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The Moore-Penrose Inverse in Rings with Involution

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Abstract. Let R be a unital ring with involution. In this paper, we first show that for an element $a \in R$, a is Moore-Penrose invertible if and only if a is well-supported if and only if a is co-supported. Moreover, several new necessary and sufficient conditions for the existence of the Moore-Penrose inverse of an element in a ring R are obtained. In addition, the formulae of the Moore-Penrose inverse of an element in a ring are presented.

1. Introduction

Let R be a $*$-ring, that is a ring with an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^*+b^*$. We say that $b \in R$ is the Moore-Penrose inverse of $a \in R$, if the following hold:

$$aba = a, \quad bab = b, \quad (ab)^* = ab \quad (ba)^* = ba.$$

There is at most one b such that above four equations hold. If such an element b exists, it is denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements will be denoted by $R^\dagger$. An element $b \in R$ is an inner inverse of $a \in R$ if $aba = a$ holds. The set of all inner inverses of a will be denoted by $a\{1\}$. An element $a \in R$ is said to be group invertible if there exists $b \in R$ such that the following equations hold:

$$aba = a, \quad bab = b, \quad ab = ba.$$

The element b which satisfies the above equations is called a group inverse of a. If such an element b exists, it is unique and denoted by $a^\#$. The set of all group invertible elements will be denoted by $R^\#$.

An element $a \in R$ is called an idempotent if $a^2 = a$. a is called a projection if $a^2 = a = a^*$. a is called normal if $aa^* = a^*a$. a is called a Hermite element if $a^* = a$. a is said to be an EP element if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$. The set of all EP elements will be denoted by R^{EP}. $\tilde{a}$ is called a $\{1,3\}$-inverse of a if we have $a\tilde{a}a = a$, $(a\tilde{a})^* = a\tilde{a}$. The set of all $\{1,3\}$-invertible elements will be denoted by $R^{\{1,3\}}$. Similarly, an element $\hat{a} \in R$ is called a $\{1,4\}$-inverse of a if $a\hat{a}a = a$, $(\hat{a}a)^* = \hat{a}a$. The set of all $\{1,4\}$-invertible elements will be denoted by $R^{\{1,4\}}$.

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We will also use the following notations: $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$, ${}^{\circ}a = \{x \in R \mid xa = 0\}$ and $a^{\circ} = \{x \in R \mid ax = 0\}$.

In [2], Chen showed that the equivalent conditions such that $a \in R$ to be an EP element are closely related with powers of the group and Moore-Penrose inverse of a. In [12], Mosić and Djordjević presented several equivalent conditions, which ensure that an element $a \in R$ is a partial isometry and EP. These conditions involve elements $a, a^*, a^{\dagger}, a^{\#}$ and also powers of these elements. In [13], more new characterizations of EP elements in rings are given by Mosić and Djordjević, which involve powers of their group and Moore-Penrose inverse. In [19], Tian and Wang presented some necessary and sufficient conditions such that $A \in \mathbb{C}_{n\times n}$ to be an EP matrix, which also involve powers of their group and Moore-Penrose inverse, where $\mathbb{C}_{n\times n}$ stands for the set of all $n \times n$ matrices over the field of complex numbers. Motivated by the above facts, in this paper, we will show that the existence of the Moore-Penrose inverse of an element in a ring R is closely related with powers of some Hermite elements, idempotents and projections.

Recently, Zhu, Chen and Patrício in [20] introduced the concepts of left $*$-regularity and right $*$-regularity. We call an element $a \in R$ is left (right) $*$-regular if there exists $x \in R$ such that $a = aa^*ax$ ($a = xaa^*a$). They proved that $a \in R^{\dagger}$ if and only if a is left $*$-regular if and only if a is right $*$-regular. Motivated by the above results, we will give more equivalent conditions for an element in a ring to be Moore-Penrose invertible.

In [4], Hartwig proved that for an element $a \in R$, a is $\{1,3\}$-invertible with $\{1,3\}$-inverse x if and only if $x^*a^*a = a$ and, similarly, a is $\{1,4\}$-invertible with $\{1,4\}$-inverse y if and only if $aa^*y^* = a$. In [14], one has the following result in complex matrices case, $a \in R^{\dagger}$ if and only if $a \in Ra^*a \cap aa^*R$. In addition, if $a = aa^*y = xa^*a$ for some $x, y \in R$, then $a^{\dagger} = y^*ax^*$.

It is well-known that an important feature of the Moore-Penrose inverse is that it can be used to represent projections. Let $a \in R^{\dagger}$, then we have two projections $p = aa^{\dagger}$ and $q = a^{\dagger}a$. In [3], Han and Chen proved that $a \in R^{\{1,3\}}$ if and only if there exists unique projection $p \in R$ such that $aR = pR$. And, it is also proved that $a \in R^{\{1,4\}}$ if and only if there exists unique projection $q \in R$ such that $Ra = Rq$. We will show that the existence of the Moore-Penrose inverse is closely related with some Hermite elements and projections.

