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A New Characterization of Generalized Browder's Theorem and a Cline's Formula for Generalized Drazin-Meromorphic Inverses

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Abstract. In this paper we give a new characterization of generalized Browder's theorem by considering equality between the generalized Drazin-meromorphic Weyl spectrum and the generalized Drazin-meromorphic spectrum. Also, we generalize Cline's formula to the case of generalized Drazin-meromorphic invertibility under the assumption that $A^kB^kA^k = A^{k+1}$ for some positive integer k.

1. Introduction and Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{C}$ denote the set of natural numbers and complex numbers, respectively. Let $B(X)$ denote the Banach algebra of all bounded linear operators acting on a complex Banach space X. For $T \in B(X)$, we denote the spectrum of T, null space of T, range of T and adjoint of T by $\sigma(T)$, $\ker(T)$, $R(T)$ and T^*, respectively. For a subset A of $\mathbb{C}$ the set of accumulation points of A and the set of interior points of A are denoted by $\mathrm{acc}(A)$ and $\mathrm{int}(A)$, respectively. Let $\alpha(T) = \dim \ker(T)$ and $\beta(T) = \operatorname{codim} R(T)$ be the nullity of T and deficiency of T, respectively. An operator $T \in B(X)$ is called a lower semi-Fredholm operator if $\beta(T) < \infty$. An operator $T \in B(X)$ is called an upper semi-Fredholm operator if $\alpha(T) < \infty$ and $R(T)$ is closed. The class of all lower semi-Fredholm operators (upper semi-Fredholm operators, respectively) is denoted by $\phi_-(X)$ ($\phi_+(X)$, respectively). An operator T is called semi-Fredholm if it is upper or lower semi-Fredholm. For a semi-Fredholm operator $T \in B(X)$, the index of T is defined by $\mathrm{ind}\,(T) := \alpha(T) - \beta(T)$. The class of all Fredholm operators is defined by $\phi(X) := \phi_+(X) \cap \phi_-(X)$. The class of all lower semi-Weyl operators (upper semi-Weyl operators, respectively) is defined by $W_-(X) = \{T \in \phi_-(X) : \mathrm{ind}\,(T) \geq 0\}$ ($W_+(X) = \{T \in \phi_+(X) : \mathrm{ind}\,(T) \leq 0\}$, respectively). An operator $T \in B(X)$ is called Weyl if $T \in \phi(X)$ and ind $(T) = 0$. The *lower semi-Fredholm*, *upper semi-Fredholm*, *Fredholm*, *lower semi-Weyl*, *upper semi-Weyl* and *Weyl*

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spectra are defined by

$$\sigma_{lf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Fredholm}\},$$
$$\sigma_{uf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Fredholm}\},$$
$$\sigma_{f}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\},$$
$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Weyl}\},$$
$$\sigma_{uw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\},$$
$$\sigma_{w}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}, \text{ respectively.}$$

A bounded linear operator T is said to be bounded below if it is injective and $R(T)$ is closed. The *approximate point* and *surjective spectra* are defined by

$$\sigma_{a}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$
$$\sigma_{s}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}, \text{ respectively.}$$

For $T \in B(X)$ the ascent denoted by $p(T)$ is the smallest non negative integer p such that $\ker T^p = \ker T^{p+1}$. If no such integer exists we set $p(T) = \infty$. For $T \in B(X)$ the descent denoted by $q(T)$ is the smallest non negative integer q such that $R(T^q) = R(T^{q+1})$. If no such integer exists we set $q(T) = \infty$. By [1, Theorem 1.20] if both $p(T)$ and $q(T)$ are finite, then $p(T) = q(T)$.

An operator $T \in B(X)$ is called Drazin invertible if there exist a positive integer n and $S \in B(X)$ such that

$$ST = TS, \ T^{n+1}S = T^n \text{ and } STS = S.$$

Also, by [1, Theorem 1.132] T is Drazin invertible if and only if $p(T) = q(T) < \infty$. An operator $T \in B(X)$ is called left Drazin invertible if $p(T) < \infty$ and $R(T^{p+1})$ is closed. An operator $T \in B(X)$ is called right Drazin invertible if $q(T) < \infty$ and $R(T^q)$ is closed. An operator $T \in B(X)$ is called upper semi-Browder if it is an upper semi-Fredholm and $p(T) < \infty$. An operator $T \in B(X)$ is called lower semi-Browder if it is a lower semi-Fredholm and $q(T) < \infty$. We say that an operator $T \in B(X)$ is Browder if it is upper semi-Browder and lower semi-Browder. The *lower semi-Browder*, *upper semi-Browder* and *Browder spectra* are defined by

$$\sigma_{lb}(T) : = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi-Browder}\},$$
$$\sigma_{ub}(T) : = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$
$$\sigma_{b}(T) : = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\}, \text{ respectively.}$$

Clearly, every Browder operator is Drazin invertible.

An operator $T \in B(X)$ is said to possess the single-valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$ if for every neighbourhood V of λ_0 the only analytic function $f : V \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ is the function $f = 0$. If an operator T has SVEP at every $\lambda \in \mathbb{C}$, then T is said to have SVEP. Moreover, the set of all points $\lambda \in \mathbb{C}$ such that T does not have SVEP at λ is an open set contained in the interior of $\sigma(T)$. Therefore, if T has SVEP at each point of an open punctured disc $\mathbb{D} \setminus \{\lambda_0\}$ centered at λ_0, T also has SVEP at λ_0.

