{m}\overline{N}(r,e_i;f(z))+\overline{N}(r,\infty;f(z))+\overline{N}(r,1;\mathcal{G})\\
& +\sum_{i=1}^{m}\overline{N}(r,e_i;f(z+\omega))+\overline{N}(r,\infty;f(z+\omega))-N_0(r,0;f'(z))-N_0(r,0;f'(z))\\
& +S(r,f(z))+S(r,f(z+\omega)).
\end{aligned}$$

Using *Lemmas 2.6* and *2.7*, we see that

$$\begin{aligned}
& \overline{N}(r,1;\mathcal{F})+\overline{N}(r,1;\mathcal{G}) \qquad (2.11)\\
\leq\ & \frac{1}{2}\Big\{N(r,1;\mathcal{F})+N(r,1;\mathcal{G})\Big\}+N(r,1;\mathcal{F}\mid=1)-\Big(q-\frac{1}{2}\Big)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
\leq\ & \frac{n}{2}\Big\{T(r,f(z))+T(r,f(z+\omega))\Big\}+\overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+\omega))\\
& +\left[\sum_{i=1}^{m}\overline{N}(r,e_i;f(z)|\geq k+1)+\overline{N}(r,\infty;f(z)|\geq k+1)\right]\\
& +\chi_n\left[\sum_{i=1}^{m}\overline{N}(r,e_i;f(z)|\leq k)+\overline{N}(r,\infty;f(z)|\leq k)\right]+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
& -\Big(q-\frac{1}{2}\Big)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_0(r,0;f'(z))+\overline{N}_0(r,0;f'(z))+S(r,f(z))\\
& +S(r,f(z+\omega))\\
\leq\ & \frac{n}{2}\Big\{T(r,f(z))+T(r,f(z+\omega))\Big\}+\overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+\omega))\\
& +\sum_{i=1}^{m}\overline{N}(r,e_i;f|\geq k+1)+\overline{N}(r,\infty;f(z)|\geq k+1)\\
& +\chi_n\left[\sum_{i=1}^{m}\overline{N}(r,e_i;f(z)|\leq k)+\overline{N}(r,\infty;f(z)|\leq k)\right]-\Big(q-\frac{3}{2}\Big)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\
& +\overline{N}_0(r,0;f'(z))+\overline{N}_0(r,0;f'(z))+S(r,f(z))+S(r,f(z+\omega)).
\end{aligned}$$

Since $f(z)$, $f(z+\omega)$ share (S_2,k), so we must have

$$\overline{N}(r,\infty;f(z))+\sum_{i=1}^{m}\overline{N}(r,e_i;f)=\overline{N}(r,\infty;f(z+\omega))+\sum_{i=1}^{m}\overline{N}(r,e_i;f(z+\omega)).$$

Using this fact and putting (2.11) in (2.10), we get the lemma. □

Lemma 2.10. ([4], Lemma 2.6) *Let* $\Phi(z)=\lambda(1-z^{n-m})^2-\mu(1-z^{n-2m})(1-z^n)$, *where* $\lambda,\mu\in\mathbb{C}-\{0\}$, $\dfrac{\lambda}{\mu}=\dfrac{n(n-2m)}{(n-m)^2}$, *then* $\phi(z)$ *has exactly one multiple zero of multiplicity* 4, *which is* 1. *i.e.,*

$$\Phi(z)=(z-1)^4\prod_{k=1}^{2n-2m-4}(z-\sigma_k),$$

where $\sigma_i\neq\sigma_j$, *for* $i\neq j$, $\sigma_k\in\mathbb{C}-\{0,1\}$, *for* $i,j\in\{1,2,\ldots,2n-2m-4\}$.

