An Extension of Darbo’s Theorem via Measure of Non-Compactness with its Application in the Solvability of a System of Integral Equations

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Abstract. In this work, we present a new extension of Darbo’s theorem for two different classes of altering distance functions via measure of non-compactness. Using two-variable contractions we obtain the well-known results in this literature (see [22]). We also use these results to discuss the existence of solutions for a system of integral equations. Finally, we provide an example to confirm the results obtained.

1. Introduction and Preliminaries

The measure of non-compactness is one of the most important and useful concepts in functional analysis. This subject which was initiated by the fundamental article of Kuratowski in [17] and has provided powerful tools for obtaining the solutions of a large variety of integral equations and systems of integral equations. In fixed point theory one of the most important results is due G. Darbo [13]. So far, many scholars have provided generalizations of Darbo’s theorem and have been helped in solving the integral equations (for example see [1–18, 20–23]). In this paper, we present a new extension of Darbo’s theorem for two different classes of altering distance functions via measure of non-compactness. Using two-variable contractions we obtain the well-known results in this literature. We also use these results to discuss the existence of solutions for a system of nonlinear integral equations and give a concrete example.

From now until the end of this work, let \( E \) be a Banach space. Let us denote the set of real numbers with \( \mathbb{R} \). Consider \( \mathbb{R}_+ = [0, +\infty) \). We will denote by \( \overline{B}_r \), the closed ball centered at \( \theta \) with radius \( r \). Considering \( X \subset E, X \neq \varnothing \), assume that \( \overline{X} \) is the closure of the set \( X \) and \( \text{co}X \) denotes the closed convex hull of \( X \). Also we symbolize by \( M_E \) the family of all non-empty and bounded sets and by \( N_E \) subfamily consisting of all relatively compact sets.

Definition 1.1. ([11]) A function \( \mu : M_E \to \mathbb{R}_+ \) is called a measure of non-compactness in \( E \) if it satisfies the following hypothesis:

\[(BM1) \quad \text{The family ker} \mu = \{X \in M_E : \mu (X) = 0\} \neq \varnothing \text{ and ker} \mu \subset N_E;\]

\[(BM2) \quad X \subset Y \Rightarrow \mu (X) \leq \mu (Y);\]

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We say that $l(X) = \mu(\text{co}X) = \mu(X)$;

(BM4) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$;

(BM5) If $(X_k)$ is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{k+1} \subset X_k$ for $k = 1, 2, ..., \text{then the set } X_\infty = \bigcap_{k=1}^{\infty} X_k \neq \emptyset$.

The subfamily ker $\mu$ defined in (BM1) represents Kernel of $\mu$ and since $\mu(X_\infty) = \mu\left(\bigcap_{k=1}^{\infty} X_k\right) \leq \mu(X_k)$, we see that $\mu(X_\infty) = 0$. Therefore $X_\infty \in \text{ker } \mu$.

**Definition 1.2.** We say that $l : [0, +\infty)^3 \rightarrow [0, +\infty)$ is a lower semi-continuous function, if for any arbitrary sequences $\{a_k\}$ and $\{b_k\}$ and $\{c_k\}$ of $[0, +\infty)$,

$$l\left(\liminf_{k \to \infty} a_k, \liminf_{k \to \infty} b_k, \liminf_{k \to \infty} c_k\right) \leq \liminf_{k \to \infty} l(a_k, b_k, c_k).$$

For example, $l_1(p, q, r) = \ln(p + q + r + 1)$ and $l_2(p, q, r) = \max\{p, q, r\}$ are lower semicontinuous.

**Theorem 1.3.** \cite{[6]} Assume that $\mu_1, \mu_2, ..., \mu_k$ are measures of non-compactness in Banach spaces $E_1, E_2, ..., E_k$ respectively. Also, suppose that the function $G : [0, +\infty]^k \rightarrow [0, +\infty)$ is convex and $G(p_1, p_2, ..., p_k) = 0 \iff p_i = 0, (i = 1, 2, 3, ..., k)$. Then

$$\bar{\mu}(X) = G(\mu_1(X_1), \mu_2(X_2), ..., \mu_k(X_k)),$$

defines a measure of non-compactness in $E_1 \times E_2 \times \cdots \times E_k$ where $X_i$ denotes the natural projections of $X$ into $E_i$, for $i = 1, 2, 3, ..., k$.

**Example 1.4.** \cite{[6]} Consider $G(p, q, r) = p + q + r$ for every $(p, q, r) \in [0, +\infty)^3$, then $G$ has all conditions of Theorem 1.3. So, $\bar{\mu}(X) = \mu(X_1) + \mu(X_2) + \mu(X_3)$ for each $X \subseteq E \times E$ is the measure of non-compactness in $E \times E \times E$.

**Theorem 1.5.** \cite{[3]} Assume that $C$ be a convex and closed subset of $E$. Then every compact, continuous map $T : C \rightarrow C$ has at least one fixed point.

**Theorem 1.6.** \cite{[3]} Assume that $\Omega$ be a non-empty, bounded, closed and convex subset of $E$. Consider the constant $\lambda \in [0, 1]$. Also, suppose that $T : \Omega \rightarrow \Omega$ is a continuous operator such that $\mu(T(X)) \leq \lambda \mu(X)$ for each $X \subset \Omega$ with $X \neq \emptyset$. Then $T$ has a fixed point in $\Omega$.

Now, we introduce three different classes of functions that we need in the next section.

**Definition 1.7.** Let $\Theta$ be the class of all functions $\theta : [0, +\infty)^3 \rightarrow [0, +\infty)$ satisfying the following hypothesis:

(A1) $\theta(p_1 + p_2, q_1 + q_2, r_1 + r_2) \leq \theta(p_1, q_1, r_1) + \theta(p_2, q_2, r_2)$ for every $p_1, p_2, q_1, q_2, r_1, r_2 \in \mathbb{R}_+$,

(A2) $\theta(p, q, r) = 0 \iff p = q = r = 0$, for every $p, q, r \in \mathbb{R}_+$,

(A3) $\theta$ is lower semicontinuous.

For example, $\theta_1(p, q, r) = \ln(p + q + r + 1)$ and $\theta_2(p, q, r) = \max\{p, q, r\}$ satisfy the above three properties.

**Definition 1.8.** Let $\Phi$ be the class of all functions $\phi : [0, +\infty)^3 \rightarrow [0, +\infty)$ satisfying the following hypothesis:

(B1) $\phi$ is continuous and nondecreasing,

(B2) $\phi(h, h, h) < h$ for every $h > 0$,

(B3) $\frac{1}{3}\left(\phi(p_1, q_1, r_1) + \phi(p_2, q_2, r_2) + \phi(p_3, q_3, r_3)\right) \leq \phi\left(\frac{p_1 + p_2 + p_3}{3}, \frac{q_1 + q_2 + q_3}{3}, \frac{r_1 + r_2 + r_3}{3}\right)$ for every
Definition 1.9. Let $\Psi$ be the class of all functions $\psi : [0, +\infty)^2 \to [0, +\infty)$ satisfying the following hypothesis:

(C1) $\psi$ is continuous,

(C2) $\psi (h,h) \geq h$ for every $h > 0$.

For example, $\psi_1 (p,q) = p + q$ and $\psi_2 (p,q) = \sqrt{p^2 + q^2}$ and $\psi_3 (p,q) = e^{\sqrt{p^2 + q^2}} - 1$, in which $p, q \in \mathbb{R}_+$ satisfy the above two properties.

Let $BC (\mathbb{R}_+)$ be the Banach space consisting of all defined, bounded and continuous functions on $\mathbb{R}_+$ equipped with the standard supremum norm

$$
||x|| = \sup \{ |x (\tau)| : \tau \geq 0 \}.
$$

Fix $X \subset BC (\mathbb{R}_+), X \neq \emptyset$ and $L > 0$ and $\tau \in \mathbb{R}_+$. For $x \in X$ and $c \geq 0$

$$
\omega_\gamma (x,c) = \sup \{ |x (\tau) - x (v)| : \tau, v \in [0,L], |\tau - v| \leq c \},
$$

$$
\omega_\gamma (X,c) = \sup \{ \omega_\gamma (x,c) : x \in X \},
$$

$$
\omega_0 (X) = \lim_{\gamma \to 0} \omega_\gamma (X,\gamma),
$$

$$
\omega_0 (X) = \sup_{L \to \infty} \omega_0 (X),
$$

$$
X (\tau) = \{ x (\tau) : x \in X \},
$$

and

$$
\mu (X) = \omega_0 (X) + \lim_{\tau \to \infty} \sup \text{diam} X (\tau),
$$

where

$$
\text{diam} X (\tau) = \sup \{ |x (\tau) - y (\tau)| : x, y \in X \}.
$$

As mentioned in [11], $\mu (X)$ is the measure of non-compactness in $BC (\mathbb{R}_+)$. 

2. Main results

Throughout the main results section, let us assume that $\Omega$ is a non-empty, bounded, closed, and convex subset of $E$. Also, assume $\mu$ is an arbitrary measure of non-compactness in $E$.