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda$$

and

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda.$$

An operator $T \in B(X)$ is called Riesz if $\lambda I - T$ is Browder for all $\lambda \in \mathbb{C} \setminus \{0\}$. An operator $T \in B(X)$ is called meromorphic if $\lambda I - T$ is Drazin invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$. Clearly, every Riesz operator is meromorphic. A subspace M of X is said to be T-*invariant* if $T(M) \subset M$. For a T-invariant subspace M of X we define $T_M : M \to M$ by $T_M(x) = T(x), x \in M$. We say T is completely reduced by the pair (M, N) (denoted by $(M, N) \in Red(T)$) if M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$.

An operator $T \in B(X)$ is called semi-regular if $R(T)$ is closed and $\ker(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. An operator $T \in B(X)$ is called nilpotent if $T^n = 0$ for some $n \in \mathbb{N}$ and called quasi-nilpotent if $\|T^n\|^{\frac{1}{n}} \to 0$, i.e $\lambda I - T$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$.

For $T \in B(X)$ and a non negative integer n, define $T_{[n]}$ to be the restriction of T to $T^n(X)$. If for some non negative integer n the range space $T^n(X)$ is closed and $T_{[n]}$ is Fredholm (a lower semi Fredholm, an upper semi Fredholm, a lower semi Browder, an upper semi Browder, Browder, respectively) then T is said to be B-Fredholm (a lower semi B-Fredholm, an upper semi B-Fredholm, a lower semi B-Browder, an upper semi B-Browder, B-Browder, respectively). For a semi B-Fredholm operator T (see [6]), the index of T is defined as index of $T_{[n]}$. The *lower semi B-Fredholm*, *upper semi B-Fredholm*, *B-Fredholm*, *lower semi B-Browder*, *upper semi B-Browder* and *B-Browder spectra* are defined by

$$\begin{aligned}
\sigma_{lsbf}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Fredholm}\},\\
\sigma_{usbf}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Fredholm}\},\\
\sigma_{bf}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Fredholm}\},\\
\sigma_{lsbb}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Browder}\},\\
\sigma_{usbb}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Browder}\},\\
\sigma_{bb}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder}\}, \text{ respectively.}
\end{aligned}$$

By [1, Theorem 3.47] an operator $T \in B(X)$ is upper semi B-Browder (lower semi B-Browder, B-Browder, respectively) if and only if T is left Drazin invertible (right Drazin invertible, Drazin invertible, respectively).

An operator $T \in B(X)$ is called a lower semi B-Weyl (an upper semi B-Weyl, respectively) if it is a lower semi B-Fredholm (an upper semi B-Fredholm, respectively) having ind $(T) \geq 0$ (ind $(T) \leq 0$, respectively). An operator $T \in B(X)$ is called B-Weyl if it is B-Fredholm and ind $(T) = 0$. The *lower semi B-Weyl*, *upper semi B-Weyl* and *B-Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{lsbw}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\},\\
\sigma_{usbw}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\},\\
\sigma_{bw}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\}, \text{ respectively.}
\end{aligned}$$

It is known that (see [6, Theorem 2.7]) $T \in B(X)$ is B-Fredholm (B-Weyl, respectively) if there exists $(M, N) \in Red(T)$ such that T_M is Fredholm (Weyl, respectively) and T_N is nilpotent. Recently, (see [15, 17]) have generalized the class of B-Fredholm and B-Weyl operators and introduced the concept of pseudo B-Fredholm and pseudo B-Weyl operators. An operator $T \in B(X)$ is said to be pseudo B-Fredholm (pseudo B-Weyl, respectively) if there exists $(M, N) \in Red(T)$ such that T_M is Fredholm (Weyl, respectively) and T_N is quasi-nilpotent. The *pseudo B-Fredholm* and *pseudo B-Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{pBf}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not pseudo B-Fredholm}\},\\
\sigma_{pBw}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not pseudo B-Weyl}\}, \text{ respectively.}
\end{aligned}$$

An operator T is said to admit a *generalized kato decomposition* (*GKD*) if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is quasi-nilpotent. In the above definition if we assume T_N to be nilpotent, then T is said to be of Kato Type (see [14]). An operator is said to admit a *Kato-Riesz decomposition* (*GKRD*), if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is Riesz.

Recently, Živković-Zlatanović and Duggal [16] introduced the notion of generalized Kato-meromorphic decomposition. An operator $T \in B(X)$ is said to admit a *generalized Kato-meromorphic decomposition* (*GKMD*), if there exists a pair $(M, N) \in Red(T)$ such that T_M is semi-regular and T_N is meromorphic. For $T \in B(X)$, the *generalized Kato-meromorphic spectrum* is defined by

$$\sigma_{gKM}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not admit a GKMD}\}.$$

Recall that an operator $T \in B(X)$ is said to be Drazin invertible if there exists $S \in B(X)$ such that $TS = ST$, $STS = S$ and $TST - T$ is nilpotent. This definition is equivalent to the fact that there exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is nilpotent. Koliha [13] generalized this concept by replacing the third condition with $TST - T$ is quasi-nilpotent. An operator is said to be generalized Drazin invertible if there exist a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is quasi-nilpotent. The *generalized Drazin spectrum* is defined by

$$\sigma_{gD}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin invertible}\}.$$

Recently, Živković-Zlatanović and Cvetković [14] introduced the concept of generalized Drazin-Riesz invertible by replacing the third condition with $TST - T$ is Riesz. They proved that an operator $T \in B(X)$ is generalized Drazin-Riesz invertible if and only if there exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is Riesz. An operator $T \in B(X)$ is called generalized Drazin-Riesz bounded below (surjective, respectively) if there exists a pair $(M, N) \in Red(T)$ such that T_M is bounded below (surjective, respectively) and T_N is Riesz. The *generalized Drazin-Riesz bounded below*, *generalized Drazin-Riesz surjective* and *generalized Drazin-Riesz invertible spectra* are defined by

$$\begin{aligned}
\sigma_{gDR\mathcal{J}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz bounded below}\},\\
\sigma_{gDR\mathcal{Q}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz surjective}\},\\
\sigma_{gDR}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz invertible}\}, \text{ respectively.}
\end{aligned}$$