In [7, Theorem 2.4], Koliha proved that $a \in \mathcal{A}^{\dagger}$ if and only if a is well-supported, where $\mathcal{A}$ is a C*-algebra. In [8, Theorem 1], Koliha, Djordjević and Cvetković proved that $a \in R^{\dagger}$ if and only if a is left $*$-cancellable and well-supported. Where an element $a \in R$ is called well-supported if there exists projection $p \in R$ such that $ap = a$ and $a^*a + 1 - p \in R^{-1}$. In Theorem 3.7, we will show that the condition that a is left $*$-cancellable in [8, Theorem 1] can be dropped. Moreover, we prove that $a \in R^{\dagger}$ if and only if there exists $e^2 = e \in R$ such that $ea = 0$ and $aa^* + e$ is left invertible. And, it is also proved that $a \in R^{\dagger}$ if and only if there exists $b \in R$ such that $ba = 0$ and $aa^* + b$ is left invertible.

In [4], Hartwig proved that $a \in R^{\{1,3\}}$ if and only if $R = aR \oplus (a^*)^{\circ}$. And, it is also proved that $a \in R^{\{1,4\}}$ if and only if $R = Ra \oplus {}^{\circ}(a^*)$. Hence $a \in R^{\dagger}$ if and only if $R = aR \oplus (a^*)^{\circ} = Ra \oplus {}^{\circ}(a^*)$. We will show that $a \in R^{\dagger}$ if and only if $R = a^{\circ} \oplus (a^*a)^nR$. It is also shown that $a \in R^{\dagger}$ if and only if $R = a^{\circ} + (a^*a)^nR$, for all choices $n \in \mathbf{N}^+$, where $\mathbf{N}^+$ stands for the set of all positive integers.

2. Preliminary

In this section, several auxiliary lemmas are presented.

Lemma 2.1. [4, p.201] *Let $a \in R$. Then we have the following results:*

(1) *a is $\{1,3\}$-invertible with $\{1,3\}$-inverse x if and only if $x^*a^*a = a$;*

(2) *a is $\{1,4\}$-invertible with $\{1,4\}$-inverse y if and only if $aa^*y^* = a$.*

The following two Lemmas can be found in [14] in the complex matrix case, one can see that these are also valid for an element in a ring with involution.

Lemma 2.2. *Let $a \in R$. Then $a \in R^{\dagger}$ if and only if there exist $x, y \in R$ such that $x^*a^*a = a$ and $aa^*y^* = a$. In this case, $a^{\dagger} = yax$.*

Lemma 2.3. *Let $a \in R^{\dagger}$. Then:*

(1) $aa^*, a^*a \in R^{EP}$ *and* $(aa^*)^{\dagger} = (a^*)^{\dagger}a^{\dagger}$ *and* $(a^*a)^{\dagger} = a^{\dagger}(a^*)^{\dagger}$;

(2) *If a is normal, then* $a \in R^{EP}$ *and* $(a^k)^{\dagger} = (a^{\dagger})^k$ *for any* $k \in \mathbf{N}^+$.

We will give a generalization of Lemma 2.3(1) in the following lemma.

Lemma 2.4. *Let $a \in R^{\dagger}$. Then* $(aa^*)^n, (a^*a)^m \in R^{EP}$ *for any* $n, m \in \mathbf{N}^+$.

Proof. Suppose $a \in R^{\dagger}$, by Lemma 2.3 and $(aa^*)^* = aa^*$, we have

$$((aa^*)^n)^{\dagger} = ((aa^*)^{\dagger})^n \text{ and } ((a^*a)^n)^{\dagger} = ((a^*a)^{\dagger})^n, \tag{1}$$
$$(aa^*)^{\dagger} = (a^*)^{\dagger}a^{\dagger} \text{ and } (a^*a)^{\dagger} = a^{\dagger}(a^*)^{\dagger}, \tag{2}$$
$$aa^*(aa^*)^{\dagger} = (aa^*)^{\dagger}aa^* \text{ and } a^*a(a^*a)^{\dagger} = (a^*a)^{\dagger}a^*a. \tag{3}$$

Thus we have

$$\begin{aligned}
&(aa^*)^n((aa^*)^n)^{\dagger}(aa^*)^n \overset{(1)}{=} (aa^*)^n((aa^*)^{\dagger})^n(aa^*)^n \\
\overset{(2)}{=}\ &(aa^*)^n((a^*)^{\dagger}a^{\dagger})^n(aa^*)^n \overset{(3)}{=} (aa^*(a^*)^{\dagger}a^{\dagger}aa^*)^n = (aa^*)^n; \\
&((aa^*)^n)^{\dagger}(aa^*)^n((aa^*)^n)^{\dagger} \overset{(1)}{=} ((aa^*)^{\dagger})^n(aa^*)^n((aa^*)^{\dagger})^n \\
\overset{(2)}{=}\ &((a^*)^{\dagger}a^{\dagger})^n(aa^*)^n((a^*)^{\dagger}a^{\dagger})^n \overset{(3)}{=} ((a^*)^{\dagger}a^{\dagger}aa^*(a^*)^{\dagger}a^{\dagger})^n \overset{(1)}{=} ((aa^*)^n)^{\dagger}; \\
&[(aa^*)^n((aa^*)^n)^{\dagger}]^* \overset{(1)}{=} [(aa^*)^n((aa^*)^{\dagger})^n]^* \\
\overset{(3)}{=}\ &[(aa^*(aa^*)^{\dagger})^n]^* \overset{(3)}{=} (aa^*)^n((aa^*)^n)^{\dagger}; \\
&[((aa^*)^n)^{\dagger}(aa^*)^n]^* \overset{(1)}{=} [((aa^*)^{\dagger})^n(aa^*)^n]^* \\
\overset{(3)}{=}\ &[((aa^*)^{\dagger}aa^*)^n]^* \overset{(3)}{=} ((aa^*)^n)^{\dagger}(aa^*)^n; \\
&(aa^*)^n((aa^*)^n)^{\dagger} \overset{(1)}{=} (aa^*)^n((aa^*)^{\dagger})^n \\
\overset{(3)}{=}\ &(aa^*(aa^*)^{\dagger})^n \overset{(3)}{=} ((aa^*)^{\dagger}aa^*)^n \overset{(3)}{=} ((aa^*)^n)^{\dagger}(aa^*)^n.
\end{aligned}$$