Theorem 2.1. Assume that $\bar{\mu}$ be a measure of non-compactness as in Example 1.4 and $\psi \in \Psi, \theta \in \Theta$. Also, suppose $G : \Omega \times \Omega \times \Omega \to \Omega \times \Omega \times \Omega$ is a continuous operator satisfying:

$$
\psi (\bar{\mu} (G (X)), \bar{\mu} (G (X))) \leq \psi (\bar{\mu} (X), \bar{\mu} (X)) - \theta (\bar{\mu} (X), \bar{\mu} (X)),
$$

for each $X \subset \Omega \times \Omega \times \Omega$ with $X \neq \emptyset$. Then $G$ has at least one fixed point in $\Omega \times \Omega \times \Omega$.

Proof. We define a sequence $\{ \Omega_1 \times \Omega_1 \times \Omega_1 \}_{k=1}^{\infty}$ inductively such that

$$
\Omega_0 \times \Omega_0 \times \Omega_0 = \Omega \times \Omega \times \Omega, \Omega_k \times \Omega_k \times \Omega_k = cG (\Omega_{k-1} \times \Omega_{k-1} \times \Omega_{k-1}),
$$
for $k = 1, 2, \ldots$ By given conditions, we get

$$G(\Omega_0 \times \Omega_0 \times \Omega_0) = G(\Omega \times \Omega \times \Omega) \subseteq \Omega \times \Omega \times \Omega = \Omega_0 \times \Omega_0 \times \Omega_0,$$

$$\Omega_1 \times \Omega_1 \times \Omega_1 = \text{co}G(\Omega_0 \times \Omega_0 \times \Omega_0) \subseteq \Omega \times \Omega \times \Omega = \Omega_0 \times \Omega_0 \times \Omega_0,$$

$$\ldots$$

$$\Omega_{k+1} \times \Omega_{k+1} \times \Omega_{k+1} \subseteq \Omega_k \times \Omega_k \times \Omega_k \subseteq \ldots \subseteq \Omega_1 \times \Omega_1 \times \Omega_1 \subseteq \Omega_0 \times \Omega_0 \times \Omega_0.$$

Next, if for an integer $K \geq 0$ we have $\mu(\Omega_k \times \Omega_k \times \Omega_k) = 0$, then $\Omega_k \times \Omega_k \times \Omega_k$ is relatively compact. Hence, the proof is completed by using Theorem 1.5. Therefore we suppose that $\mu(\Omega_k \times \Omega_k \times \Omega_k) > 0$ for every $k \geq 0$. Also with given assumptions, we obtain

$$\psi(\mu(\Omega_{k-1} \times \Omega_{k-1} \times \Omega_{k-1})) \leq \psi(\mu(\Omega_k \times \Omega_k \times \Omega_k)) - \theta(\mu(\Omega_k \times \Omega_k \times \Omega_k)).$$

Since the sequence $[\mu(\Omega_k \times \Omega_k \times \Omega_k)]_{k=1}^{\infty}$ is a nonincreasing and positive sequence, therefore, there is an $\alpha \geq 0$ such that $\mu(\Omega_k \times \Omega_k \times \Omega_k) \to \alpha$, as $k \to \infty$. Moreover, we have

$$\psi(\alpha, \alpha) = \lim_{k \to \infty} \sup_{k \to \infty} \left( \mu(\Omega_{k+1} \times \Omega_{k+1} \times \Omega_{k+1}) \right) \mu(\Omega_k \times \Omega_k \times \Omega_k),$$

$$\leq \lim_{k \to \infty} \inf_{k \to \infty} \theta(\frac{\mu(\Omega_k \times \Omega_k \times \Omega_k)}{\mu(\Omega_{k+1} \times \Omega_{k+1} \times \Omega_{k+1}))} \right) \leq \lim_{k \to \infty} \mu(\Omega_k \times \Omega_k \times \Omega_k),$$

$$= \psi(\alpha, \alpha) - \theta(\alpha, \alpha, \alpha).$$

So, $\theta(\alpha, \alpha, \alpha) = 0$, and hence $\alpha = 0$. So, we conclude that $\mu(\Omega_k \times \Omega_k \times \Omega_k) \to 0$, as $k \to \infty$. Now, since $\Omega_{k+1} \times \Omega_{k+1} \times \Omega_{k+1} \subseteq \Omega_k \times \Omega_k \times \Omega_k$, then from (BM5), we conclude that $\Omega_+ \times \Omega_+ \times \Omega_+ = \cap_{k=0}^\infty \Omega_k \times \Omega_k \times \Omega_k$ is a non-empty, convex, closed set, invariant under $G$ and $\Omega_+ \times \Omega_+ \times \Omega_+ \subseteq \ker \mu$. So, from Theorem 1.5 we deduce that $G$ has a fixed point in $\Omega_+ \times \Omega_+ \times \Omega_+$. Since $\Omega_+ \times \Omega_+ \times \Omega_+ \subseteq \Omega \times \Omega \times \Omega$, then the proof is completed.

\[ \square \]

**Theorem 2.2.** Suppose $\psi \in \Psi$ is nondecreasing with $\psi(p_1 + p_2, q_1 + q_2) \leq \psi(p_1, q_1) + \psi(p_2, q_2)$ for every $p_1, p_2, q_1, q_2 \in \mathbb{R}_+$ and $\theta \in \Theta$. Also assume that $G_i : \Omega \times \Omega \times \Omega \to \Omega$ ($i = 1, 2, 3$) are continuous operators satisfying:

$$\psi(\mu(\mu(G_1(X_1 \times X_2 \times X_3)), \mu(G_1(X_1 \times X_2 \times X_3))) \leq \frac{1}{3}\psi \left( \begin{array} {c} \mu(X_1) + \mu(X_2) + \mu(X_3) \\ \mu(X_1) + \mu(X_2) + \mu(X_3) \\ \mu(X_1) + \mu(X_2) + \mu(X_3) \end{array} \right)$$

$$- \theta(\mu(X_1), \mu(X_2), \mu(X_3)),$$

$$\psi(\mu(G_2(X_1 \times X_2 \times X_3)), \mu(G_2(X_1 \times X_2 \times X_3))) \leq \frac{1}{3}\psi \left( \begin{array} {c} \mu(X_1) + \mu(X_2) + \mu(X_3) \\ \mu(X_1) + \mu(X_2) + \mu(X_3) \\ \mu(X_1) + \mu(X_2) + \mu(X_3) \end{array} \right)$$

$$- \theta(\mu(X_1), \mu(X_2), \mu(X_3)),$$

$$\psi(\mu(G_3(X_1 \times X_2 \times X_3)), \mu(G_3(X_1 \times X_2 \times X_3))) \leq \frac{1}{3}\psi \left( \begin{array} {c} \mu(X_1) + \mu(X_2) + \mu(X_3) \\ \mu(X_1) + \mu(X_2) + \mu(X_3) \\ \mu(X_1) + \mu(X_2) + \mu(X_3) \end{array} \right)$$

$$- \theta(\mu(X_1), \mu(X_2), \mu(X_3)),$$

(3)
for each $X_1, X_2, X_3 \subset \Omega$. Then there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

\[
\begin{align*}
G_1(\tau^*, v^*, \rho^*) &= \tau^* \\
G_2(\tau^*, v^*, \rho^*) &= v^* \\
G_3(\tau^*, v^*, \rho^*) &= \rho^*
\end{align*}
\] (4)

**Proof.** Consider $\mu$ as defined in Example 1.4. We define $\tilde{G}$ on $\Omega \times \Omega \times \Omega$ as following:

\[
\tilde{G}(\tau, v, \rho) = (G_1(\tau, v, \rho), G_2(\tau, v, \rho), G_3(\tau, v, \rho)).
\]

Clearly, $\tilde{G}$ is continuous on $\Omega \times \Omega \times \Omega$ by its definition. We will show that $\tilde{G}$ satisfies all the hypothesis of Theorem 2.1. For this purpose, let $X \subset \Omega \times \Omega \times \Omega, X \neq \emptyset$. Then, by axiom (BM2) of Definition 1.1 and relation (3) we obtain

\[
\begin{align*}
\psi\left(\tilde{\mu}(\tilde{G}(X)), \mu(\tilde{G}(X))\right) &\leq \psi\left(\begin{array}{c}
\tilde{\mu} \left(\begin{array}{c}
G_1(X_1 \times X_2 \times X_3) \times G_2(X_1 \times X_2 \times X_3) \\
\times G_3(X_1 \times X_2 \times X_3)
\end{array}\right) \\
\tilde{\mu} \left(\begin{array}{c}
\mu(G_1(X_1 \times X_2 \times X_3)) + \mu(G_2(X_1 \times X_2 \times X_3)) \\
+ \mu(G_3(X_1 \times X_2 \times X_3))
\end{array}\right)
\end{array}\right) \\
= \psi\left(\begin{array}{c}
\mu(G_1(X_1 \times X_2 \times X_3)) + \mu(G_2(X_1 \times X_2 \times X_3)) \\
+ \mu(G_3(X_1 \times X_2 \times X_3))
\end{array}\right) \\
\leq \psi\left(\mu(G_1(X_1 \times X_2 \times X_3)) \times \mu(G_2(X_1 \times X_2 \times X_3)) \times \mu(G_3(X_1 \times X_2 \times X_3))\right)
\end{align*}
\]