Also, they introduced the notion of operators which are direct sum of a Riesz and a Fredholm (lower (upper) semi-Fredholm, lower (upper) semi-Weyl, Weyl). An operator is called generalized Drazin-Riesz Fredholm (generalized Drazin-Riesz lower (upper) semi-Fredholm, generalized Drazin-Riesz lower (upper) semi-Weyl, generalized Drazin-Riesz Weyl, respectively) if there exists $(M, N) \in Red(T)$ such that T_M is Fredholm (lower (upper) semi-Fredholm, lower (upper) semi-Weyl, Weyl, respectively) and T_N is Riesz. The *generalized Drazin-Riesz lower (upper) semi-Fredholm*, *generalized Drazin-Riesz Fredholm*, *generalized Drazin-Riesz upper(lower) semi-Weyl* and *generalized Drazin-Riesz Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{gDR\phi_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz lower semi-Fredholm}\},\\
\sigma_{gDR\phi_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz upper semi-Fredholm}\},\\
\sigma_{gDR\phi}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz Fredholm}\},\\
\sigma_{gDRW_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz lower semi-Weyl}\},\\
\sigma_{gDRW_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz upper semi-Weyl}\},\\
\sigma_{gDRW}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-Riesz Weyl}\}, \text{ respectively.}
\end{aligned}$$

Also, Živković-Zlatanović and Duggal [16] introduced the notion of generalized Drazin-meromorphic invertible by replacing the third condition with $TST - T$ is meromorphic. They proved that the an operator $T \in B(X)$ is generalized Drazin-meromorphic invertible if and only if there exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is meromorphic. An operator $T \in B(X)$ is said to be generalized Drazin-meromorphic bounded below (surjective, respectively) if there exists a pair $(M, N) \in Red(T)$ such that T_M is bounded below (surjective, respectively) and T_N is meromorphic. The *generalized Drazin-meromorphic bounded below*, *generalized Drazin-meromorphic surjective* and *generalized Drazin-meromorphic invertible spectra* are defined by

$$\begin{aligned}
\sigma_{gDM\mathcal{J}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic bounded below}\},\\
\sigma_{gDM\mathcal{Q}}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic surjective}\},\\
\sigma_{gDM}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic invertible}\}, \text{ respectively.}
\end{aligned}$$

Also, they introduced the notion of operators which are direct sum of a meromorphic and Fredholm (lower (upper) semi-Fredholm, lower (upper) semi-Weyl, Weyl). An operator is called generalized Drazin-meromorphic Fredholm (generalized Drazin-meromorphic lower (upper) semi-Fredholm, generalized Drazin-meromorphic lower (upper) semi-Weyl, generalized Drazin-meromorphic Weyl) if there exists $(M, N) \in Red(T)$ such that T_M is Fredholm (lower (upper) semi-Fredholm, lower (upper) semi-Weyl,

Weyl) and T_N is meromorphic. The *generalized Drazin-meromorphic lower (upper) semi-Fredholm*, *generalized Drazin-meromorphic Fredholm*, *generalized Drazin-meromorphic lower (upper) semi-Weyl* and *generalized Drazin-meromorphic Weyl spectra* are defined by

$$\begin{aligned}
\sigma_{gDM\phi_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic lower semi-Fredholm}\},\\
\sigma_{gDM\phi_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic upper semi-Fredholm}\},\\
\sigma_{gDM\phi}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic Fredholm}\},\\
\sigma_{gDMW_-}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic lower semi-Weyl}\},\\
\sigma_{gDMW_+}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic upper semi-Weyl}\},\\
\sigma_{gDMW}(T) &:= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not generalized Drazin-meromorphic Weyl}\}, \text{ respectively.}
\end{aligned}$$

From [14, 16] we have

$$\begin{aligned}
&\sigma_{gD*\phi}(T) = \sigma_{gD*\phi_+}(T) \cup \sigma_{gD*\phi_-}(T),\\
&\sigma_{gK*}(T) \subset \sigma_{gD*\phi_+}(T) \subset \sigma_{gD*W_+}(T) \subset \sigma_{gD*\mathcal{J}}(T),\\
&\sigma_{gK*}(T) \subset \sigma_{gD*\phi_-}(T) \subset \sigma_{gD*W_-}(T) \subset \sigma_{gD*\mathcal{Q}}(T),\\
&\sigma_{gK*}(T) \subset \sigma_{gD*\phi}(T) \subset \sigma_{gD*W} \subset \sigma_{gD*}(T),
\end{aligned}$$

where $*$ stands for Riesz or meromorphic operators.

Recall that an operator T satisfies Browder's theorem if $\sigma_b(T) = \sigma_w(T)$ and generalized Browder's theorem if $\sigma_{bb}(T) = \sigma_{bw}(T)$. Amouch et al. [7] and Karmouni and Tajmouati [12] gave a new characterization of Browder's theorem using spectra arised from Fredholm theory and Drazin invertibilty. Motivated by them, we give a new characterization of operators satisfying generalized Browder's theorem. We prove that an operator T satisfies generalized Browder's theorem if and only if $\sigma_{gDMW}(T) = \sigma_{gDM}(T)$. In the last section, we generalize the Cline's formula for the case of generalized Drazin-meromorphic invertibility under the assumption that $A^kB^kA^k = A^{k+1}$ for some positive integer k.