By the definition of the EP element, we have $(aa^*)^n \in R^{EP}$. Similarly, $(a^*a)^m \in R^{EP}$. □

Definition 2.5. *An element $a \in R$ is $*$-cancellable if $a^*ax = 0$ implies $ax = 0$ and $yaa^* = 0$ implies $ya = 0$.*

The equivalence of conditions (1), (3) and (5) in the following lemma was also proved by Puystjens and Robinson [16, Lemma 3] in categories with involution.

Lemma 2.6. [9, Theorem 5.4] *Let $a \in R$. Then the following conditions are equivalent:*
(1) $a \in R^{\dagger}$;
(2) $a^* \in R^{\dagger}$;
(3) *a is $*$-cancellable and aa^* and a^*a are regular;*
(4) *a is $*$-cancellable and a^*aa^* is regular;*
(5) $a \in Ra^*a \cap aa^*R$.

Lemma 2.7. *Let $a \in R^{\dagger}$. Then for any $n, m \in \mathbf{N}^+$, we have*

(1) $(aa^*)^n((aa^*)^n)^{\dagger}a = a$;

(2) $a((a^*a)^m)^{\dagger}(a^*a)^m = a$.

Proof. (1) If $n = 1$ and $aa^*(aa^*)^\dagger aa^* = aa^*$, by a is $*$-cancellable, we have

$$aa^*(aa^*)^\dagger a = a.$$

Suppose the result hold for $n = k$, ie.,

$$(aa^*)^k((aa^*)^k)^\dagger a = a. \tag{4}$$

By Lemma 2.3, we have

$$\begin{aligned}
&(aa^*)^{k+1}[(aa^*)^{k+1}]^\dagger a \\
&= aa^*(aa^*)^k[(aa^*)^\dagger]^{k+1}a = aa^*(aa^*)^k[(aa^*)^\dagger]^k(aa^*)^\dagger a \\
&= aa^*(aa^*)^k[(aa^*)^\dagger]^k(a^*)^\dagger a^\dagger a = aa^*(aa^*)^k[(aa^*)^\dagger]^k(a^\dagger)^*a^\dagger a \\
&= aa^*(aa^*)^k[(aa^*)^\dagger]^k(a^\dagger aa^\dagger)^*a^\dagger a = aa^*(aa^*)^k[(aa^*)^\dagger]^k aa^\dagger(a^\dagger)^*a^\dagger a \\
&\overset{(4)}{=} aa^*aa^\dagger(a^\dagger)^*a^\dagger a = aa^*(aa^\dagger)^*(a^\dagger)^*(a^\dagger a)^* \\
&= a(aa^\dagger a)^*(a^\dagger aa^\dagger)^* = aa^*(a^\dagger)^* \\
&= a(a^\dagger a)^* = aa^\dagger a = a.
\end{aligned} \tag{5}$$

Thus, the result follows by induction.

(2) It is similar to (1). □

Lemma 2.8. [20, Theorem 2.16, 2.19 and 2.20] *Let $a \in R$. The following conditions are equivalent:*

(1) $a \in R^\dagger$;

(2) $a \in aa^*aR$;

(3) $a \in Raa^*a$.

In this case, $a^\dagger = (ax)^*axa^* = a^*ya(ya)^*$, *where* $a = aa^*ax = yaa^*a$.

Lemma 2.9. [15, Proposition 2] *Let $a \in R$. If $aR = a^*R$, then the following are equivalent:*

(1) $a \in R^{EP}$;

(2) $a \in R^\dagger$;

(3) $a \in R^\#$.

3. Main results

In this section, several necessary and sufficient conditions for the existence of the Moore-Penrose inverse of an element in a ring R are given.