3. Proof of the theorem

Proof. [Proof of Theorem 1.8] Suppose $\mathcal{F}$ and $\mathcal{G}$ be given by (2.1). Since $E_{f(z)}(\mathcal{S}_1, q) = E_{f(z+\omega)}(\mathcal{S}_1, q)$ from (2.1) it follows that $\mathcal{F}$ and $\mathcal{G}$ share $(1, q)$.
Case 1. Suppose $\mathcal{H} \not\equiv 0$.
Case 1.1. Let us first suppose that $n = 2m + 3$.
Case 1.1.1. Let $m = 1$. Then from the definition we see that $\chi_n = 0$. So using *Lemma 2.5* in *Lemma 2.9* for $n = 5$ we get

$$\begin{aligned}
&\frac{5}{2}\Big\{T(r, f(z)) + T(r, f(z+\omega))\Big\} \qquad (3.1)\\
\leq\ & \overline{N}(r, 0; f(z)) + \overline{N}(r, 0; f(z+\omega)) + 2\Big[\overline{N}(r, \infty; f(z)) + \overline{N}(r, e_1; f(z))\Big]\\
&+\overline{N}(r, \infty; f(z)| \geq k+1) + \overline{N}(r, e_1; f(z)| \geq k+1) - \Big(q - \frac{3}{2}\Big)\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G})\\
&+S(r, f(z)) + S(r, f(z+\omega))\\
\leq\ & \overline{N}(r, 0; f(z)) + \overline{N}(r, 0; f(z+\omega)) + \overline{N}(r, \infty; f(z)| \geq k+1) - \Big(q - \frac{3}{2}\Big)\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G})\\
&+2\left[\frac{1}{2}\Big\{\overline{N}(r, 0; f(z)) + \overline{N}(r, 0; f(z+\omega)) + \overline{N}_*(r, 1; \mathcal{F}, \mathcal{G})\Big\}\right]\\
&+\sum_{i=1}^{m}\overline{N}(r, e_i; f(z)| \geq k+1) + S(r, f(z)) + S(r, f(z+\omega))\\
\leq\ & \left[2 + \frac{1}{3k+2}\right]\Big[\overline{N}(r, 0; f(z)) + \overline{N}(r, 0; f(z+\omega))\Big] + \left[1 + \frac{1}{3k+2} - q + \frac{3}{2}\right]\times\\
&\overline{N}_*(r, 1; \mathcal{F}, \mathcal{G}) + S(r, f(z)) + S(r, f(z+\omega)).
\end{aligned}$$

Next for $q = 3$, $k = 1$; (3.1) implies that

$$\begin{aligned}
&\frac{5}{2}\Big\{T(r, f(z) + T(r, f(z+\omega)\Big\}\\
\leq\ & \left[2 + \frac{1}{5}\right]\Big\{T(r, f(z)) + T(r, f(z+\omega))\Big\} + S(r, f(z)) + S(r, f(z+\omega)),
\end{aligned}$$

which is not possible.
For $q = 2$, $k = 2$, using *Lemma 2.8* in (3.1), we get

$$\begin{aligned}
&\frac{5}{2}\Big\{T(r, f(z)) + T(r, f(z+\omega))\Big\}\\
\leq\ & \Big(2 + \frac{1}{3k+2}\Big)\Big\{T(r, f(z)) + T(r, f(z+\omega))\Big\}\\
&+\Big(1 + \frac{1}{3k+2} - q + \frac{3}{2}\Big)\frac{1}{(2q-1)}\Big\{T(r, f(z)) + T(r, f(z+\omega))\Big\} + S(r, f(z))\\
&+S(r, f(z+\omega))\\
\leq\ & \Big(2 + \frac{1}{3}\Big)\Big\{T(r, f(z)) + T(r, f(z+\omega))\Big\} + S(r, f(z)) + S(r, f(z+\omega)),
\end{aligned}$$

which is again a contradiction.
Case 1.1.2. Let $m \geq 2$.