2. Main Results

The following result will be used in the sequel:

Theorem 2.1. [16, Theorem 2.1] *Let $T \in B(X)$, then T is generalized Drazin-meromorphic upper semi-Weyl (lower semi-Weyl, upper semi-Fredholm, lower semi-Fredholm, Weyl, respectively) if and only if T admits a GKMD and $0 \notin \mathrm{acc}\sigma_{usbw}(T)$ $(\mathrm{acc}\sigma_{lsbw}(T), \mathrm{acc}\sigma_{usbf}(T), \mathrm{acc}\sigma_{lsbf}(T), \mathrm{acc}\sigma_{bw}(T)$, respectively).*

The following example shows that the inclusions $\sigma_{gDMW_+}(T) \subset \sigma_{gDM\mathcal{J}}(T)$ and $\sigma_{gDMW_-}(T) \subset \sigma_{gDM\mathcal{Q}}(T)$ can be proper.

Example 2.2. [14, Example 3.3] Let $X = c_0(\mathbb{N}), c(\mathbb{N}), l^\infty(\mathbb{N})$ or $l^p(\mathbb{N})$, $p \geq 1$. Let U and V be the forward and the backward unilateral shifts on X, respectively. Let $T = U \oplus V$. Then $\sigma_a(T) = \sigma_s(T) = \mathbb{D}$, where $\mathbb{D}$ denotes the closed unit disc. Therefore, $0 \in \mathrm{int}\sigma_a(T)$ and $0 \in \mathrm{int}\sigma_s(T)$. Thus, by [16, Theorems 2.5 and 2.6] $0 \in \sigma_{gDM\mathcal{J}}(T)$ and $0 \in \sigma_{gDM\mathcal{Q}}(T)$. Since $0 \notin \sigma_{gDRW_+}(T)$ and we know that $\sigma_{gDMW_+}(T) \subset \sigma_{gDRW_+}(T)$, $0 \notin \sigma_{gDMW_+}(T)$. Thus, $0 \in \sigma_{gDM\mathcal{J}}(T) \setminus \sigma_{gDMW_+}(T)$. Similarly, $0 \in \sigma_{gDM\mathcal{Q}}(T) \setminus \sigma_{gDMW_-}(T)$.

In the following results we obtain necessary and sufficient conditions to get equality.

Proposition 2.3. *Let $T \in B(X)$, then $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDMW_+}(T)$ if and only if T has SVEP at every $\lambda \notin \sigma_{gDMW_+}(T)$.*

Proof. Suppose that $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDMW_+}(T)$. Let $\lambda \notin \sigma_{gDMW_+}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic bounded below. Therefore, by [16, Theorem 2.5] T has SVEP at λ. Conversely, suppose that T has SVEP at every $\lambda \notin \sigma_{gDMW_+}(T)$. It suffices to show that $\sigma_{gDM\mathcal{J}}(T) \subset \sigma_{gDMW_+}(T)$. Let $\lambda \notin \sigma_{gDMW_+}(T)$ which implies that $\lambda I - T$ is generalized Drazin-meromorphic upper semi-Weyl. Therefore, by Theorem 2.1 $\lambda I - T$ admits a *GKMD*. Thus, there exists $(M, N) \in Red(\lambda I - T)$ such that $(\lambda I - T)_M$ is semi-regular and

$(\lambda I - T)_N$ is meromorphic. Since T has SVEP at every $\lambda \notin \sigma_{gDMW_+}(T)$, $(\lambda I - T)$ has SVEP at 0. As SVEP at a point is inherited by the restrictions on closed invariant subspaces, $(\lambda I - T)_M$ has SVEP at 0. Therefore, by [1, Theorem 2.91] $(\lambda I - T)_M$ is bounded below. Thus, by [16, Theorem 2.6] we have $\lambda I - T$ is generalized Drazin-meromorphic bounded below. Hence, $\lambda \notin \sigma_{gDM\mathcal{J}}(T)$. □

Proposition 2.4. *Let $T \in B(X)$, then $\sigma_{gDM\mathcal{Q}}(T) = \sigma_{gDMW_-}(T)$ if and only if T^* has SVEP at every $\lambda \notin \sigma_{gDMW_-}(T)$.*

Proof. Suppose that $\sigma_{gDM\mathcal{Q}}(T) = \sigma_{gDMW_-}(T)$. Let $\lambda \notin \sigma_{gDMW_-}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic surjective. Therefore, by [16, Theorem 2.6] T^* has SVEP at λ. Conversely, suppose that T^* has SVEP at every $\lambda \notin \sigma_{gDMW_-}(T)$. It suffices to show that $\sigma_{gDM\mathcal{Q}}(T) \subset \sigma_{gDMW_-}(T)$. Let $\lambda \notin \sigma_{gDMW_-}(T)$ which implies that $\lambda I - T$ is generalized Drazin-meromorphic lower semi-Weyl. Then by Theorem 2.1 $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{lsbw}(T)$. Since T^* has SVEP at every $\lambda \notin \sigma_{gDMW_-}(T)$ and $\sigma_{gDMW_-}(T) \subset \sigma_{lw}(T)$ then T^* has SVEP at every $\lambda \notin \sigma_{lw}(T) = \sigma_{uw}(T^*)$. Therefore, by [1, Theorem 5.27] we have $\sigma_{lw}(T) = \sigma_{uw}(T^*) = \sigma_{ub}(T^*) = \sigma_{lb}(T)$. Thus, by [1, Theorem 5.38] we have $\sigma_{lsbw}(T) = \sigma_{lsbb}(T)$. This implies that $\lambda \notin \mathrm{acc}\sigma_{lsbb}(T)$. Therefore, by [16, Theorem 2.6] $\lambda I - T$ is generalized Drazin-meromorprhic surjective and it follows that $\lambda \notin \sigma_{gDM\mathcal{Q}}(T)$. □