Theorem 3.1. *Let $a \in R$. Then the following conditions are equivalent for any $m, n \in \mathbf{N}^+$:*

(1) $a \in R^\dagger$;

(2) $a \in R(a^*a)^m \cap (aa^*)^nR$;

(3) $a \in a(a^*a)^nR$;

(4) $a \in R(aa^*)^na$;

(5) $(aa^*)^n \in R^\dagger$ *and* $(aa^*)^n[(aa^*)^n]^\dagger a = a$;

(6) $(a^*a)^n \in R^\dagger$ *and* $a[(a^*a)^n]^\dagger(a^*a)^n = a$;

(7) a *is* $*$*-cancellable and* $(aa^*)^m$ *and* $(a^*a)^n$ *are regular;*

(8) a *is* $*$*-cancellable and* $(a^*a)^n a^*$ *is regular;*

(9) a *is* $*$*-cancellable and* $a^*(aa^*)^n$ *is regular;*

(10) a *is* $*$*-cancellable and* $(aa^*)^n \in R^\#$;

(11) a *is* $*$*-cancellable and* $(a^*a)^n \in R^\#$;

(12) a *is* $*$*-cancellable and* $(aa^*)^n \in R^\dagger$;

(13) a *is* $*$*-cancellable and* $(a^*a)^n \in R^\dagger$.

In this case, $a^\dagger = y_1^*(aa^*)^{m+n-2}ax_1^* = x_2^*(a^*a)^{2n-1}x_2a^* = a^*y_2(aa^*)^{2n-1}y_2^*$, *where* $a = x_1(a^*a)^m$, $a = (aa^*)^n y_1$, $a = a(a^*a)^n x_2$, $a = y_2(aa^*)^n a$, *for some* $x_1, x_2, y_1, y_2 \in R$.

Proof. (1) ⇒ (2) By Lemma 2.7 we can get

$$(aa^*)^n((aa^*)^n)^\dagger a = a \tag{6}$$

and

$$a((a^*a)^m)^\dagger(a^*a)^m = a. \tag{7}$$

By (6) and (7), we have $a \in R(a^*a)^m \cap (aa^*)^n R$.

(2) ⇒ (1) Suppose $a \in R(a^*a)^m \cap (aa^*)^n R$, then for some $x_1, y_1 \in R$, we have

$$a = x_1(a^*a)^m \text{ and } a = (aa^*)^n y_1. \tag{8}$$

If $m = n = 1$, it is easy to see that $a \in R^\dagger$ by Lemma 2.6. Next, we suppose $m, n > 1$. By (8) and Lemma 2.1, we have

$$[x_1(a^*a)^{m-1}]^* \in a\{1,3\} \text{ and } [(aa^*)^{n-1}y_1]^* \in a\{1,4\}. \tag{9}$$

Thus by (9) and Lemma 2.2, we have $a \in R^\dagger$ and

$$\begin{aligned} a^\dagger &= a^{(1,4)}aa^{(1,3)} \\ &= [(aa^*)^{n-1}y_1]^* a [x_1(a^*a)^{m-1}]^* \\ &= y_1^*(aa^*)^{n-1}a(a^*a)^{m-1}x_1^* \\ &= y_1^*(aa^*)^{m+n-2}ax_1^*. \end{aligned}$$

(1) ⇒ (3) By Lemma 2.3, we have

$$a^*a = a^*aa^\dagger(a^\dagger)^*a^*a \tag{10}$$

and

$$a^\dagger(a^\dagger)^*a^*a = a^*aa^\dagger(a^\dagger)^*. \tag{11}$$

Thus

$$\begin{aligned}
a &= aa^\dagger a = (aa^\dagger)^* a = (a^\dagger)^* a^* a \\
&\overset{(10)}{=} (a^\dagger)^* a^* aa^\dagger (a^\dagger)^* a^* a \overset{(11)}{=} (a^\dagger)^* (a^* a)^2 a^\dagger (a^\dagger)^* \\
&= ((a^\dagger)^* a^* a) a^* aa^\dagger (a^\dagger)^* \ = (aa^\dagger a) a^* aa^\dagger (a^\dagger)^* \\
&= aa^* aa^\dagger (a^\dagger)^* \\
&\overset{(10)}{=} a(a^* aa^\dagger (a^\dagger)^* a^* a) a^\dagger (a^\dagger)^* \overset{(11)}{=} a(a^* a)^2 (a^\dagger (a^\dagger)^*)^2 \\
&= \cdots\cdots \\
&= a(a^* a)^n (a^\dagger (a^\dagger)^*)^n.
\end{aligned}$$

Hence $a \in a(a^*a)^n R$.

(3) ⇒ (1) Suppose $a \in a(a^*a)^n R$, then for some $x_2 \in R$ we have $a \in a(a^*a)^n x_2 = aa^*a(a^*a)^{n-1}x_2 \in aa^*aR$. Thus by Lemma 2.8, we have $a \in R^\dagger$ and

$$\begin{aligned}
a^\dagger &= [a(a^*a)^{n-1}x_2]^* a(a^*a)^{n-1}x_2 a^* \\
&= x_2^*(a^*a)^{n-1}a^*a(a^*a)^{n-1}x_2 a^* \\
&= x_2^*(a^*a)^{2n-1}x_2 a^*.
\end{aligned}$$

(1) ⇔ (4) It is similar to (1) ⇔ (3) and suppose $a = y_2(aa^*)^n a$ for some $y_2 \in R$, by Lemma 2.8, we have

$$\begin{aligned}
a^\dagger &= a^* y_2(aa^*)^{n-1}a[y_2(aa^*)^{n-1}a]^* \\
&= a^* y_2(aa^*)^{n-1}aa^*(aa^*)^{n-1}y_2^* \\
&= a^* y_2(aa^*)^{2n-1}y_2^*.
\end{aligned}$$

(1) ⇒ (5) It is easy to see that by Lemma 2.4 and Lemma 2.7.