Then we see that $n \geq 7$. So it is clear that $\chi_n = 1$. Hence proceeding in the same way as above and using *Lemma 2.5*, *Lemma 2.8* in *Lemma 2.9*, we get

$$\begin{aligned}
&\left(\frac{n}{2}+m-1\right)\left\{T(r,f(z))+T(r,f(z+\omega))\right\} \\
\leq\;& \overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+\omega))+3\left[\overline{N}(r,\infty;f(z))+\sum_{i=1}^{m}\overline{N}(r,e_i;f(z))\right] \\
&-\left(q-\frac{3}{2}\right)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+S(r,f(z))+S(r,f(z+\omega)) \\
\leq\;& \frac{5}{2}\left\{\overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+\omega))\right\}+(3-q)\overline{N}_*(r,1;\mathcal{F},\mathcal{G}),
\end{aligned}$$

which is a contradiction for $(q,k)=(3,1)$.

When $(q,k)=(2,2)$, noting that

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \leq \frac{1}{3}\left\{\sum_{i=1}^{m}\overline{N}(r,e_i;f(z))+\sum_{i=1}^{m}\overline{N}(r,e_i;f(z+\omega))\right\},$$

we again get a contradiction.

Case 2 Let $\mathcal{H} \equiv 0$. So we get

$$\frac{1}{\mathcal{F}-1}=\frac{\mathcal{A}}{\mathcal{G}-1}+\mathcal{B}, \tag{3.2}$$

where $\mathcal{A}(\neq 0),\mathcal{B}$ are complex constants. In view of *Lemma 2.4,* obviously (3.2) implies

$$T(r,f(z))=T(r,f(z+\omega))+S(r,f(z+\omega)). \tag{3.3}$$

Now we can write (3.2) as

$$\mathcal{F}=\frac{(\mathcal{B}+1)\mathcal{G}+\mathcal{A}-\mathcal{B}-1}{\mathcal{B}\mathcal{G}+\mathcal{A}-\mathcal{B}}. \tag{3.4}$$

Hence let us consider the following subcases.

Subcase-2.1. Let $\mathcal{B} \neq 0$.

Subcase 2.1.1. Let $\mathcal{B} \neq -1$.

Subcase 2.1.1.1. Let $\mathcal{A}-\mathcal{B}-1 \neq 0$. Obviously $\dfrac{\mathcal{A}-\mathcal{B}-1}{\mathcal{B}+1} \neq \dfrac{\mathcal{A}-\mathcal{B}}{\mathcal{B}}$. For if $\dfrac{\mathcal{A}-\mathcal{B}-1}{\mathcal{B}+1}=\dfrac{\mathcal{A}-\mathcal{B}}{\mathcal{B}}$, then $1=0$, which is absurd. Therefore

$$\overline{N}\left(r,\frac{\mathcal{B}+1-\mathcal{A}}{\mathcal{B}+1};\mathcal{G}\right)=\overline{N}(r,0;\mathcal{F}). \tag{3.5}$$

Now we consider the following subcases.

Subcase 2.1.1.1.1. Suppose $\dfrac{\mathcal{B}+1-\mathcal{A}}{\mathcal{B}+1} \neq \dfrac{a}{\gamma_i}$. Then $\mathcal{G}-\dfrac{\mathcal{B}+1-\mathcal{A}}{\mathcal{B}+1}$ has n distinct simple zeros λ_i (say) and from (3.4) we get each of these zeros is of multiplicity at least n. Therefore using (3.3) and the *Second Fundamental Theorem* for the finite order meromorphic function $f(z+\omega)$ and using *Lemma 2.4*, we get

$$\begin{aligned}
(n-2)T(r,f(z+\omega)) &\leq \sum_{i=1}^{n}\overline{N}(r,\lambda_i;f(z+\omega))+S(r,f(z+\omega)) \\
&\leq T(r,f(z+\omega))+S(r,f(z+\omega)),
\end{aligned}$$

which is a contradiction for $n \geq 2m+3$.