So, from Theorem 2.1 we deduce that $\tilde{G}$ has a fixed point, that is, there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

\[
(\tau^*, v^*, \rho^*) = \tilde{G}(\tau^*, v^*, \rho^*) = (G_1(\tau^*, v^*, \rho^*), G_2(\tau^*, v^*, \rho^*), G_3(\tau^*, v^*, \rho^*))
\]

which means (4) is satisfied. \(\square\)
Corollary 2.3. Suppose $\lambda_1, \lambda_2, \lambda_3$ are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Also assume that $G_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ ($i = 1, 2, 3$) are continuous operators satisfying:

$$
\mu(G_i(X_1 \times X_2 \times X_3)) \leq \frac{\lambda_1}{3} \mu(X_1) + \frac{\lambda_2}{3} \mu(X_2) + \frac{\lambda_3}{3} \mu(X_3),
$$

for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

$$
\left\{
\begin{array}{ll}
G_1(\tau^*, v^*, \rho^*) &= \tau^* \\
G_2(\tau^*, v^*, \rho^*) &= v^* \\
G_3(\tau^*, v^*, \rho^*) &= \rho^*
\end{array}
\right.
$$

Proof. Considering $\psi(p, q) = p + q$ and $\theta(p, q, r) = \frac{2}{3} [(1 - \lambda_1)p + (1 - \lambda_2)q + (1 - \lambda_3)r]$ in Theorem 2.2 the result is desirable. \qed

Corollary 2.4. Consider the constant $\lambda$ with $0 \leq \lambda < 1$. Also assume that $G_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ ($i = 1, 2, 3$) are continuous operators satisfying:

$$
\mu(G_i(X_1 \times X_2 \times X_3)) \leq \lambda \max \{\mu(X_1), \mu(X_2), \mu(X_3)\},
$$

for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

$$
\left\{
\begin{array}{ll}
G_1(\tau^*, v^*, \rho^*) &= \tau^* \\
G_2(\tau^*, v^*, \rho^*) &= v^* \\
G_3(\tau^*, v^*, \rho^*) &= \rho^*
\end{array}
\right.
$$

Proof. Considering $\psi(p, q) = p + q$ and $\theta(p, q, r) = 2(1 - \lambda) \max \{p, q, r\}$ in Theorem 2.2 the result is desirable. \qed

Corollary 2.5. Suppose $G_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ ($i = 1, 2, 3$) are continuous operators satisfying:

$$
\mu(G_i(X_1 \times X_2 \times X_3)) \leq \frac{\mu(X_1) + \mu(X_2) + \mu(X_3)}{3} - \ln(\mu(X_1) + \mu(X_2) + \mu(X_3) + 1),
$$

for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

$$
\left\{
\begin{array}{ll}
G_1(\tau^*, v^*, \rho^*) &= \tau^* \\
G_2(\tau^*, v^*, \rho^*) &= v^* \\
G_3(\tau^*, v^*, \rho^*) &= \rho^*
\end{array}
\right.
$$

Proof. Considering $\psi(p, q) = p + q$ and $\theta(p, q, r) = 2 \ln(p + q + r + 1)$ in Theorem 2.2 the result is desirable. \qed

Corollary 2.6. Consider the constant $\lambda$ with $0 \leq \lambda \leq 1$. Also assume that $G_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ ($i = 1, 2, 3$) are continuous operators satisfying:

$$
\mu(G_i(X_1 \times X_2 \times X_3)) \leq (1 - \lambda^2)\left(\frac{\mu(X_1) + \mu(X_2) + \mu(X_3)}{9}\right),
$$

for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

$$
\left\{
\begin{array}{ll}
G_1(\tau^*, v^*, \rho^*) &= \tau^* \\
G_2(\tau^*, v^*, \rho^*) &= v^* \\
G_3(\tau^*, v^*, \rho^*) &= \rho^*
\end{array}
\right.
$$
Proof. Considering $\psi(p, q) = \sqrt{p + q}$ and $\theta(p, q, r) = \frac{1}{2} \sqrt{p + q + r}$ in Theorem 2.2 the result is desirable. □

**Theorem 2.7.** Assume that $\overline{\mu}$ is a measure of non-compactness as in Example 1.4 and $\phi \in \Phi$, $\psi \in \Psi$. Also suppose $F : \Omega \times \Omega \times \Omega \rightarrow \Omega \times \Omega \times \Omega$ be a continuous operator satisfying:

$$
\psi\left(\overline{\mu}(F(X))\right) \leq \psi\left(\mu(X), \overline{\mu}(X), \overline{\mu}(X)\right),
$$

(5)

for each $X \in \Omega \times \Omega \times \Omega$ with $X \neq \emptyset$. Then $F$ has at least one fixed point in $\Omega \times \Omega \times \Omega$.

Proof. We define a sequence $\{\Omega_k \times \Omega_k \times \Omega_k\}_{k=1}^{\infty}$ inductively such that

$$
\Omega_0 \times \Omega_0 \times \Omega_0 = \Omega \times \Omega \times \Omega, \Omega_k \times \Omega_k \times \Omega_k = \text{co} F(\Omega_{k-1} \times \Omega_{k-1} \times \Omega_{k-1}),
$$

for $k = 1, 2, \ldots$. By given conditions, we obtain

$$
\begin{align*}
F(\Omega_0 \times \Omega_0 \times \Omega_0) &= \text{co} F(\Omega_0 \times \Omega_0 \times \Omega_0) = \Omega_0 \times \Omega_0 \times \Omega_0, \\
\Omega_1 \times \Omega_1 \times \Omega_1 &= \text{co} F(\Omega_0 \times \Omega_0 \times \Omega_0) = \Omega_0 \times \Omega_0 \times \Omega_0, \\
& \quad \cdots
\end{align*}
$$

If for an integer $K \geq 0$ we have $\overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k) = 0$, then $\Omega_k \times \Omega_k \times \Omega_k$ is relatively compact. Hence, the proof is completed by using Theorem 1.5. Therefore, we suppose that $\overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k) > 0$ for each $k \geq 0$. Now, by given conditions, we get

$$
\begin{align*}
\psi(\overline{\mu}(\Omega_{k+1} \times \Omega_{k+1} \times \Omega_{k+1}), \overline{\mu}(\Omega_{k+1} \times \Omega_{k+1} \times \Omega_{k+1})) &\leq \psi(\mu(\Omega_k \times \Omega_k \times \Omega_k), \overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k)), \\
&= \psi(\mu(F(\Omega_k \times \Omega_k \times \Omega_k)), \overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k)), \\
&\leq \phi(\mu(\Omega_k \times \Omega_k \times \Omega_k), \overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k), \overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k)).
\end{align*}
$$

(6)

Moreover, the sequence $\{\overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k)\}_{k=1}^{\infty}$ is a nonincreasing and positive sequence of real numbers, therefore, there is an $\alpha \geq 0$, $\overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k) \rightarrow \alpha$, as $k \rightarrow \infty$. We show that $\alpha = 0$. If we assume that $\alpha > 0$, then from (6), we get

$$
\begin{align*}
\psi(\alpha, \alpha) &= \psi\left(\lim_{k \to \infty} \overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k), \lim_{k \to \infty} \overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k)\right) \\
&\leq \phi(\mu(\Omega_k \times \Omega_k \times \Omega_k), \mu(\Omega_k \times \Omega_k \times \Omega_k), \mu(\Omega_k \times \Omega_k \times \Omega_k)), \\
&= \phi(\alpha, \alpha, \alpha) < \alpha.
\end{align*}
$$

Which is contradiction. Therefore, we conclude that $\overline{\mu}(\Omega_k \times \Omega_k \times \Omega_k) \rightarrow 0$, as $k \rightarrow \infty$. Now, since $\Omega_k \times \Omega_k \times \Omega_k \subseteq \Omega \times \Omega \times \Omega$, then from (BM5), we conclude that $\Omega_\infty \times \Omega_\infty \times \Omega_\infty = \cap_{k=1}^{\infty} \Omega_k \times \Omega_k \times \Omega_k$ is a non-empty, convex, closed set, invariant under $F$ and $\Omega_\infty \times \Omega_\infty \times \Omega_\infty \in \ker \overline{\mu}$. So, from Theorem 1.5 we conclude that $F$ has a fixed point in $\Omega_\infty \times \Omega_\infty \times \Omega_\infty$. Since $\Omega_\infty \times \Omega_\infty \times \Omega_\infty \subset \Omega \times \Omega \times \Omega$, then the proof is completed. □

**Theorem 2.8.** Suppose $\psi \in \Psi$ is nondecreasing with $\psi(p_1 + p_2, q_1 + q_2) \leq \psi(p_1, q_1) + \psi(p_2, q_2)$ for every $p_1, p_2, q_1, q_2 \in \mathbb{R}_+$ and $\phi \in \Phi$. Also assume that $F_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ ($i = 1, 2, 3$) are continuous operators satisfying:

$$
\begin{align*}
\psi(\mu(F_1(X_1 \times X_2 \times X_3)), \mu(F_1(X_1 \times X_2 \times X_3))) &\leq \phi(\mu(X_1), \mu(X_2), \mu(X_3)), \\
\psi(\mu(F_2(X_1 \times X_2 \times X_3)), \mu(F_2(X_1 \times X_2 \times X_3))) &\leq \phi(\mu(X_2), \mu(X_2), \mu(X_1)), \\
\psi(\mu(F_3(X_1 \times X_2 \times X_3)), \mu(F_3(X_1 \times X_2 \times X_3))) &\leq \phi(\mu(X_3), \mu(X_1), \mu(X_2)),
\end{align*}
$$