(1) ⇒ (6) It is similar to (1) ⇒ (5).

(5) ⇒ (4) Suppose $(aa^*)^n \in R^\dagger$ and $(aa^*)^n((aa^*)^n)^\dagger a = a$. Let $b = (aa^*)^n[(aa^*)^n]^\dagger$, then $b^* = b$ and $ba = a$. Thus

$$a = ba = b^*a = [(aa^*)^n((aa^*)^n)^\dagger]^*a = ((aa^*)^n)^\dagger(aa^*)^n a \in R(aa^*)^n a.$$

(6) ⇒ (3) It is similar to (5) ⇒ (4).

(1) ⇒ (7) It is easy to see that by Lemma 2.4.

(7) ⇒ (1) Let $m = n = 1$, then by Lemma 2.6, we have $a \in R^\dagger$.

(1) ⇒ (8) By Lemma 2.4, we have $(a^*a)^n \in R^{EP}$ and $((a^*a)^n)^\dagger = (a^\dagger(a^*)^\dagger)^n$. Let $c = (a^\dagger)^*((a^*a)^\dagger)^n$, then

$$\begin{aligned}
(a^*a)^n a^* c (a^*a)^n a^* &= (a^*a)^n a^*(a^\dagger)^*((a^*a)^\dagger)^n(a^*a)^n a^* \\
&= (a^*a)^n[a^*(a^\dagger)^*(a^*a)^\dagger]((a^*a)^\dagger)^{n-1}(a^*a)^n a^* \\
&= (a^*a)^n[a^*(a^\dagger)^*a^\dagger(a^*)^\dagger]((a^*a)^\dagger)^{n-1}(a^*a)^n a^* \\
&= (a^*a)^n[a^\dagger aa^\dagger(a^*)^\dagger]((a^*a)^\dagger)^{n-1}(a^*a)^n a^* \\
&= (a^*a)^n(a^*a)^\dagger((a^*a)^\dagger)^{n-1}(a^*a)^n a^* \\
&= (a^*a)^n((a^*a)^\dagger)^n(a^*a)^n a^* \\
&= (a^*a)^n((a^*a)^n)^\dagger(a^*a)^n a^* \\
&= (a^*a)^n a^*.
\end{aligned}$$

Thus $(a^*a)^n a^*$ is regular.

(8) ⇒ (1) Suppose a is ∗-cancellable and $(a^*a)^n a^*$ is regular. Let $n = 1$, then by Lemma 2.6, we have $a \in R^\dagger$.

(1) ⇔ (9) It is similar to (1) ⇔ (8).

(1) $\Rightarrow$ (10)-(13) It is easy to see that by Lemma 2.4.
The equivalence between (10)-(13) can be seen by Lemma 2.9.
(12) $\Rightarrow$ (9) Suppose a is $*$-cancellable and $(aa^*)^n \in R^\#$, then

$$(aa^*)^n = (aa^*)^n[(aa^*)^n]^\#(aa^*)^n. \tag{12}$$

Pre-multiplication of (12) by a^* now yields

$$\begin{aligned} a^*(aa^*)^n &= a^*(aa^*)^n[(aa^*)^n]^\#(aa^*)^n \\ &= a^*(aa^*)^n[(aa^*)^n]^\#[(aa^*)^n]^\#(aa^*)^n(aa^*)^n \\ &= a^*(aa^*)^n[(aa^*)^n]^\#[(aa^*)^n]^\#(aa^*)^{n-1}a[a^*(aa^*)^n]. \end{aligned}$$

Thus $a^*(aa^*)^n$ is regular. □

Definition 3.2. [17] *Let $a, b \in R$, we say that a is a multiple of b if $a \in Rb \cap bR$.*

Definition 3.3. *Let $a, b \in R$, we say that a is a left (right) multiple of b if $a \in Rb$ ($a \in bR$).*

The existence of the Moore-Penrose inverse of an element in a ring is priori related to a Hermite element. If we take $n = 1$, the condition (2) in the following theorem can be found in [17, Theorem 1] in the category case.