Subcase 2.1.1.1.2. Suppose $\dfrac{\mathcal{B}+1-\mathcal{A}}{\mathcal{B}+1} = \dfrac{a}{\gamma_i}$. Then

$$\mathcal{G} - \frac{\mathcal{B}+1-\mathcal{A}}{\mathcal{B}+1} = \frac{a\prod_{i=1}^{m}(f(z+\omega)-e_i)^3\prod_{k=1}^{n-3m}(f(z+\omega)-\beta_k)}{b\prod_{j=1}^{2m}(f(z+\omega)-\alpha_j)} \tag{3.6}$$

where β_k and e_i are the distinct zeros of $\mathcal{G} - \dfrac{\mathcal{B}+1-\mathcal{A}}{\mathcal{B}+1}$. Since $f(z)$ and $f(z+\omega)$ share (S_2, k), therefore with the help of (3.5) and (3.6), we get from (3.4) that $\overline{N}(r, e_i; f(z+\omega)) = S(r, f(z+\omega))$ for $i = 1, 2, \ldots, 2m$ and each β_k point of $f(z+\omega)$ is of multiplicity at least n.

Hence using (3.3) and the *Second Fundamental Theorem* for the finite order meromorphic function $f(z+\omega)$ and using *Lemma 2.4*, we get that

$$\begin{aligned}
& (n-2m-2)T(r, f(z+\omega)) \\
\leq\ & \sum_{i=1}^{m}\overline{N}(r, e_i; f(z+\omega)) + \sum_{i=1}^{n-3m}\overline{N}(r, \beta_i; f(z+\omega)) + S(r, f(z+\omega)) \\
\leq\ & \frac{(n-3m)}{n}T(r, f(z+\omega)) + S(r, f(z+\omega)),
\end{aligned}$$

which is a contradiction for $n \geq 2m+3$.

Subcase 2.1.1.2. Let $\mathcal{A} - \mathcal{B} - 1 = 0$. Then (3.4) reduces to

$$\mathcal{F} = \frac{(\mathcal{B}+1)\mathcal{G}}{\mathcal{B}\mathcal{G}+1}. \tag{3.7}$$

Subcase 2.1.1.2.1. Let $\dfrac{1}{\mathcal{B}} = -\dfrac{a}{\gamma_i}$, for some $i \in \{1, 2, \ldots, m\}$. Now we can rewrite (3.7) as

$$\mathcal{G} = \frac{\mathcal{F}}{\mathcal{B}+1-\mathcal{B}\mathcal{F}}. \tag{3.8}$$

Obviously $\dfrac{\mathcal{B}+1}{\mathcal{B}} \neq \dfrac{a}{\gamma_i}$. For if $\dfrac{\mathcal{B}+1}{\mathcal{B}} = \dfrac{a}{\gamma_i}$ i.e., $\dfrac{\mathcal{B}+1}{\mathcal{B}} = -\dfrac{1}{\mathcal{B}}$ i.e., $\mathcal{B} = -2$ i.e., $\dfrac{a}{\gamma_i} = \dfrac{1}{2}$ i.e., $a = \dfrac{\gamma_i}{2}$, which is a contradiction. Therefore

$$\mathcal{B}\mathcal{F} - (\mathcal{B}+1) = \mathcal{B}\frac{a\prod_{i=1}^{n}(f(z)-\rho_i)}{b\prod_{i=1}^{2m}(f(z)-\alpha_i)}. \tag{3.9}$$

where ρ_i's are distinct zeros of $\mathcal{B}\mathcal{F} - (\mathcal{B}+1)$ for $i = 1, 2, \ldots, n$. From (3.8) clearly $\overline{N}(r, \infty; \mathcal{G}) = \overline{N}(r, 0; \mathcal{B}+1-\mathcal{B}\mathcal{F})$. i.e.,

$$\overline{N}(r, \infty; f(z+\omega)) + \sum_{i=1}^{2m}\overline{N}(r, \alpha_i; f(z+\omega)) = \sum_{i=1}^{n}\overline{N}(r, \rho_i; f). \tag{3.10}$$