Corollary 2.5. *Let $T \in B(X)$, then $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$ if and only if T and T^* have SVEP at every $\lambda \notin \sigma_{gDMW}(T)$.*

Proof. Suppose that $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$. Let $\lambda \notin \sigma_{gDMW}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic invertible. Therefore, by [16, Theorem 2.4] T and T^* have SVEP at λ. Conversely, let $\lambda \notin \sigma_{gDMW}(T) = \sigma_{gDMW_+}(T) \cup \sigma_{gDMW_-}(T)$. Then by proofs of Proposition 2.3 and Proposition 2.4 we have $\lambda \notin \sigma_{gDM\mathcal{J}}(T) \cup \sigma_{gDM\mathcal{Q}}(T) = \sigma_{gDM}(T)$. □

Theorem 2.6. *Let $T \in B(X)$, then following statements are equivalent:*
(i) $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$,
(ii) T or T^* have SVEP at every $\lambda \notin \sigma_{gDMW}(T)$.

Proof. Suppose that T has SVEP at every $\lambda \notin \sigma_{gDMW}(T)$. It suffices to prove that $\sigma_{gDM}(T) \subset \sigma_{gDMW}(T)$. Let $\lambda \notin \sigma_{gDMW}(T)$ then $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{bw}(T)$. Since $\sigma_{gDRW}(T) \subset \sigma_{bw}(T)$, T has SVEP at every $\lambda \notin \sigma_{bw}(T)$. Therefore, $\sigma_{bw}(T) = \sigma_{bb}(T)$. Thus, $\lambda \notin \mathrm{acc}\sigma_{bb}(T)$ which implies that $\lambda I - T$ is generalized Drazin-meromorphic invertible.

Now suppose that T^* has SVEP at every $\lambda \notin \sigma_{gDRW}(T)$. Since $\sigma_{bb}(T) = \sigma_{bb}(T^*)$ and $\sigma_{bw}(T) = \sigma_{bw}(T^*)$ we have $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$. The converse is an immediate consequence of Corollary 2.5. □

Recall that an operator $T \in B(X)$ is said satisfy generalized a-Browder's theorem if $\sigma_{usbb}(T) = \sigma_{usbw}(T)$. An operator $T \in B(X)$ satisfies a-Browder's theorem if $\sigma_{ub}(T) = \sigma_{uw}(T)$. By [4, Theorem 2.2] we know that a-Browder's theorem is equivalent to generalized a-Browder's theorem.

Theorem 2.7. *Let $T \in B(X)$, then the following holds:*
(i) generalized a-Browder's theorem holds for T if and only if $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDMW_+}(T)$,
(ii) generalized a-Browder's theorem holds for T^* if and only if $\sigma_{gDM\mathcal{Q}}(T) = \sigma_{gDMW_-}(T)$,
(iii) generalized Browder's theorem holds for T if and only if $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$.

Proof. (i) Suppose that generalized a-Browder's theorem holds for T which implies that $\sigma_{usbb}(T) = \sigma_{usbw}(T)$. It suffices to prove that $\sigma_{gDM\mathcal{J}}(T) \subset \sigma_{gDMW_+}(T)$. Let $\lambda \notin \sigma_{gDMW_+}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic upper semi-Weyl. By Theorem 2.1 it follows that $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{usbw}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{usbb}(T)$. Therefore, by [16, Theorem 2.5] $\lambda I - T$ is generalized Drazin-meromorphic bounded below which gives $\lambda \notin \sigma_{gDM\mathcal{J}}(T)$. Conversely, suppose that $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDMW_+}(T)$. Using Proposition 2.3 we deduce that T has SVEP at every $\lambda \notin \sigma_{gDMW_+}(T)$. Since $\sigma_{gDMW_+}(T) \subset \sigma_{uw}(T)$, T has SVEP at every $\lambda \notin \sigma_{uw}(T)$. By [1, Theorem 5.27] T satisfies a-Browder's theorem. Therefore, generalized a-Browder's theorem holds for T.
(ii) Suppose that generalized a-Browder's theorem holds for T^* which implies that $\sigma_{lsbb}(T) = \sigma_{lsbw}(T)$.

It suffices to prove that $\sigma_{gDMQ}(T) \subset \sigma_{gDMW_-}(T)$. Let $\lambda \notin \sigma_{gDMW_-}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic lower semi-Weyl. By Theorem 2.1 it follows that $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{lsbw}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{lsbb}(T)$. Therefore, by [16, Theorem 2.6] $\lambda I - T$ is generalized Drazin-meromorphic surjective which gives $\lambda \notin \sigma_{gDMQ}(T)$. Conversely, suppose that $\sigma_{gDMQ}(T) = \sigma_{gDMW_-}(T)$. Using Proposition 2.4 we deduce that T^* has SVEP at every $\lambda \notin \sigma_{gDMW_-}(T)$. Since $\sigma_{gDMW_-}(T) \subset \sigma_{lw}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lw}(T) = \sigma_{uw}(T^*)$. Therefore, generalized a-Browder's theorem holds for T^*.
(iii) Suppose that generalized Browder's theorem holds for T which implies that $\sigma_{bb}(T) = \sigma_{bw}(T)$. It suffices to prove that $\sigma_{gDM}(T) \subset \sigma_{gDMW}(T)$. Let $\lambda \notin \sigma_{gDMW}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic Weyl. By Theorem 2.1 it follows that $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{bw}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{bb}(T)$. Therefore, by [16, Theorem 2.4] $\lambda I - T$ is generalized Drazin-meromorphic invertible which gives $\lambda \notin \sigma_{gDM}(T)$. Conversely, suppose that $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$. Using Corollary 2.5 we deduce that T and T^* have SVEP at every $\lambda \notin \sigma_{gDMW}(T)$. Since $\sigma_{gDMW}(T) \subset \sigma_{bw}(T)$, T and T^* have SVEP at every $\lambda \notin \sigma_{bw}(T)$. Therefore, by [1, Theorem 5.14] generalized Browder's theorem holds for T. □