Theorem 3.4. *Let $a \in R$. Then the following conditions are equivalent for any $n \in \mathbf{N}^+$:*

(1) *$a \in R^\dagger$;*

(2) *There exists a projection $p \in R$ such that $pa = a$ and p is a multiple of $(aa^*)^n$;*

(3) *There exists a Hermite element $q \in R$ such that $qa = a$ and q is a left multiple of $(aa^*)^n$;*

(4) *There exists a Hermite element $r \in R$ such that $ra = a$ and r is a right multiple of $(aa^*)^n$;*

(5) *There exists $b \in R$ such that $ba = a$ and b is a left multiple of $(aa^*)^n$.*

Proof. (1) $\Rightarrow$ (2) Suppose $a \in R^\dagger$ and let $p = aa^\dagger$, then $p^2 = p = p^*$ and $pa = a$. By Lemma 2.3, we have

$$aa^*(a^\dagger)^*a^\dagger aa^* = aa^* \tag{13}$$

and

$$aa^*(a^\dagger)^*a^\dagger = (a^\dagger)^*a^\dagger aa^*. \tag{14}$$

$$\begin{aligned} p = aa^\dagger &= (aa^\dagger)^* = (a^\dagger)^*a^* \\ &= (a^\dagger)^*(aa^\dagger a)^* = (a^\dagger)^*a^\dagger aa^* \\ &\overset{(13)}{=} (a^\dagger)^*a^\dagger aa^*(a^\dagger)^*a^\dagger aa^* \overset{(14)}{=} [(a^\dagger)^*a^\dagger]^2(aa^*)^2 \\ &= \cdots\cdots \\ &= [(a^\dagger)^*a^\dagger]^n(aa^*)^n. \end{aligned} \tag{15}$$

By $p = p^*$ and (15), we have

$$p = p^* = [[(a^\dagger)^*a^\dagger]^n(aa^*)^n]^* = (aa^*)^n[[(a^\dagger)^*a^\dagger]^n]^*. \tag{16}$$

By (15) and (16), we have p is a multiple of $(aa^*)^n$.
(2) $\Rightarrow$ (3) It is obvious.
(3) $\Rightarrow$ (4) Let $r = q^*$.
(4) $\Rightarrow$ (5) Suppose $r^* = r$, $ra = a$ and r is a right multiple of $(aa^*)^n$, then

$$r = (aa^*)^n w \text{ for some } w \in R. \tag{17}$$

Let $b = r$, then $ba = a$ and by $r^* = r$, we have

$$b = r = r^* \overset{(17)}{=} ((aa^*)^n w)^* = w^*(aa^*)^n. \tag{18}$$

That is b is a left multiple of $(aa^*)^n$.

(5) ⇒ (1) Since b is a left multiple of $(aa^*)^n$, then $b \in R(aa^*)^n$, post-multiplication of $b \in R(aa^*)^n$ by a now yields $ba \in R(aa^*)^n a$. Then by $ba = a$, which gives $a \in R(aa^*)^n a$, thus the condition (4) in Theorem 3.1 is satisfied. □

Similarly, we have the following theorem.

Theorem 3.5. *Let $a \in R$. Then the following conditions are equivalent for any $n \in \mathbf{N}^+$:*

(1) $a \in R^\dagger$;

(2) *There exist a projection $w \in R$ such that $aw = a$ and w is a multiple of $(a^*a)^n$;*

(3) *There exist a Hermite element $u \in R$ such that $au = a$ and u is a right multiple of $(a^*a)^n$;*

(4) *There exist a Hermite element $v \in R$ such that $av = a$ and v is a left multiple of $(a^*a)^n$;*

(5) *There exist $c \in R$ such that $ac = a$ and c is a right multiple of $(a^*a)^n$.*

If we take $n = 1$, the condition (2) in the following theorem can be found in [17, Theorem 1] in the category case.

Theorem 3.6. *Let $a \in R$. Then the following conditions are equivalent for any $n \in \mathbf{N}^+$:*

(1) $a \in R^\dagger$;

(2) *There exists a projection $q \in R$ such that $qa = 0$ and $(aa^*)^n + q$ is invertible;*

(3) *There exists a projection $q \in R$ such that $qa = 0$ and $(aa^*)^n + q$ is left invertible;*

(4) *There exists an idempotent $f \in R$ such that $fa = 0$ and $(aa^*)^n + f$ is invertible;*

(5) *There exists an idempotent $f \in R$ such that $fa = 0$ and $(aa^*)^n + f$ is left invertible;*

(6) *There exists $c \in R$ such that $ca = 0$ and $(aa^*)^n + c$ is invertible;*

(7) *There exists $c \in R$ such that $ca = 0$ and $(aa^*)^n + c$ is left invertible.*

In this case, $a^\dagger = a^* y_i (aa^*)^{2n-1} y_i^*$, $i \in \{1, 2, 3\}$, where $1 = y_1((aa^*)^n + q) = y_2((aa^*)^n + f) = y_3((aa^*)^n + c)$, for some $y_1, y_2, y_3 \in R$.