(7)
for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

$$
\begin{align*}
F_1 (\tau^*, v^*, \rho^*) &= \tau^*, \\
F_2 (\tau^*, v^*, \rho^*) &= v^*, \\
F_3 (\tau^*, v^*, \rho^*) &= \rho^*.
\end{align*}
$$

(8)

**Proof.** Consider $\bar{\mu}$ as defined in Example 1.4. We define $\bar{F}$ on $\Omega \times \Omega \times \Omega$ as follows:

$$
\bar{F}(\tau, v, \rho) = (F_1 (\tau, v, \rho), F_2 (\tau, v, \rho), F_3 (\tau, v, \rho)).
$$

Clearly, $\bar{F}$ is continuous on $\Omega \times \Omega \times \Omega$ by its definition. We will show that $\bar{F}$ satisfies all the hypothesis of Theorem 2.7. For this purpose, let $X \subseteq \Omega \times \Omega \times \Omega, X \neq \emptyset$. Then, by (BM2) and (7) we obtain

$$
\psi \left( \bar{\mu} (\bar{F}(X)), \bar{\mu}(\bar{F}(X)) \right) \leq \psi \left( \bar{\mu} \left( \begin{array}{c}
F_1 (X_1 \times X_2 \times X_3) \times F_2 (X_1 \times X_2 \times X_3) \\
\times F_3 (X_1 \times X_2 \times X_3)
\end{array} \right) \right),
$$

(9)

Thus, by Theorem 2.7 $\bar{F}$ has a fixed point, i.e., there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

$$
(\tau^*, v^*, \rho^*) = \bar{F}(\tau^*, v^*, \rho^*) = (F_1 (\tau^*, v^*, \rho^*), F_2 (\tau^*, v^*, \rho^*), F_3 (\tau^*, v^*, \rho^*)),
$$

which means (8) is satisfied. \( \square \)

**Corollary 2.9.** Assume that $F_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ $(i = 1, 2, 3)$ are continuous operators satisfying:

$$
\begin{align*}
\mu (F_1 (X_1 \times X_2 \times X_3)) &\leq \phi (\mu (X_1), \mu (X_2), \mu (X_3)), \\
\mu (F_2 (X_1 \times X_2 \times X_3)) &\leq \phi (\mu (X_2), \mu (X_3), \mu (X_1)), \\
\mu (F_3 (X_1 \times X_2 \times X_3)) &\leq \phi (\mu (X_3), \mu (X_1), \mu (X_2)),
\end{align*}
$$

for each $X_1, X_2, X_3 \subseteq \Omega$, where $\phi \in \Phi$. Then there exist $\tau^*, v^*, \rho^* \in \Omega$ such that

$$
\begin{align*}
F_1 (\tau^*, v^*, \rho^*) &= \tau^*, \\
F_2 (\tau^*, v^*, \rho^*) &= v^*, \\
F_3 (\tau^*, v^*, \rho^*) &= \rho^*.
\end{align*}
$$
Proof. Considering $\psi(p, q) = \frac{p + q}{2}$ in Theorem 2.8 the result is desirable. □

Corollary 2.10. Suppose $\lambda_1, \lambda_2, \lambda_3$ are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Also assume that $F_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ (i = 1, 2, 3) are continuous operators satisfying:

$$
\begin{align*}
\mu (F_1 (X_1 \times X_2 \times X_3)) & \leq \lambda_1 \mu (X_1) + \lambda_2 \mu (X_2) + \lambda_3 \mu (X_3), \\
\mu (F_2 (X_1 \times X_2 \times X_3)) & \leq \lambda_1 \mu (X_2) + \lambda_2 \mu (X_3) + \lambda_3 \mu (X_1), \\
\mu (F_3 (X_1 \times X_2 \times X_3)) & \leq \lambda_1 \mu (X_3) + \lambda_2 \mu (X_1) + \lambda_3 \mu (X_2),
\end{align*}
$$

for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, \nu^*, \rho^* \in \Omega$ such that

$$
\begin{align*}
F_1 (\tau^*, \nu^*, \rho^*) &= \tau^* \\
F_2 (\tau^*, \nu^*, \rho^*) &= \nu^* \\
F_3 (\tau^*, \nu^*, \rho^*) &= \rho^*
\end{align*}
$$

Proof. Considering $\psi(p, q) = p + q$ and $\phi(p, q, r) = 2\lambda_1 p + 2\lambda_2 q + 2\lambda_3 r$ in Theorem 2.8 the result is desirable. □

Corollary 2.11. Assume that $F_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ (i = 1, 2, 3) are continuous operators satisfying:

$$
\mu (F_i (X_1 \times X_2 \times X_3)) \leq \frac{1}{\sqrt{2}} \ln \left( 1 + \frac{\mu (X_1) + \mu (X_2) + \mu (X_3)}{3} \right),
$$

for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, \nu^*, \rho^* \in \Omega$ such that

$$
\begin{align*}
F_1 (\tau^*, \nu^*, \rho^*) &= \tau^* \\
F_2 (\tau^*, \nu^*, \rho^*) &= \nu^* \\
F_3 (\tau^*, \nu^*, \rho^*) &= \rho^*
\end{align*}
$$

Proof. Considering $\psi(p, q) = \sqrt{p^2 + q^2}$ and $\phi(p, q, r) = \ln \left( 1 + \frac{p + q + r}{3} \right)$ in Theorem 2.8 the result is desirable. □

Corollary 2.12. Suppose $\lambda_1, \lambda_2, \lambda_3$ are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Also assume that $F_i : \Omega \times \Omega \times \Omega \rightarrow \Omega$ (i = 1, 2, 3) are continuous operators satisfying:

$$
\begin{align*}
\mu (F_1 (X_1 \times X_2 \times X_3)) + \ln (1 + \mu (F_1 (X_1 \times X_2 \times X_3))) & \leq \lambda_1 \mu (X_1) + \lambda_2 \mu (X_2) + \lambda_3 \mu (X_3), \\
\mu (F_2 (X_1 \times X_2 \times X_3)) + \ln (1 + \mu (F_2 (X_1 \times X_2 \times X_3))) & \leq \lambda_1 \mu (X_2) + \lambda_2 \mu (X_3) + \lambda_3 \mu (X_1), \\
\mu (F_3 (X_1 \times X_2 \times X_3)) + \ln (1 + \mu (F_3 (X_1 \times X_2 \times X_3))) & \leq \lambda_1 \mu (X_3) + \lambda_2 \mu (X_1) + \lambda_3 \mu (X_2),
\end{align*}
$$

for each $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $\tau^*, \nu^*, \rho^* \in \Omega$ such that

$$
\begin{align*}
F_1 (\tau^*, \nu^*, \rho^*) &= \tau^* \\
F_2 (\tau^*, \nu^*, \rho^*) &= \nu^* \\
F_3 (\tau^*, \nu^*, \rho^*) &= \rho^*
\end{align*}
$$

Proof. Considering $\psi(p, q) = \frac{p^q + \ln \left( 1 + \frac{p + q}{2} \right)}{2}$ and $\phi(p, q, r) = \lambda_1 p + \lambda_2 q + \lambda_3 r$ in Theorem 2.8 the result is desirable. □
3. Application and Example

Consider the following system of integral equations:

\[
\begin{align*}
\tau, x(\tau), y(\epsilon_1(\tau)), z(\epsilon_1(\tau)) \\
+ f_1 \left( \theta_1 \left( \int_0^{\beta(\tau)} g_1 (\tau, \nu, x(a_1(\nu)), y(a_1(\nu)), z(a_1(\nu))) d\nu \right) \right), \\
\tau, x(\epsilon_2(\tau)), y(\epsilon_2(\tau)), z(\epsilon_2(\tau)) \\
+ f_2 \left( \theta_2 \left( \int_0^{\beta(\tau)} g_2 (\tau, \nu, x(a_2(\nu)), y(a_2(\nu)), z(a_2(\nu))) d\nu \right) \right), \\
\tau, x(\epsilon_3(\tau)), y(\epsilon_3(\tau)), z(\epsilon_3(\tau)) \\
+ f_3 \left( \theta_3 \left( \int_0^{\beta(\tau)} g_3 (\tau, \nu, x(a_3(\nu)), y(a_3(\nu)), z(a_3(\nu))) d\nu \right) \right) \\
\end{align*}
\]

\( (11) \)

**Theorem 3.1.** Let

(I) \( A_i : \mathbb{R}_+ \rightarrow \mathbb{R}, i = 1, 2, 3 \) are continuous and bounded functions with

\[ M_i = \sup \{|A_i(\tau)| : \tau \in \mathbb{R}_+\}. \]

(II) \( \epsilon_i, \sigma_i, \beta_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) are continuous functions and \( \epsilon_i(\tau) \rightarrow \infty \) as \( \tau \rightarrow \infty \), for \( i = 1, 2, 3 \),