Since $f(z)$ and $f(z+\omega)$ share (S_2,k), therefore from (3.9) and (3.10), we get '∞' is an e.v.P. of $f(z+\omega)$ and hence (3.10) reduces to

$$\sum_{i=1}^{2m} \overline{N}(r,\alpha_i;f(z+\omega)) = \sum_{i=1}^{n} \overline{N}(r,\rho_i;f). \tag{3.11}$$

Hence by the *Second Fundamental Theorem* for the finite order meromorphic function $f(z)$ and *Lemma 2.4*, (3.3), we get

$$\begin{aligned}(n-2)T(r,f(z)) &\leq \sum_{i=1}^{n} \overline{N}(r,\rho_i;f(z)) + S(r,f(z))\\ &\leq \sum_{i=1}^{2m} \overline{N}(r,\alpha_i;f(z+\omega)) + S(r,f(z)),\end{aligned}$$

which is a contradiction for $n \geq 2m+3$.

Subcase 2.1.1.2.2. Let $\dfrac{1}{\mathcal{B}} \neq -\dfrac{a}{\gamma_i}$. Then from (3.7), we get poles of $f(z)$ are e.v.P. Therefore $\sum_{i=1}^{2m} \overline{N}(r,\alpha_i;f) = \overline{N}(r,\infty;\mathcal{F}) = \overline{N}\left(r,\dfrac{-1}{\mathcal{B}};\mathcal{G}\right) = \sum_{i=1}^{n} \overline{N}(r,\eta_i;f(z+\omega))$ where $(f(z+\omega)-\eta_i)$ are distinct factors of $\mathcal{G}+\dfrac{1}{\mathcal{B}}$.

Hence using the *Second Fundamental Theorem* for the finite order meromorphic function $f(z+\omega)$ and (3.3) we get

$$\begin{aligned}(n-2)T(r,f(z+\omega)) &\leq \sum_{i=1}^{n} \overline{N}(r,\eta_i;f(z+\omega)) + S(r,f(z+\omega))\\ &\leq \sum_{i=1}^{2m} \overline{N}(r,\alpha_i;f(z)) + S(r,f(z+\omega)),\end{aligned}$$

which is a contradiction $n \geq 2m+3$.

Subcase 2.1.2. Let $\mathcal{B} = -1$. So from (3.4), we get

$$\mathcal{F} = \frac{\mathcal{A}}{-\mathcal{G}+\mathcal{A}+1}. \tag{3.12}$$

Obviously poles of $\mathcal{G}$ are zeros of $\mathcal{F}$. Since poles of $f(z+\omega)$ are poles of $\mathcal{G}$ and we have $f(z)$ and $f(z+\omega)$ share (S_2,k) therefore from (3.12) it is clear that ∞ is an e.v.P. of $f(z+\omega)$. Other poles of $\mathcal{G}$ are α_i $(i=1,2,\ldots,2m)$ points of $f(z+\omega)$. From (3.12) each α_i point of $f(z+\omega)$ is of multiplicity at least n.

Therefore using the *Second Fundamental Theorem* for the finite order meromorphic function $f(z+\omega)$ and *Lemma 2.4*, (3.3), we get

$$\begin{aligned}(2m-1)T(r,f(z+\omega)) &\leq \overline{N}(r,\infty;f(z+\omega)) + \sum_{i=1}^{2m} \overline{N}(r,\alpha_1;f(z+\omega)) + S(r,f(z+\omega))\\ &\leq \frac{2m}{n}T(r,f(z+\omega)) + S(r,f(z+\omega)),\end{aligned}$$

which is a contradiction for $n \geq 2m+3$.