Proof. (1) ⇒ (2) Suppose $a \in R^\dagger$ and let $q = 1 - aa^\dagger$, then $q^2 = q = q^*$ and $qa = (1 - aa^\dagger)a = 0$. By Lemma 2.3, we have

$$aa^*(a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger aa^*. \tag{19}$$

Moreover,

$$
\begin{aligned}
&(aa^* + q)((a^\dagger)^* a^\dagger + 1 - aa^\dagger) \\
&= (aa^* + 1 - aa^\dagger)((a^\dagger)^* a^\dagger + 1 - aa^\dagger) \\
&= aa^*(a^\dagger)^* a^\dagger + aa^*(1 - aa^\dagger) + (1 - aa^\dagger)(a^\dagger)^* a^\dagger + (1 - aa^\dagger)^2 \\
&= aa^*(a^\dagger)^* a^\dagger + 1 - aa^\dagger \\
&= aa^\dagger + 1 - aa^\dagger = 1. \\
&((aa^*)^2 + q)[((a^\dagger)^* a^\dagger)^2 + 1 - aa^\dagger] \\
&= ((aa^*)^2 + 1 - aa^\dagger)[((a^\dagger)^* a^\dagger)^2 + 1 - aa^\dagger] \\
&\overset{(19)}{=} (aa^*(a^\dagger)^* a^\dagger)^2 + (aa^*)^2(1 - aa^\dagger) + (1 - aa^\dagger)((a^\dagger)^* a^\dagger)^2 + (1 - aa^\dagger)^2 \\
&= (aa^\dagger)^2 + 1 - aa^\dagger \\
&= aa^\dagger + 1 - aa^\dagger = 1. \\
&\cdots\cdots \\
&((aa^*)^n + q)[((a^\dagger)^* a^\dagger)^n + 1 - aa^\dagger] = 1.
\end{aligned}
$$

Similarly, we also have $[((a^\dagger)^* a^\dagger)^n + 1 - aa^\dagger]((aa^*)^n + q) = 1$. Thus, $(aa^*)^n + p$ is invertible.

(2) ⇒ (3) It is clear.

(3) ⇒ (1) Suppose $q^2 = q = q^*$, $pa = 0$ and $(aa^*)^n + q$ is left invertible, then $1 = y_1((aa^*)^n + q)$ for some $y_1 \in R$. By $pa = 0$, we have

$$a = y_1((aa^*)^n + q)a = y_1(aa^*)^n a \in R(aa^*)^n a.$$

That is the condition (4) in Theorem 3.1 is satisfied and

$$
\begin{aligned}
a^\dagger &= a^* y_1(aa^*)^{n-1} a[y_1(aa^*)^{n-1} a]^* \\
&= a^* y_1(aa^*)^{n-1} aa^*(aa^*)^{n-1} y_1^* \\
&= a^* y_1(aa^*)^{2n-1} y_1^*.
\end{aligned}
$$

(1) ⇒ (4) Let $f = q = 1 - aa^\dagger$, then by (1) ⇒ (2), we have $f^2 = f \in R$, $fa = 0$ and $(aa^*)^n + f$ is invertible.

(4) ⇒ (5) It is clear.

(5) ⇒ (1) Suppose $f^2 = f \in R$, $fa = 0$ and $(aa^*)^n + f$ is left invertible, then $1 = y_2((aa^*)^n + f)$ for some $y_2 \in R$. By $fa = 0$, we have

$$a = y_2((aa^*)^n + f)a = y_2(aa^*)^n a \in R(aa^*)^n a.$$

That is the condition (4) in Theorem 3.1 is satisfied and

$$
\begin{aligned}
a^\dagger &= a^* y_2(aa^*)^{n-1} a[y_2(aa^*)^{n-1} a]^* \\
&= a^* y_2(aa^*)^{n-1} aa^*(aa^*)^{n-1} y_2^* \\
&= a^* y_2(aa^*)^{2n-1} y_2^*.
\end{aligned}
$$

(1) ⇒ (6) Let $c = q = 1 - aa^\dagger$, then by (1) ⇒ (2), we have $ca = 0$ and $(aa^*)^n + c$ is invertible. Since $c = q$ and $q^2 = q = q^*$, thus $(aa^*)^n + q$ is invertible implies $(aa^*)^n + c$ is invertible.

(6) ⇒ (7) It is clear.

(7) ⇒ (1) Suppose $ca = 0$ and $(aa^*)^n + c$ is left invertible, then $1 = y_3((aa^*)^n + c)$ for some $y_3 \in R$. By $ca = 0$, we have

$$a = y_3((aa^*)^n + c)a = y_3(aa^*)^n a \in R(aa^*)^n a.$$

That is the condition (4) in Theorem 3.1 is satisfied and

$$
\begin{aligned}
a^\dagger &= a^* y_3(aa^*)^{n-1} a[y_3(aa^*)^{n-1} a]^* \\
&= a^* y_3(aa^*)^{n-1} aa^*(aa^*)^{n-1} y_3^* \\
&= a^* y_3(aa^*)^{2n-1} y_3^*.
\end{aligned}
$$

□

Similarly, we have the following theorem.

Theorem 3.7. *Let $a \in R$. Then the following conditions are equivalent for any $n \in \mathbf{N}^+$:*

(1) $a \in R^\dagger$;

(2) *There exists a projection $p \in R$ such that $ap = 0$ and $(a^*a)^n + p$ is invertible;*

(3) *There exists a projection $p \in R$ such that $ap = 0$ and $(a^*a)^n + p$ is right invertible;*

(4) *There exists an idempotent $e \in R$ such that $ae = 0$ and $(a^*a)^n + e$ is invertible;*

(5) *There exists an idempotent $e \in R$ such that $ae = 0$ and $(a^*a)^n + e$ is right invertible;*

(6) *There exists $b \in R$ such that $ab = 0$ and $(a^*a)^n + b$ is invertible;*

(7) *There exists $b \in R$ such that $ab = 0$ and $(a^*a)^n + b$ is right invertible.*

In this case, $a^\dagger = x_i^*(a^*a)^{2n-1}x_ia^*$, $i \in \{1,2,3\}$, where $1 = ((aa^*)^n + p)x_1 = ((aa^*)^n + e)x_2 = ((aa^*)^n + b)x_3$, for some $x_1, x_2, x_3 \in R$.