Using Theorem 2.7, [2, Theorem 2.3], [4, Theorem 2.1], [5, Proposition 2.2] and [12, Theorem 2.6] we have the following theorem:

Theorem 2.8. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) Browder's theorem holds for T,
(ii) Browder's theorem holds for T^*,
(iii) T has SVEP at every $\lambda \notin \sigma_w(T)$,
(iv) T^* has SVEP at every $\lambda \notin \sigma_w(T)$,
(v) T has SVEP at every $\lambda \notin \sigma_{bw}(T)$,
(vi) generalized Browder's theorem holds for T,
(vii) T or T^* has SVEP at every $\lambda \notin \sigma_{gDRW}(T)$,
(viii) $\sigma_{gDR}(T) = \sigma_{gDRW}(T)$,
(ix) T or T^* has SVEP at every $\lambda \notin \sigma_{gDMW}(T)$,
(x) $\sigma_{gDM}(T) = \sigma_{gDMW}(T)$,
(xi) $\sigma_{gD}(T) = \sigma_{pBW}(T)$.

Using [4, Theorem 2.2] and [12, Theorem 2.7] a similar result for a-Browder's theorem can be stated as follows:

Theorem 2.9. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) a-Browder's theorem holds for T,
(ii) generalized a-Browder's theorem holds for T,
(iii) T has SVEP at every $\lambda \notin \sigma_{gDRW_+}(T)$,
(iv) $\sigma_{gDR\mathcal{J}}(T) = \sigma_{gDRW_+}(T)$,
(v) T has SVEP at every $\lambda \notin \sigma_{gDMW_+}(T)$,
(vi) $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDMW_+}(T)$.

Lemma 2.10. *Let $T \in B(X)$, then*
(i) $\sigma_{uf}(T) = \sigma_{ub}(T) \Leftrightarrow \sigma_{usbf}(T) = \sigma_{usbb}(T)$,
(ii) $\sigma_{lf}(T) = \sigma_{lb}(T) \Leftrightarrow \sigma_{lsbf}(T) = \sigma_{lsbb}(T)$.

Proof. (i) Let $\sigma_{uf}(T) = \sigma_{ub}(T)$. It suffices to show that $\sigma_{usbb}(T) \subset \sigma_{usbf}(T)$. Let $\lambda_0 \notin \sigma_{usbf}(T)$. Then $\lambda_0 I - T$ is upper semi B-Fredholm. Therefore, by [1, Theorem 1.117] there exists an open disc $\mathbb{D}$ centered at λ_0 such that $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$. Since $\sigma_{uf}(T) = \sigma_{ub}(T)$, $\lambda I - T$ is upper semi-Browder for all $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$. Therefore, $p(\lambda I - T) < \infty$ for all $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$. Thus, T has SVEP at every $\lambda \in \mathbb{D} \setminus \{\lambda_0\}$ which gives T has SVEP at λ_0. Thus, by [3, Theorem 2.5] it follows that $\lambda \notin \sigma_{usbb}(T)$. Conversely, let $\sigma_{usbb}(T) = \sigma_{usbf}(T)$. It suffices to show that $\sigma_{ub}(T) \subset \sigma_{uf}(T)$. Let $\lambda \notin \sigma_{uf}(T)$. Then $\lambda \notin \sigma_{usbf}(T) = \sigma_{usbb}(T)$. Therefore, $p(\lambda I - T) < \infty$ which implies that $\lambda \notin \sigma_{ub}(T)$.

(ii) Using a similar argument as above we can get the desired result. □

Remark 2.11. From [16, Example 3.7] it is seen that the inclusions $\sigma_{gDM\phi_+}(T) \subset \sigma_{gDM\mathcal{J}}(T)$, $\sigma_{gDM\phi_-}(T) \subset \sigma_{gDM\mathcal{Q}}(T)$ and $\sigma_{gDM\phi}(T) \subset \sigma_{gDM}(T)$ can be proper. In the following theorems we give necessary and sufficient conditions to get equality.

Theorem 2.12. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) $\sigma_{usbf}(T) = \sigma_{usbb}(T)$,
(ii) T has SVEP at every $\lambda \notin \sigma_{usbf}(T)$,
(iii) T has SVEP at every $\lambda \notin \sigma_{gDM\phi_+}(T)$,
(iv) $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDM\phi_+}(T)$.