(III) \( \theta_i : \mathbb{R} \rightarrow \mathbb{R} \) with \( \theta_i(0) = 0 \) are continuous functions and consider the positive constants \( \alpha_i, \beta_i \) with

\[ |\theta_i(\tau_1) - \theta_i(\tau_2)| \leq \beta_i |\tau_1 - \tau_2|^{\alpha_i}, \]

for every \( \tau_1, \tau_2 \in \mathbb{R}_+, i = 1, 2, 3 \),

(IV) \( |f_i(\tau, 0, 0, 0, 0)| \) and \( |h_i(\tau, 0, 0, 0)|, (i = 1, 2, 3) \) are bounded on \( \mathbb{R}_+ \), that is,

\[ M_i' = \sup \{|f_i(\tau, 0, 0, 0, 0)| : \tau \in \mathbb{R}_+\} < \infty, \]

\[ M_i'' = \sup \{|h_i(\tau, 0, 0, 0)| : \tau \in \mathbb{R}_+\} < \infty. \]

(V) \( f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) and \( h_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions, \( \phi_i \in \Phi \) and \( \phi_i : \mathbb{R}_+ \rightarrow \mathbb{R} \) are nondecreasing continuous functions with \( \phi_i(0) = 0 \), for \( i = 1, 2, 3 \),

\[ |h_i(\tau, x, y, z) - h_i(\tau, u, v, w)| \leq \frac{1}{2} \phi_i \left( |x - u|, |y - v|, |z - w| \right), \]

\[ |f_i(\tau, x, y, z, m) - f_i(\tau, u, v, w, n)| \leq \frac{1}{2} \phi_i \left( |x - u|, |y - v|, |z - w| \right) + \phi_i \left( |m - n| \right), \]

for every \( \tau \geq 0, x, y, z, m, n, u, v, w \in \mathbb{R} \),

(VI) \( \gamma_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}, (i = 1, 2, 3) \) are continuous and also we have

\[ \lim_{\tau \rightarrow \infty} \int_0^{\beta(\tau)} |g_i(\tau, \nu, x(a_1^i(\nu)), y(a_1^i(\nu)), z(a_1^i(\nu))) - g_i(\tau, \nu, x(a_1^i(\nu)), y(a_1^i(\nu)), z(a_1^i(\nu)))| d\nu = 0, \]

and,
\[ M_i^+ = \sup \left\{ \left| \int_0^{\delta_i(x)} g_i(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v))) \, dv \right| : \tau \in \mathbb{R}_+ \right\}, \quad (15) \]

(VII) The following inequality for a \( p > 0 \) is valid.

\[ M_i + \phi_i(\kappa, \kappa, \kappa) + M_i^+ + \phi_i(\delta_iM_i^+) < \kappa, (i = 1, 2, 3). \quad (16) \]

Then the system (11) has at least one solution in \( BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \).

\textbf{Proof.} Consider the operators \( T_i : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \to BC(\mathbb{R}_+) \), \( (i = 1, 2, 3) \) by the formula:

\[ T_i(x, y, z)(\tau) = \left\{ \begin{array}{l}
A_i(\tau) + h_i(\tau, x(\epsilon_i(\tau)), y(\epsilon_i(\tau)), z(\epsilon_i(\tau))) \\
+ f_i \left( \phi_i \left( \int_0^{\delta_i(x)} g_i(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v))) \, dv \right) \right) \end{array} \right. \quad (17) \]

Since \( A_i, h_i \) and \( f_i \), \( (i = 1, 2, 3) \) are continuous, then \( T_i, (i = 1, 2, 3) \) are continuous. Also with given assumptions, we get

\[ \left| T_i(x, y, z)(\tau) \right| \leq \left| A_i(\tau) \right| + \left| h_i(\tau, x(\epsilon_i(\tau)), y(\epsilon_i(\tau)), z(\epsilon_i(\tau))) \right| - h_i(\tau, 0, 0, 0) \]

\[ + \left| f_i \left( \phi_i \left( \int_0^{\delta_i(x)} g_i(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v))) \, dv \right) \right) \right| \]

\leq M_i + M_i^+ + \frac{1}{2} \phi_i \left( \left| x(\epsilon_i(\tau)) \right|, \left| y(\epsilon_i(\tau)) \right|, \left| z(\epsilon_i(\tau)) \right| \right) \]

\[ + \frac{1}{2} \phi_i \left( \left| x(\epsilon_i(\tau)) \right|, \left| y(\epsilon_i(\tau)) \right|, \left| z(\epsilon_i(\tau)) \right| \right) \]

\[ + \phi_i \left( \delta_i \left( \int_0^{\delta_i(x)} g_i(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v))) \, dv \right) \right) \]

\[ \leq M_i + M_i^+ + \phi_i \left( \left| x(\epsilon_i(\tau)) \right|, \left| y(\epsilon_i(\tau)) \right|, \left| z(\epsilon_i(\tau)) \right| \right) \]

\[ + \phi_i \left( \delta_i \left( \int_0^{\delta_i(x)} g_i(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v))) \, dv \right) \right) \]

\[ \leq M_i + M_i^+ + \phi_i \left( \| x \|, \| y \|, \| z \| \right) \]

\[ + \phi_i \left( \delta_i \left( \int_0^{\delta_i(x)} g_i(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v))) \, dv \right) \right) \]

\[ \leq M_i + M_i^+ + \phi_i \left( \| x \|, \| y \|, \| z \| \right) + \phi_i \left( \delta_iM_i^+ \right), \quad (18) \]

that shows \( T_i, (i = 1, 2, 3) \), are well defined. Also, condition (VII) and relation (18) imply that \( T_i \left( B_\rho \times B_\rho \times B_\rho \right) \subseteq B_\rho \).

Now, we show that \( T_i, i = 1, 2, 3 \), are continuous on \( B_\rho \times B_\rho \times B_\rho \). Fix arbitrarily \( \varepsilon > 0 \). Consider
Furthermore, from relation (14), we have

\[
\left| T_i (x, y, z) (\tau) - T_i (u, v, w) (\tau) \right| \leq \left| h_1 (\tau, x (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right| + \left| f_1 \left( \theta \left( \int_0^{\theta_0 (\epsilon_i)} \Phi_0 (\tau, \nu, x (\sigma_i (\nu)), y (\sigma_i (\nu)), z (\sigma_i (\nu))) d \nu \right) \right) \right| + \left| f_1 \left( \theta \left( \int_0^{\theta_0 (\epsilon_i)} \Phi_0 (\tau, \nu, u (\sigma_i (\nu)), v (\sigma_i (\nu)), w (\sigma_i (\nu))) d \nu \right) \right) \right|
\]

\[
\leq \frac{1}{2} \Phi_1 \left( \left\| x - u \right\|, \left\| y - v \right\|, \left\| z - w \right\| \right) + \frac{1}{2} \Phi_2 \left( \left\| x - u \right\|, \left\| y - v \right\|, \left\| z - w \right\| \right) + \Phi_1 \left( \int_0^{\theta_0 (\epsilon_i)} \left| g_i (\tau, \nu, x (\sigma_i (\nu)), y (\sigma_i (\nu)), z (\sigma_i (\nu))) \right| d \nu \right) + \Phi_1 \left( \int_0^{\theta_0 (\epsilon_i)} \left| -g_i (\tau, \nu, u (\sigma_i (\nu)), v (\sigma_i (\nu)), w (\sigma_i (\nu))) \right| d \nu \right)
\]

\[
\leq \frac{1}{2} \Phi_1 \left( \left\| x - u \right\|, \left\| y - v \right\|, \left\| z - w \right\| \right) + \frac{1}{2} \Phi_2 \left( \left\| x - u \right\|, \left\| y - v \right\|, \left\| z - w \right\| \right) + \Phi_1 \left( \int_0^{\theta_0 (\epsilon_i)} \left| g_i (\tau, \nu, x (\sigma_i (\nu)), y (\sigma_i (\nu)), z (\sigma_i (\nu))) \right| d \nu \right) + \Phi_1 \left( \int_0^{\theta_0 (\epsilon_i)} \left| -g_i (\tau, \nu, u (\sigma_i (\nu)), v (\sigma_i (\nu)), w (\sigma_i (\nu))) \right| d \nu \right).
\]

\[
\text{(19)}
\]

Furthermore, from relation (14), we have

\[
\Phi_1 \left( \delta_1 \left( \int_0^{\theta_0 (\epsilon_i)} \left| g_i (\tau, \nu, x (\sigma_i (\nu)), y (\sigma_i (\nu)), z (\sigma_i (\nu))) \right| d \nu \right) \right) \leq \frac{\epsilon}{2}
\]

\[
\text{(20)}
\]

for every \( x, y, z, u, v, w \in BC (\mathbb{R}_+) \).