Subcase 2.2. Let $\mathcal{B} = 0$. Then (3.2) implies

$$\frac{1}{\mathcal{F}-1} = \frac{\mathcal{A}}{\mathcal{G}-1} \tag{3.13}$$

i.e.,

$$\mathcal{AF} = \mathcal{G} + \mathcal{A} - 1. \tag{3.14}$$

Subcase 2.2.1. Suppose that $\mathcal{A} - 1 \neq 0$. Then this case can be dealt in a similar fashion as of *Subcase 2.1.1.1*.
Subcase 2.2.2. Suppose that $\mathcal{A} - 1 = 0$. Then we get $\mathcal{F} \equiv \mathcal{G}$. i.e.,

$$\frac{f^n(z)}{bf^{2m}(z) + cf^m(z) + d} \equiv \frac{f^n(z+\omega)}{bf^{2m}(z+\omega) + cf^m(z+\omega) + d}. \tag{3.15}$$

Let $h(z) = \dfrac{f(z+\omega)}{f(z)}$.
Subcase 2.2.2.1. Let $h(z)$ be non-constant. Then we see that (3.15) reduces to

$$\begin{aligned} & bf^{2m}(z)h^{2m}(z)\left(1 - h^{n-2m}(z)\right) + cf^m(z)h^m(z)\left(1 - h^{n-m}(z)\right) \\ & + d\left(1 - h^n(z)\right) = 0 \end{aligned} \tag{3.16}$$

and which in turn takes the following form by applying *Lemma 2.10* with $\lambda = c^2$, $\mu = 4bd$

$$\begin{aligned} & \left\{bf^m(z)h^m(z)\left(1 - h^{n-2m}(z)\right) + \frac{c}{2}\left(1 - h^{n-m}(z)\right)\right\}^2 \\ = & \frac{c^2\left(1 - h^{n-m}(z)\right)^2 - 4bd\left(1 - h^{n-2m}(z)\right)\left(1 - h^n(z)\right)}{4} \\ = & \frac{\Phi(h)}{4} \\ = & \frac{(h(z)-1)^4 \displaystyle\prod_{k=1}^{2n-2m-4}(h(z) - \sigma_k)}{4}. \end{aligned} \tag{3.17}$$

From (3.15), we see that $h(z)$ has no pole i.e., we have $\overline{N}(r, \infty; h(z)) = 0$.

Now by the *Second Fundamental Theorem* for the finite order meromorphic function and *Lemma 2.4*, from (3.17), we get

$$\begin{aligned} (2n - 2m - 4)T(r, h) & \leq \overline{N}(r, 0; h) + \overline{N}(r, \infty; h) + \sum_{k=1}^{2n-2m-4} \overline{N}(r, \sigma_k; h) + S(r, h) \\ & \leq \overline{N}(r, 0; h) + \frac{1}{2}\sum_{k=1}^{2n-2m-4} N(r, \sigma_k; h) + S(r, h) \\ & \leq (n - m - 1)T(r, h) + S(r, h), \end{aligned}$$

which contradicts $n \geq 2m + 3$.
Subcase 2.2.2.2. Let $h(z)$ be constant, then since $f(z)$ is non-constant, so we get form (3.16) that $h^n - 1 = 0$, $h^{n-m} - 1 = 0$ and $h^{n-2m} - 1 = 0$. i.e., $h^d - 1 = 0$, where $d = gcd(n, n-m, n-2m) = 1$ as $gcd(n, m) = 1$. In other words $h(z) = 1$. i.e., $f(z) \equiv f(z+\omega)$. □

4. Some relevant discussions and examples

With $n = 5$, we are now going to prove the following proposition to show that the condition $a \neq \dfrac{\delta_{c,d}^5}{2} = \dfrac{8c^5}{9375d^4}$ in *Corollary 1.9* is essential.

Proposition 4.1. *Under the supposition of* Corollary 1.9, *for* $n = 5$, *let*

$$f(z+\omega) = \frac{\frac{-5d}{2c} f(z)}{f(z) + \frac{5d}{2c}} = \frac{-5df(z)}{2cf(z)+5d}$$

and $a = \frac{\delta^5_{c,d}}{2}$, *then we have* $2\mathcal{G} = \frac{\mathcal{F}}{\mathcal{F} - \frac{1}{2}}$, *where* $\mathcal{G} = \mathcal{R}(g)$ *and* $\mathcal{F} = \mathcal{R}(f)$.