Proof. (i) $\Leftrightarrow$ (ii) Suppose that $\sigma_{usbf}(T) = \sigma_{usbb}(T)$. Let $\lambda \notin \sigma_{usbf}(T)$, then $\lambda \notin \sigma_{usbb}(T)$ which gives $p(\lambda I - T) < \infty$. Therefore, T has SVEP at λ. Now suppose that T has SVEP at every $\lambda \notin \sigma_{usbf}(T)$. It suffices to prove that $\sigma_{usbb}(T) \subset \sigma_{usbf}(T)$. Let $\lambda \notin \sigma_{usbf}(T)$, then $\lambda I - T$ is upper semi B-Fredholm operator. Since T has SVEP at λ then by [3, Theorem 2.5] it follows that $\lambda \notin \sigma_{usbb}(T)$.
(iii) $\Leftrightarrow$ (iv) Suppose that T has SVEP at every $\lambda \notin \sigma_{gDM\phi_+}(T)$ which implies that $\lambda I - T$ is generalized Drazin-meromorphic upper semi-Fredholm. It suffices to show that $\sigma_{gDM\mathcal{J}}(T) \subset \sigma_{gDM\phi_+}(T)$. Let $\lambda \notin \sigma_{gDM\phi_+}(T)$, then by Theorem 2.1 there exists $(M, N) \in Red(\lambda I - T)$ such that $(\lambda I - T)_M$ is semi-regular and $(\lambda I - T)_N$ is meromorphic. Since T has SVEP at λ, $(\lambda I - T)_M$ has SVEP at 0. Therefore, by [1, Theorem 2.91] $(\lambda I - T)_M$ is bounded below. Thus, $\lambda \notin \sigma_{gDM\mathcal{J}}(T)$. Conversely, suppose that $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDM\phi_+}(T)$. Let $\lambda \notin \sigma_{gDR\phi_+}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic bounded below. Therefore, by [16, Theorem 2.5] it follows that T has SVEP at λ.
(i) $\Leftrightarrow$ (iv) Suppose that $\sigma_{usbf}(T) = \sigma_{usbb}(T)$. It suffices to prove that $\sigma_{gDM\mathcal{J}}(T) \subset \sigma_{gDM\phi_+}(T)$. Let $\lambda \notin \sigma_{gDM\phi_+}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic upper semi-Fredholm. By Theorem 2.1 it follows that $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{usbf}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{usbb}(T)$. Therefore, by [16, Theorem 2.5] $\lambda I - T$ is generalized Drazin-meromorphic bounded below which gives $\lambda \notin \sigma_{gDM\mathcal{J}}(T)$. Conversely, suppose that $\sigma_{gDM\mathcal{J}}(T) = \sigma_{gDM\phi_+}(T)$. Then by (iv) $\Rightarrow$ (iii) T has SVEP at every $\lambda \notin \sigma_{gDM\phi_+}(T)$. Since $\sigma_{gDM\phi_+}(T) \subset \sigma_{uf}(T)$, T has SVEP at every $\lambda \notin \sigma_{uf}(T)$. Therefore, by [12, Theorem 2.8] we have $\sigma_{uf}(T) = \sigma_{ub}(T)$. Thus, by Lemma 2.10 $\sigma_{usbf}(T) = \sigma_{usbb}(T)$. □

Theorem 2.13. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) $\sigma_{lsbf}(T) = \sigma_{lsbb}(T)$,
(ii) T^* has SVEP at every $\lambda \notin \sigma_{lsbf}(T)$,
(iii) T^* has SVEP at every $\lambda \notin \sigma_{gDM\phi_-}(T)$,
(iv) $\sigma_{gDM\mathcal{Q}}(T) = \sigma_{gDM\phi_-}(T)$.

Proof. (i) $\Leftrightarrow$ (ii) Suppose that $\sigma_{lsbf}(T) = \sigma_{lsbb}(T)$. Let $\lambda \notin \sigma_{lsbf}(T)$, then $\lambda \notin \sigma_{lsbb}(T)$ which gives $q(\lambda I - T) < \infty$. Therefore, T^* has SVEP at λ. Now suppose that T^* has SVEP at every $\lambda \notin \sigma_{lsbf}(T)$. It suffices to prove that $\sigma_{lsbb}(T) \subset \sigma_{lsbf}(T)$. Let $\lambda \notin \sigma_{lsbf}(T)$, then $\lambda I - T$ is lower semi B-Fredholm operator. Since T^* has SVEP at λ then by [3, Theorem 2.5] we have $\lambda \notin \sigma_{lsbb}(T)$.
(iii) $\Leftrightarrow$ (iv) Suppose that T^* has SVEP at every $\lambda \notin \sigma_{gDM\phi_-}(T)$ which implies that $\lambda I - T$ is generalized Drazin-meromorphic lower semi-Fredholm. It suffices to show that $\sigma_{gDM\mathcal{Q}}(T) \subset \sigma_{gDM\phi_-}(T)$. By Theorem 2.1 it follows that $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{lsbf}(T)$. Since $\sigma_{gDM\phi_-}(T) \subset \sigma_{lf}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$. Therefore, by [12, Theorem 2.9] we have $\sigma_{lf}(T) = \sigma_{lb}(T)$. Thus, by Lemma 2.10 $\sigma_{lsbf}(T) = \sigma_{lsbb}(T)$ which implies that $\lambda \notin \mathrm{acc}\sigma_{lsbb}(T)$. Hence, $\lambda \notin \sigma_{gDM\mathcal{Q}}(T)$. Conversely, suppose that $\sigma_{gDM\mathcal{Q}}(T) = \sigma_{gDM\phi_-}(T)$. Let $\lambda \notin \sigma_{gDM\phi_-}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic surjective. Therefore by [16, Theorem 2.6] it follows that T^* has SVEP at λ.
(i) $\Leftrightarrow$ (iv) Suppose that $\sigma_{lsbf}(T) = \sigma_{lsbb}(T)$. It suffices to prove that $\sigma_{gDM\mathcal{Q}}(T) \subset \sigma_{gDM\phi_-}(T)$. Let $\lambda \notin \sigma_{gDM\phi_-}(T)$, then $\lambda I - T$ is generalized Drazin-meromorphic lower semi-Fredholm. By Theorem 2.1 it follows that $\lambda I - T$ admits a *GKMD* and $\lambda \notin \mathrm{acc}\sigma_{lsbf}(T)$. This gives $\lambda \notin \mathrm{acc}\sigma_{lsbb}(T)$. Therefore, by [16, Theorem 2.6] $\lambda I - T$ is generalized Drazin-meromorphic surjective which gives $\lambda \notin \sigma_{gDM\mathcal{Q}}(T)$. Conversely, suppose that $\sigma_{gDM\mathcal{Q}}(T) = \sigma_{gDM\phi_-}(T)$. Then by (iv) $\Rightarrow$ (iii) T^* has SVEP at every $\lambda \notin \sigma_{gDM\phi_-}(T)$. Since $\sigma_{gDM\phi_-}(T) \subset \sigma_{lf}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$. This gives $\sigma_{lsbf}(T) = \sigma_{lsbb}(T)$. □