If \( \tau > L \), then from relations (19) and (20), we obtain

\[
\left| T_i (x, y, z) (\tau) - T_i (u, v, w) (\tau) \right| \leq \Phi_1 \left( \frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2} \right) + \frac{\epsilon}{2}
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

\[
\text{(21)}
\]

If \( \tau \in [0, L] \), then we get

\[
\left| T_i (x, y, z) (\tau) - T_i (u, v, w) (\tau) \right| \leq \Phi_1 \left( \frac{\epsilon}{2}, \frac{\epsilon}{2}, \frac{\epsilon}{2} \right) + \Phi_1 \left( \delta_1 \left( \int_0^{\theta_0 (\epsilon_i)} \left| g_i (\tau, \nu, x (\sigma_i (\nu)), y (\sigma_i (\nu)), z (\sigma_i (\nu))) \right| d \nu \right) \right)
\]

\[
< \frac{\epsilon}{2} + \Phi_1 \left( \delta_1 \left( \Phi_0 (\tau, \nu, \epsilon_i (\nu)) \right) \right),
\]

\[
\text{(22)}
\]
where
\[
\omega(\epsilon) = \sup \left\{ \left| g_i(\tau, v, x, y, z) - g_i(\tau, v, u, v, w) \right| : \tau \in [0, L], v \in \left[0, \frac{\beta_i}{L}\right], \right. \\
x, y, z, u, v, w \in [-\rho, \rho], \left. \left\| (x, y, z) - (u, v, w) \right\| < \frac{\epsilon}{2} \right\},
\]
\[
\beta_i = \sup \{ \beta_i(\tau : \tau \in [0, L]) \}.
\]

Using the continuity of \( g_i, i = 1, 2, 3 \) on \([0, L] \times \left[0, \frac{\beta_i}{L}\right] \times [-\rho, \rho] \times [-\rho, \rho] \), we have \( \omega(\epsilon) \to 0 \), as \( \epsilon \to 0 \)

and by continuity \( \varphi_i, i = 1, 2, 3 \), we obtain
\[
\varphi_i \left( \beta_i \left( \frac{\beta_i}{\epsilon} \right) \right) \to 0,
\]

as \( \epsilon \to 0 \). Therefore, from relations (21) and (22), we conclude that \( T_i, i = 1, 2, 3 \), are continuous functions from \( \bar{B}_\rho \times \bar{B}_\rho \times \bar{B}_\rho \) into \( \bar{B}_\rho \). Next, we show that \( T_i, i = 1, 2, 3 \), satisfies the conditions of Corollary 2.9. For this purpose, suppose \( L, \epsilon \in \mathbb{R}, \tau_1, \tau_2 \in [0, L] \) with \( |\tau_1 - \tau_2| \leq \epsilon \) and \( X_1, X_2, X_3 \) are arbitrary non-empty subsets of \( \bar{B}_\rho \).

Let \((x, y, z) \in X_1 \times X_2 \times X_3 \). We can assume that \( \beta_i(\tau_1) < \beta_i(\tau_2) \). Consequently,
where

\[
\begin{align*}
\alpha^I(A, \varepsilon, \epsilon) & \leq \alpha^I(A, \varepsilon) + \alpha_{\beta,\alpha}^I(f, \varepsilon) + \alpha_{\beta}^I(h, \varepsilon) \\
\leq & \frac{1}{2} \phi_1 \left( |x(e_i(t_2) - x(e_i(t_1)))|, |y(e_i(t_2) - y(e_i(t_1)))|, |z(e_i(t_2) - z(e_i(t_1)))| \right) \\
+ & \frac{1}{2} \phi_2 \left( |x(e_i(t_2) - x(e_i(t_1)))|, |y(e_i(t_2) - y(e_i(t_1)))|, |z(e_i(t_2) - z(e_i(t_1)))| \right) \\
+ & \phi_3 \left( \theta_1 \left( \int_{\beta} \left( \int_{\tau} \left( g_i(\tau, v, x(e_i(v)), y(e_i(v)), z(e_i(v))) \cdot du \right) \cdot \right) \right) \right) \\
+ & \phi_4 \left( \theta_2 \left( \int_{\beta} \left( \int_{\tau} \left( g_i(\tau, v, x(e_i(v)), y(e_i(v)), z(e_i(v))) \cdot du \right) \right) \right) \right) \\
\end{align*}
\]

\[
\alpha^f(A, \varepsilon) + \alpha^f_{\beta,\alpha} (f, \varepsilon) + \alpha^f_{\beta} (h, \varepsilon)
\]

\[
\phi_3 \left( \alpha^f(x, \alpha^f(e_i, \varepsilon)), \alpha^f(y, \alpha^f(e_i, \varepsilon)), \alpha^f(z, \alpha^f(e_i, \varepsilon)) \right) + \phi_4 \left( \hat{\delta}(\beta) \alpha_{\beta}^f (g, \varepsilon) \right)^{\alpha \varepsilon} \right) \right) + \phi_5 \left( \delta(\hat{\delta}(\beta) \alpha_{\beta}^f (g, \varepsilon) \right)^{\alpha \varepsilon} \right) \right) \right)
\]

\[
(23)
\]

with

\[
\alpha^I(A, \varepsilon, \epsilon) = \sup \{|A_i(t_1) - A_i(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon\}
\]

\[
\alpha^f_{\beta,\alpha} (f, \varepsilon) = \sup \left\{ \left| f_i(t_2, x, y, z) - f_i(t_1, x, y, z) \right| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon, x, y, z \in [-\rho, \rho] \right\}
\]

\[
\alpha^f_{\beta} (g, \varepsilon) = \sup \left\{ \left| g_i(t_2, x, y, z) - g_i(t_1, x, y, z) \right| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon, x, y, z \in [-\rho, \rho] \right\}
\]

\[
\alpha^f(\beta, \varepsilon) = \sup \left\{ |\hat{\beta}_1(t_1) - \beta(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon \right\}
\]

Since \((x, y, z)\) was an arbitrary element of the set \(X_1 \times X_2 \times X_3\) in relation (23), so we get

\[
\alpha^I(T_i(X_1 \times X_2 \times X_3), \varepsilon) \leq \alpha^I(A, \varepsilon) + \alpha^f_{\beta,\alpha} (f, \varepsilon) + \alpha^f_{\beta} (g, \varepsilon) + \phi_5 \left( \hat{\delta}(\beta) \alpha_{\beta}^f (g, \varepsilon) \right)^{\alpha \varepsilon} \right) \right) + \phi_5 \left( \delta(\hat{\delta}(\beta) \alpha_{\beta}^f (g, \varepsilon) \right)^{\alpha \varepsilon} \right) \right) \right)
\]

\[
(24)
\]

Using continuity of \(f_i, g_i, h_i\) on \([0, L] \times [-\rho, \rho] \times [-\rho, \rho] \times [-\delta, \hat{\delta}(H^\varepsilon, H^\alpha)\varepsilon], [0, L] \times [0, \beta_1^\varepsilon] \times [-\rho, \rho] \times [-\rho, \rho] \times [-\delta, \hat{\delta}(H^\varepsilon, H^\alpha)\varepsilon], [0, L] \times [-\rho, \rho] \times [-\rho, \rho] \times [-\rho, \rho] \times [-\rho, \rho]\), we have

\[
\alpha^f_{\beta,\alpha} (f, \varepsilon) \rightarrow 0,
\]

\[
\alpha^f_{\beta} (g, \varepsilon) \rightarrow 0,
\]

\[
\alpha^f_{\beta} (h, \varepsilon) \rightarrow 0.
\]

Moreover, using continuity of \(e_i, \beta_i\) and \(A_i\), we conclude that

\[
\alpha^f(\varepsilon_i, \varepsilon) \rightarrow 0, \alpha^f(\beta_i, \varepsilon) \rightarrow 0, \alpha^I(A, \varepsilon) \rightarrow 0
\]
as \( e \to 0 \), Therefore we obtain

\[
\phi_1 \left( \delta \left( \int_{\overline{L}^2} (g_i, e) \right) \right) + \phi_2 \left( \delta \left( H \alpha^2 (\beta_i, e) \right) \right) \to 0
\]

as \( e \to 0 \). Now by letting \( e \to 0 \) in relation (24), we obtain

\[
\alpha^2 (T_i (X_1 \times X_2 \times X_3)) \leq \phi_1 \left( \alpha^2 (X_1), \alpha^2 (X_2), \alpha^2 (X_3) \right).
\]

(25)

Also, by letting \( L \to \infty \) in relation (25), we get

\[
\alpha^2 (T_i (X_1 \times X_2 \times X_3)) \leq \phi_1 \left( \alpha^2 (X_1), \alpha^2 (X_2), \alpha^2 (X_3) \right).
\]

(26)