Proof. We have

$$\mathcal{G} = -\frac{af^5(z+\omega)}{bf^2(z+\omega) + cf(z+\omega) + d}. \tag{4.1}$$

Since $\frac{c^2}{4bd} = \frac{n(n-2)}{(n-1)^2} = \frac{15}{16}$, we get $b = \frac{4c^2}{15d}$. Let $\beta_1, \beta_2 = \frac{(-15 \pm \sqrt{15}i)d}{8c}$. From (4.1) we see that

$$\begin{aligned}
2\mathcal{G} &= -\frac{2af^5(z+\omega)}{\frac{4c^2}{15d} f^2(z+\omega) + cf(z+\omega) + d} \\
&= -\frac{30ad\ f^5(z+\omega)}{4c^2 f^2(z+\omega) + 15cd\ f(z+\omega) + 15d^2} \\
&= -\frac{30ad\ f^5(z+\omega)}{4c^2(f(z+\omega) - \beta_1)(f(z+\omega)-\beta_2)} \\
&= -\frac{30ad^6(-5)^5 f^5(z)}{4c^2(2cf(z)+5d)^5 \left(\frac{-5df(z)}{2cf(z)+5d} - \beta_1\right)\left(\frac{-5df(z)}{2cf(z)+5d} - \beta_2\right)} \\
&= \frac{30ad^6 5^5\ f^5(z)}{4c^2(2cf(z)+5d)^3 \left(f(z) + \frac{5d\beta_1}{2c\beta_1 + 5d}\right)\left(f(z) + \frac{5d\beta_2}{2c\beta_2+5d}\right)(5d+2c\beta_1)(5d+2c\beta_2)}.
\end{aligned}$$

Now as $5d + 2c\beta_1 = \frac{(5+\sqrt{15}i)d}{4}$ and $5d + 2c\beta_2 = \frac{(5-\sqrt{15}i)d}{4}$, we deduce that $(5d+2c\beta_1)(5d+2c\beta_2) = \frac{5d^2}{2}$,

$$\frac{5d\beta_1}{5d+2c\beta_1} = \frac{5d}{2c}\left(\frac{-3+\sqrt{15}i}{2}\right), \quad \frac{5d\beta_2}{5d+2c\beta_2} = \frac{5d}{2c}\left(\frac{-3-\sqrt{15}i}{2}\right).$$

So $\frac{5d\beta_1}{5d+2c\beta_1} + \frac{5d\beta_2}{5d+2c\beta_2} = -\frac{15d}{2c}$ and $\frac{5\beta_1}{5+2c\beta_1}\frac{5\beta_2}{5+2c\beta_2} = \frac{75d^2}{2c^2}$. Therefore

$$\begin{aligned}
2\mathcal{G} &= \frac{30ad^6 5^5\ f^5(z)}{4c^2(2cf(z)+5d)^3\left(f^2(z) - \frac{15d}{2c}f(z) + \frac{75d^2}{2c^2}\right)\frac{5d^2}{2}} \\
&= \frac{30ad^4 5^4\ f^5(z)}{(8c^3f^3(z) + 60c^2df^2(z) + 150cd^2f(z) + 125d^3)(2c^2f^2(z) - 15cdf(z) + 75d^2)} \\
&= \frac{30ad^4 5^4\ f^5(z)}{16c^5f^5(z) + 2500c^2d^3f^2(z) + 9375cd^4f(z) + 9375d^5}.
\end{aligned}$$