Definition 3.8. [8, Definition 5 and p.374] *Let $a \in R$, we call a is well-supported if there exist a projection $p \in R$ such that $ap = 0$ and $a^*a + p$ is invertible. we call a is co-supported if there exist a projection $q \in R$ such that $qa = 0$ and $aa^* + q$ is invertible.*

Let $a \in R$, we call a is weak-supported if there exists $b \in R$ such that $ab = 0$ and $a^*a + b$ is invertible. We call a is coweak-supported if there exists $c \in R$ such that $ac = 0$ and $aa^* + c$ is invertible. Let $a \in R$, we call a is right weak-supported if there exists $b \in R$ such that $ab = 0$ and $a^*a + b$ is right invertible. We call a is left coweak-supported if there exists $c \in R$ such that $ac = 0$ and $aa^* + c$ is left invertible.

Theorem 3.9. *Let $a \in R$. Then the following conditions are equivalent:*

(1) $a \in R^\dagger$;

(2) *a is weak-supported;*

(3) *a is right weak-supported;*

(4) *a is coweak-supported;*

(5) *a is left coweak-supported.*

Proof. By the proof of Theorem 3.6 and Theorem 3.7. □

If we take $n = 1$ in the equivalent condition (2) in Theorem 3.7, one can see that the condition a is left $*$-cancellable in [8, Theorem 1] can be dropped. In [8], Koliha, Djordjević and Cvetkvić also proved that $a \in R^\dagger$ if and only if a is right $*$-cancellable and co-supported. If we take $n = 1$ in the equivalent condition (2) in Theorem 3.6, one can see that the condition a is right $*$-cancellable can be dropped. Thus we have the following corollary.

Theorem 3.10. *Let $a \in R$. Then the following conditions are equivalent:*
(1) $a \in R^\dagger$;
(2) *a is well-supported;*
(3) *a is co-supported.*

Theorem 3.11. *Let $a \in R$. Then the following conditions are equivalent for any $n \in \mathbf{N}^+$:*

(1) $a \in R^{\dagger}$;

(2) $R = a^{\circ} \oplus (a^*a)^n R$;

(3) $R = a^{\circ} + (a^*a)^n R$;

(4) $R = (a^*)^{\circ} \oplus (aa^*)^n R$;

(5) $R = (a^*)^{\circ} + (aa^*)^n R$;

(6) $R = {}^{\circ}a \oplus R(aa^*)^n$;

(7) $R = {}^{\circ}a + R(aa^*)^n$;

(8) $R = {}^{\circ}(a^*) \oplus R(a^*a)^n$;

(9) $R = {}^{\circ}(a^*) + R(a^*a)^n$.

Proof. (1) ⇒ (2) Suppose $a \in R^{\dagger}$, then by Theorem 3.1 we have $a \in a(a^*a)^n R$, that is

$$a = a(a^*a)^n b \text{ for some } b \in R. \tag{20}$$

Thus $a[1 - (a^*a)^n b] = 0$, which is equivalent to $1 - (a^*a)^n b \in a^{\circ}$.

By $1 = 1 - (a^*a)^n b + (a^*a)^n b \in a^{\circ} + (a^*a)^n R$, we have

$$R = a^{\circ} + (a^*a)^n R. \tag{21}$$

Let $u \in a^{\circ} \cap (a^*a)^n R$, then we have

$$au = 0 \text{ and } u = (a^*a)^n v, \text{ for some } v \in R. \tag{22}$$

Hence

$$\begin{aligned} u &= (a^*a)^n v = a^*a(a^*a)^{n-1} v = (a(a^*a)^n b)^* a(a^*a)^{n-1} v \\ &= b^*(a^*a)^n a^*a(a^*a)^{n-1} v = b^*(a^*a)^n (a^*a)^n v \\ &= b^*(a^*a)^n u = b^*(a^*a)^{n-1} a^*(au) \\ &= 0. \end{aligned}$$

Whence $R = a^{\circ} \oplus (a^*a)^n R$.

(2) ⇒ (3) It is clear.

(3) ⇒ (1) Suppose $R = a^{\circ} + (a^*a)^n R$, Pre-multiplication of $R = a^{\circ} + (a^*a)^n R$ by a now yields

$$aR = aa^{\circ} + a(a^*a)^n R. \tag{23}$$

By $aa^{\circ} = 0$, we have $a \in a(a^*a)^n R$, that is the condition (3) in Theorem 3.1 is satisfied.

By the equivalence between (1), (2) and (3) and Lemma 2.6, which implies the equivalence between (1), (4) and (5). The equivalence between (1), (6)-(9) is similar to the equivalence between (1), (2)-(5). □

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