Furthermore, for every \((x, y, z), (u, v, w) \in X_1 \times X_2 \times X_3, t \in \mathbb{R}_+\), we get

\[
\left| T_i (x, y, z) (\tau) - T_i (u, v, w) (\tau) \right| \leq \left| h_1 (\tau, x (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right| - h_1 (\tau, u (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau)))
\]

\[
+ \left| f_i \left( \frac{1}{\delta_i} \int_{\overline{L}^2} g_i (\tau, \epsilon_i (\tau), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right) \right|
\]

\[
+ \left| f_i \left( \frac{1}{\delta_i} \int_{\overline{L}^2} g_i (\tau, v (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right) \right|
\]

\[
\leq \frac{1}{2} \phi_1 \left( \left| x (\epsilon_i (\tau)) - u (\epsilon_i (\tau)) \right|, \left| y (\epsilon_i (\tau)) - v (\epsilon_i (\tau)) \right|, \left| z (\epsilon_i (\tau)) \right| \right)\]

\[
+ \frac{1}{2} \phi_1 \left( \left| x (\epsilon_i (\tau)) - u (\epsilon_i (\tau)) \right|, \left| y (\epsilon_i (\tau)) - v (\epsilon_i (\tau)) \right|, \left| z (\epsilon_i (\tau)) \right| \right)
\]

\[
+ \phi_1 \left( \left| f_i \left( \frac{1}{\delta_i} \int_{\overline{L}^2} g_i (\tau, \epsilon_i (\tau), v (\epsilon_i (\tau)), w (\epsilon_i (\tau))) \right) \right| \right)
\]

\[
\leq \frac{1}{2} \phi_1 (diam X_1 (\epsilon_i (\tau)), diam X_2 (\epsilon_i (\tau)), diam X_3 (\epsilon_i (\tau)))
\]

\[
+ \frac{1}{2} \phi_1 (diam X_1 (\epsilon_i (\tau)), diam X_2 (\epsilon_i (\tau)), diam X_3 (\epsilon_i (\tau)))
\]

\[
+ \phi_1 \left( \left| f_i \left( \frac{1}{\delta_i} \int_{\overline{L}^2} g_i (\tau, v (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right) \right| \right)
\]

\[
\leq \phi_1 (diam X_1 (\epsilon_i (\tau)), diam X_2 (\epsilon_i (\tau)), diam X_3 (\epsilon_i (\tau)))
\]

\[
+ \phi_1 \left( \left| f_i \left( \frac{1}{\delta_i} \int_{\overline{L}^2} g_i (\tau, v (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right) \right| \right)
\]

(27)

Because \((x, y, z)\) and \((u, v, w)\) and \(\tau\), were chosen arbitrary in (27), we will have

\[
diam T_i (X_1 \times X_2 \times X_3) (\tau)
\]

\[
\leq \phi_1 (diam X_1 (\epsilon_i (\tau)), diam X_2 (\epsilon_i (\tau)), diam X_3 (\epsilon_i (\tau)))
\]

\[
+ \phi_1 \left( \left| \delta_i \int_{\overline{L}^2} g_i (\tau, v (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right| \right)
\]

\[
+ \phi_1 \left( \left| \delta_i \int_{\overline{L}^2} g_i (\tau, v (\epsilon_i (\tau)), y (\epsilon_i (\tau)), z (\epsilon_i (\tau))) \right| \right)
\]

(28)

By taking \( \tau \to \infty \) in relation (28), then using (14) we obtain

\[
\limsup_{\tau \to \infty} diam T_i (X_1 \times X_2 \times X_3) (\tau) \leq \phi_1 \left( \limsup_{\tau \to \infty} diam X_1 (\epsilon_i (\tau)) \right)
\]

\[
\leq \phi_1 \left( \limsup_{\tau \to \infty} diam X_2 (\epsilon_i (\tau)) \right)
\]

\[
\leq \phi_1 \left( \limsup_{\tau \to \infty} diam X_3 (\epsilon_i (\tau)) \right).
\]

(29)
From relation (26) together with relation (29), we obtain

\[
\omega_0(T_i(X_1 \times X_2 \times X_3)) + \limsup_{\tau \to \infty} \text{diam} T_i(X_1 \times X_2 \times X_3)(\tau)
\leq \phi_i(\omega_0(X_1), \omega_0(X_2), \omega_0(X_3))
+ \phi_i\left(\limsup_{\tau \to \infty} \text{diam} X_1(\varepsilon_i(\tau)), \limsup_{\tau \to \infty} \text{diam} X_2(\varepsilon_i(\tau)), \limsup_{\tau \to \infty} \text{diam} X_3(\varepsilon_i(\tau))\right)
\leq 3\phi_i\left(\frac{\omega_0(X_1) + \limsup_{\tau \to \infty} \text{diam} X_1(\varepsilon_i(\tau))}{3}, \frac{\omega_0(X_2) + \limsup_{\tau \to \infty} \text{diam} X_2(\varepsilon_i(\tau))}{3}, \frac{\omega_0(X_3) + \limsup_{\tau \to \infty} \text{diam} X_3(\varepsilon_i(\tau))}{3}\right),
\]

(30)

So, from relation (30), we conclude that

\[
\frac{1}{3}\mu(T_i(X_1 \times X_2 \times X_3)) \leq \phi_i\left(\frac{\mu(X_1)}{3}, \frac{\mu(X_2)}{3}, \frac{\mu(X_3)}{3}\right),
\]

and by taking \(\overline{\mu} = \frac{1}{3}\mu\), we get

\[
\overline{\mu}(T_i(X_1 \times X_2 \times X_3)) \leq \phi_i(\overline{\mu}(X_1), \overline{\mu}(X_2), \overline{\mu}(X_3)),
\]

Thus, by applying Corollary 2.9, the proof is complete. \(\square\)

Finally, we present the following example and we investigate the conditions of Theorem 3.1 for existence of a solution.

**Example 3.2.** Let us consider the following system of integral equations

\[
\begin{align*}
    x(\tau) &= \frac{1}{2}e^{-\tau^2} + \frac{1}{8(1+\tau^2)} \left(\cos x\left(\sqrt{\tau}\right) + \ln\left(1 + |y\left(\sqrt{\tau}\right)|\right) + \sin z\left(\sqrt{\tau}\right)\right) + \frac{1}{4}e^{-\tau^2} \\
    & \quad + \frac{1}{8(1+\tau^2)} \left(x\left(\sqrt{\tau}\right) + y\left(\sqrt{\tau}\right) + z\left(\sqrt{\tau}\right)\right) \\
    & \quad + \arctan\left(\frac{1}{\sqrt{\tau}}\right)\left(\frac{\mu}{\left(1+x^2(\tau^2)\right)^{1/2}}\right) d\tau, \\
    y(\tau) &= \frac{1}{5}\int_0^\tau \left(x(\tau) + y(\tau) + z(\tau)\right) d\tau \\
    & \quad + \frac{1}{8(1+\tau^2)} \left(x(\tau) + y(\tau) + z(\tau)\right) \\
    & \quad + \sin\left(\int_0^\tau \left(\frac{\mu}{\left(1+x^2(\tau^2)\right)^{1/2}}\right) d\tau\right), \\
    z(\tau) &= \frac{1}{5}\int_0^\tau \left(x(\tau) + y(\tau) + z(\tau)\right) d\tau \\
    & \quad + \frac{1}{8(1+\tau^2)} \left(x(\tau) + y(\tau) + z(\tau)\right) \\
    & \quad + \ln\left(1 + \int_0^\tau \left(\frac{\mu}{\left(1+x^2(\tau^2)\right)^{1/2}}\right) d\tau\right)
\end{align*}
\]

(31)
Here

\[ h_1 (\tau, x, y, z) = \frac{1}{8 (1 + \tau^2)} \left( \cos x + \ln \left( 1 + |y| \right) + \sin z \right), \]

\[ h_2 (\tau, x, y, z) = \frac{\tau^2}{8 (1 + \tau^2)} \left( \cos x + \ln \left( 1 + |y| \right) + \sin z \right), \]

\[ h_3 (\tau, x, y, z) = \frac{\tau^2}{8 (1 + \tau^3)} \left( \cos x + \ln \left( 1 + |y| \right) + \sin z \right), \]

\[ f_1 (\tau, x, y, z, m) = \frac{1}{7} e^{-\tau^2} + \frac{1}{8 (1 + \tau^2)} (x + y + z) + \frac{m}{2}, \]

\[ f_2 (\tau, x, y, z, m) = \frac{1}{7} e^{-\tau^2} + \frac{\tau^2}{8 (1 + \tau^4)} (x + y + z) + \frac{m}{2}, \]

\[ f_3 (\tau, x, y, z, m) = \frac{1}{7} e^{-\tau^2} + \frac{\tau^2}{8 (1 + \tau^3)} (x + y + z) + \frac{m}{2}, \]

\[ g_1 (\tau, v, x, y, z) = \left( \frac{v}{\epsilon^2} \right) \frac{x |\sin y| |\cos z|}{(1 + x^2) \left( 1 + \sin^2 y \right) (1 + \cos^2 z)}, \]

\[ g_2 (\tau, v, x, y, z) = \left( \frac{v}{\epsilon^2} \right) \frac{y^2 (1 + \cos^2 x) (1 + \sin^2 z)}{(1 + y^2) \left( 1 + \sin^2 y \right) (1 + \cos^2 z)}, \]

\[ g_3 (\tau, v, x, y, z) = \left( \frac{\sqrt{\nu}}{\epsilon^2} \right) \frac{x^2 |\cos y| + y^2 |\cos z| + z^2 |\cos x|}{(1 + x^2) (1 + y^2) (1 + z^2)}, \]

and

\[ A_1 (\tau) = \frac{1}{5} e^{-\tau^2}, \quad A_2 (\tau) = \frac{\tau^2}{5 (1 + \tau^2)}, \quad A_3 (\tau) = \frac{1}{5 \sqrt{1 + \tau^2}}, \quad \epsilon_1 (\tau) = \sqrt{\tau}, \quad \epsilon_2 (\tau) = \tau, \]

\[ \varepsilon_3 (\tau) = \tau, \quad \sigma_1 (\tau) = \tau^2, \quad \sigma_2 (\tau) = \tau, \quad \sigma_3 (\tau) = \tau, \quad \beta_1 (\tau) = \sqrt{\tau}, \quad \beta_2 (\tau) = \tau, \quad \beta_3 (\tau) = \tau^2, \]

\[ \theta_1 (\tau) = \arctan \tau, \quad \theta_2 (\tau) = \sin \tau, \quad \theta_3 (\tau) = \ln (1 + \tau), \]

\[ \phi_1 (\tau, v, u) = \frac{1}{4} (\tau + v + u), \quad \phi_2 (\tau, v, u) = \frac{1}{4} (\tau + v + u), \quad \phi_3 (\tau, v, u) = \frac{1}{4} (\tau + v + u), \]

\[ \varphi_1 (\tau) = \frac{\tau}{2}, \quad \varphi_2 (\tau) = \frac{\tau}{2}, \quad \varphi_3 (\tau) = \frac{\tau}{2}. \]

Clearly conditions (I) and (II) and (III) are valid. Obviously we have, \( M_i = \frac{1}{2}, \quad \delta_i = 1 \) and \( \alpha_i = 1, i = 1, 2, 3 \).