For $n = 5$ and $m = 1$, we set $a = \frac{1}{2}\gamma_1 = -\frac{2be_1 + c}{10e_1^4}$, where $e_1 = -\frac{5d}{2c}$. So a simple calculation yields $a = \frac{16c^5}{30(5d)^4}$. i.e., we have $30a(5d)^4 = 16c^5$ and with the help of this, we get

$$\begin{aligned}
2\mathcal{G} &= \frac{16c^5 f^5(z)}{16c^5 f^5(z) + 2500c^2d^3 f^2(z) + 9375cd^4 f(z) + 9375d^5} \\
&= \frac{af^5(z)}{af^5(z) + \frac{1}{2}\left(\frac{2500ad^3}{8c^3} f^2(z) + \frac{9375ad^4}{8c^4} f(z) + \frac{9375ad^5}{8c^5}\right)} \\
&= \frac{af^5(z)}{af^5(z) + \frac{1}{2}(bf^2(z) + cf(z) + d)} \\
&= \frac{\frac{af^5(z)}{bf^2(z) + cf(z) + d}}{\frac{af^5(z)}{bf^2(z) + cf(z) + d} + \frac{1}{2}} \\
&= \frac{\mathcal{F}}{\mathcal{F} - \frac{1}{2}}.
\end{aligned}$$

Thus we see that $\mathcal{G} - 1 = -\frac{\mathcal{F} - 1}{2\mathcal{F} - 1}$. In other words, we observe that $E_{f(z)}(\mathcal{S}_i, \infty) = E_{f(z+\omega)}(\mathcal{S}_i, \infty)$, for $i = 1, 2$; but $f(z) \not\equiv f(z + \omega)$. □

In view of the *Proposition 4.1*, for $e^w = -1$, we see, from the following series of examples, rather to say from the following counter examples that

$$f(z) \neq f(z + \omega) = \frac{\frac{-5d}{2c} f(z)}{f(z) + \frac{5d}{2c}}$$

implies $E_{f(z)}(\mathcal{S}_i, \infty) = E_{f(z+\omega)}(\mathcal{S}_i, \infty)$, $i = 1, 2$ and satisfy all the conditions of *Proposition 4.1*.

Example 4.2. *Let* $f(z) = \dfrac{\beta e^{qz}}{\sin^2\left(\frac{\pi z}{\omega}\right) - \frac{c\beta}{5d} e^{qz}}$, *where q be an odd positive integer and $\beta \in \mathbb{C}^*$.*

Example 4.3. *Let* $f(z) = \dfrac{\frac{-5d\alpha}{c} e^{qz}}{(\alpha e^{qz} - \beta)}$, *where q be an odd positive integer and $\alpha, \beta \in \mathbb{C}^*$.*

Example 4.4. *Let* $f(z) = \dfrac{-\frac{5d}{2c} e^{qz} \cos^2\left(\frac{\pi z}{\omega}\right)}{\sin^2\left(\frac{\pi z}{\omega}\right) + \frac{1}{2} e^{qz} \cos^2\left(\frac{\pi z}{\omega}\right)}$, *where q be an odd positive integer.*

Example 4.5. *Let* $f(z) = \dfrac{-\frac{5d}{c} g(e^z)\left(e^{pz} + \sin^2\left(\frac{\pi z}{\omega}\right)\right)}{g(e^z)\left(e^{pz} + \sin^2\left(\frac{\pi z}{\omega}\right)\right) + e^{qz} \log c}$, *where $g(z)$ is an even function and p be an even positive integer.*

From the above discussion, we finally pose the following open question for future investigations.

Question 4.6. *Under the supposition of* Corollary 1.9*, can one find a counter example for a infinite ordered meromorphic function for which the conclusion of* Corollary 1.9 *ceases to hold ?*

However we have been able to find a counter example for infinite ordered meromorphic function to show that when $a = \frac{\delta^5_{c,d}}{2}$ the conclusion of *Corollary 1.9* does hold.

Example 4.7. *Let* $f(z) = \dfrac{-\frac{5d}{c}P(e^z)e^{e^{pz}}}{P(e^z)e^{e^{pz}} + e^{qz}\cos^2\left(\frac{\pi z}{\omega}\right)}$*, where p and q are respectively even and odd positive integers and* $P(\zeta) = \sum_{j=1}^{s} a_j\zeta^{2j}$*, where s be a positive integer,* $a_j \in \mathbb{C}^*$ $(j = 1, 2, \ldots, s)$*,* $a_s \neq 0$.

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