Clearly, \( f_i (\tau, 0, 0, 0, 0) = \frac{1}{4} e^{-\tau^2}, \quad i = 1, 2, 3 \), are bounded and \( M_i = \frac{\nu}{2} \). Also \( h_i (\tau, 0, 0, 0), i = 1, 2, 3 \), are bounded and \( M_i^* = \frac{1}{2} \). Therefore, the condition (IV) is valid.

Obviouly, \( f_i \) and \( h_i, i = 1, 2, 3 \), are continuous. Let \( \tau \in \mathbb{R}_+ \), then we get

\[
|f_1 (\tau, x, y, z, m) - f_1 (\tau, u, v, w, n)| \leq \frac{1}{8 (1 + \tau^2)} \left( |x - u| + |y - v| + |z - w| + \frac{1}{2} |m - n| \right)
\]

\[
\leq \frac{1}{8} \left( |x - u| + |y - v| + |z - w| + \frac{1}{2} |m - n| \right)
\]

\[
= \frac{1}{2} \left( |x - u| + |y - v| + |z - w| + \frac{1}{2} |m - n| \right)
\]

\[
= \frac{1}{2} \phi_1 (|x - u|, |y - v|, |z - w|) + \varphi_1 (|m - n|).
\]
Similarly, we obtain the following two relations:

\[
\begin{align*}
|f_2(\tau, x, y, z, m) - f_2(\tau, v, w, n)| & \leq \frac{1}{2}\phi_2(|x - u|, |y - v|, |z - w|) + \phi_2(|m - n|), \\
|f_3(\tau, x, y, z, m) - f_3(\tau, u, v, w, n)| & \leq \frac{1}{2}\phi_3(|x - u|, |y - v|, |z - w|) + \phi_3(|m - n|).
\end{align*}
\]

If \( \tau \in \mathbb{R} \) and \( x, y, z, u, v, w \in \mathbb{R} \) with \( |y| \geq |v| \), then we get

\[
|h_1(\tau, x, y, z) - h_1(\tau, u, v, w)| \leq \frac{1}{8(1 + \tau^2)} |\cos x - \cos u| + \frac{1}{8(1 + \tau^2)} |\sin z - \sin w| + \frac{1}{8(1 + \tau^2)} |\tau|
\]

\[
\leq \frac{1}{8}|x - u| + \frac{1}{8} \ln \left(1 + \frac{|y|}{1 + |v|}\right) + \frac{1}{8}|z - w|
\]

\[
\leq \frac{1}{8}|x - u| + \ln \left(1 + \frac{|y|}{1 + |v|}\right) + \frac{1}{8}|z - w|
\]

\[
\leq \frac{1}{8}\left(|x - u| + |y - v| + |z - w|\right)
\]

\[
= \frac{1}{4}\left(|x - u| + |y - v| + |z - w|\right)
\]

\[
= \frac{1}{2}\phi_4(|x - u|, |y - v|, |z - w|).
\]

Similarly, we obtain the following two relations:

\[
|h_2(\tau, x, y, z) - h_2(\tau, u, v, w)| \leq \frac{1}{2}\phi_2(|x - u|, |y - v|, |z - w|),
\]

\[
|h_3(\tau, x, y, z) - h_3(\tau, u, v, w)| \leq \frac{1}{2}\phi_3(|x - u|, |y - v|, |z - w|).
\]

Therefore, the condition (V) is valid.

Clearly, \( g_1, i = 1, 2, 3, \) are continuous. For every \( \tau, v \in \mathbb{R} \) and \( x, y, z \in \mathbb{R} \), by easy calculations we get

\[
|g_1(\tau, v, x, y, z) - g_1(\tau, v, u, v, w)| \leq \frac{2v}{\epsilon},
\]

\[
|g_2(\tau, v, x, y, z) - g_2(\tau, v, u, v, w)| \leq \frac{8v^2}{\epsilon^2},
\]

\[
|g_3(\tau, v, x, y, z) - g_3(\tau, v, u, v, w)| \leq \frac{6v^3}{\epsilon^3},
\]

Hence,

\[
\lim_{\tau \to \infty} \int_0^{\phi_1(\tau)} g_1(\tau, v, x(\sigma_1(v)), y(\sigma_1(v)), z(\sigma_1(v))) g_1(\tau, v, u(\sigma_1(v)), v(\sigma_1(v)), w(\sigma_1(v))) dv \leq \lim_{\tau \to \infty} \frac{\sqrt{\tau}}{\epsilon^2} dv = \lim_{\tau \to \infty} \frac{\tau}{\epsilon^2} v = 0,
\]

\[
\lim_{\tau \to \infty} \int_0^{\phi_2(\tau)} g_2(\tau, v, x(\sigma_2(v)), y(\sigma_2(v)), z(\sigma_2(v))) g_2(\tau, v, u(\sigma_2(v)), v(\sigma_2(v)), w(\sigma_2(v))) dv \leq \lim_{\tau \to \infty} \int_0^{\sqrt{\tau}} \frac{8v}{\epsilon^2} dv = \lim_{\tau \to \infty} \frac{8v^2}{\epsilon^2} = 0,
\]

\[
\lim_{\tau \to \infty} \int_0^{\phi_3(\tau)} g_3(\tau, v, x(\sigma_3(v)), y(\sigma_3(v)), z(\sigma_3(v))) g_3(\tau, v, u(\sigma_3(v)), v(\sigma_3(v)), w(\sigma_3(v))) dv \leq \lim_{\tau \to \infty} \int_0^{\sqrt{\tau}} \frac{6v^3}{\epsilon^3} dv = \lim_{\tau \to \infty} \frac{6v^3}{\epsilon^3} = 0,
\]
Also, we get
\[ \left| \int_0^{\beta(t)} g_1(t, v, x(\alpha_1(v)), y(\alpha_1(v)), z(\alpha_1(v)))\, dv \right| \leq \int_0^{\beta(t)} \frac{4v}{e^v} \, dv = \frac{2e^2}{e^2}, \]
\[ \left| \int_0^{\beta(t)} g_2(t, v, x(\alpha_2(v)), y(\alpha_2(v)), z(\alpha_2(v)))\, dv \right| \leq \int_0^{\beta(t)} \frac{4v}{e^v} \, dv = \frac{2e^2}{e^2}, \]
\[ \left| \int_0^{\beta(t)} g_3(t, v, x(\alpha_3(v)), y(\alpha_3(v)), z(\alpha_3(v)))\, dv \right| \leq \int_0^{\beta(t)} \frac{3v}{e^v} \, dv = \frac{2e^3}{e^3}. \]

Hence
\[ M_1^- = \sup \left\{ \frac{\tau}{2e^\tau} : \tau \in \mathbb{R}_+ \right\} = \frac{1}{2e}, \]
\[ M_2^- = \sup \left\{ \frac{\tau^2}{2e^{\tau^2}} : \tau \in \mathbb{R}_+ \right\} = \frac{2}{e}, \]
\[ M_2^- = \sup \left\{ \frac{2\tau^3}{e^{\tau^3}} : \tau \in \mathbb{R}_+ \right\} = \frac{2}{e}. \]

Therefore, the condition (VI) is valid.

Now from (33) along with \( M_1 = \frac{1}{5}, M_2 = \frac{1}{7}, M_3 = \frac{1}{8} \) and \( \delta_i = 1, (i = 1, 2, 3) \) in (16), we get
\[ \frac{1}{5} + \frac{1}{7} + \frac{1}{8} + \frac{1}{4e} < \frac{\kappa}{4}, \]
\[ \frac{1}{5} + \frac{1}{7} + \frac{1}{8} + \frac{1}{e} < \frac{\kappa}{4}. \]

Hence, the condition (VII) is valid for each \( \kappa > \frac{131}{70} + \frac{4}{e} \).

Thus, all the assumptions from (I) – (VII) are satisfied. Hence by Theorem 3.1 we conclude that the system (11) has a solution in \( BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \).

References