Using [12, Corollary 2.10] and Theorems 2.12, 2.13 we have the following result:

Corollary 2.14. *Let $T \in B(X)$, then the following statements are equivalent:*
(i) $\sigma_f(T) = \sigma_b(T)$,
(ii) T and T^* have SVEP at every $\lambda \notin \sigma_f(T)$,
(iii) $\sigma_{bf}(T) = \sigma_{bb}(T)$,
(iv) T and T^* have SVEP at every $\lambda \notin \sigma_{bf}(T)$,
(v) $\sigma_{gD}(T) = \sigma_{pbf}(T)$,
(vi) T and T^* have SVEP at every $\lambda \notin \sigma_{pbf}(T)$,
(viii) $\sigma_{gDR}(T) = \sigma_{gDR\phi}(T)$,
(viii) T and T^* have SVEP at every $\lambda \notin \sigma_{gDR\phi}(T)$,
(ix) $\sigma_{gDM}(T) = \sigma_{gDM\phi}(T)$,
(x) T and T^* have SVEP at every $\lambda \notin \sigma_{gDM\phi}(T)$.

3. Cline's Formula for the generalized Drazin-meromorphic invertibility

Let R be a ring with identity. Drazin[9] introduced the concept of Drazin inverses in a ring. An element $a \in R$ is said to be *Drazin invertible* if there exist an element $b \in R$ and $r \in \mathbb{N}$ such that

$$ab = ba,\ bab = b,\ a^{r+1}b = a^r.$$

If such b exists then it is unique and is called *Drazin inverse* of a and denoted by a^D. For $a, b \in R$, Cline [8] proved that if ab is Drazin invertible, then ba is Drazin invertible and $(ba)^D = b((ab)^D)^2a$. Recently, Gupta and Kumar [10] generalized Cline's formula for Drazin inverses in a ring with identity to the case when $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$ and obtained the following result:

Theorem 3.1. ([10, Theorem 2.20]) *Let R be a ring with identity and suppose that $a^kb^ka^k = a^{k+1}$ for some $k \in \mathbb{N}$. Then a is Drazin invertible if and only if b^ka^k is Drazin invertible. Moreover, $(b^ka^k)^D = b^k(a^D)^2a^k$ and $a^D = a^k((b^ka^k)^D)^{k+1}$.*

Recently, Karmouni and Tajmouati [11] investigated for bounded linear operators A, B, C satisfying the operator equation $ABA = ACA$ and obtained that AC is generalized Drazin-Riesz invertible if and only if BA is generalized Drazin-Riesz invertible. Also, they generalized Cline's formula to the case of generalized Drazin-Riesz invertibility. In this section, we establish Cline's formula for the generalized Drazin-Riesz invertibility for bounded linear operators A and B under the condition $A^kB^kA^k = A^{k+1}$. By [10, Theorem 2.1, Theorem 2.2, Proposition 2.4 and Lemma 2.1] and a result [1, Corollary 3.99] we can deduce the following result:

Proposition 3.2. *Let $A, B \in B(X)$ satisfies $A^kB^kA^k = A^{k+1}$ for some $k \in \mathbb{N}$, then A is meromorphic if and only if B^kA^k is meromorphic.*

Theorem 3.3. *Suppose that $A, B \in B(X)$ and $A^kB^kA^k = A^{k+1}$ for some $k \in \mathbb{N}$. Then A is generalized Drazin-meromorphic invertible if and only if B^kA^k is generalized Drazin-meromorphic invertible.*

Proof. Suppose that A is generalized Drazin-meromorphic invertible, then there exists $T \in B(X)$ such that

$$TA = AT, \quad TAT = T \quad \text{and} \quad ATA - A \text{ is meromorphic.}$$

Let $S = B^kT^2A^k$. Then

$$(B^kA^k)S = (B^kA^k)(B^kT^2A^k) = B^k(A^kB^kA^k)T^2 = B^kA^{k+1}T^2 = B^kA^kT$$

and

$$S(B^kA^k) = (B^kT^2A^k)(B^kA^k) = B^kT^2A^{k+1} = B^kA^kT.$$

Therefore, $S(B^kA^k) = (B^kA^k)S$. Consider

$$S(B^kA^k)S = B^kT^2A^k(B^kA^k)B^kT^2A^k = (B^kT^2A^k)(B^kA^kT) = B^kT^2A^{k+1}T = B^kT^2A^k = S.$$

Let $Q = I - AT$, then Q is a bounded projection commuting with A. Therefore, $Q^n = Q$ for all $n \in \mathbb{N}$. We observe that

$$(QA)^kB^k(QA)^k = Q^kA^kB^kQ^kA^k = Q^kA^{k+1}Q^k = Q^{k+1}A^{k+1} = (QA)^{k+1}$$

and

$$\begin{aligned} B^kA^k - (B^kA^k)^2S &= B^kA^k - (B^kA^k)^2B^kT^2A^K = B^kA^k - B^k(A^kB^kA^k)B^kT^2A^k \\ &= B^kA^k - B^kA^{k+2}T^2 = B^k(I - A^2T^2)A^k = B^k(I - AT)A^k \\ &= B^kQA^k = B^kQ^kA^k = B^k(QA)^k. \end{aligned}$$

Since QA is meromorphic and$(QA)^kB^k(QA)^k = (QA)^{k+1}$, by Proposition 3.2 $B^kA^k - (B^kA^k)^2S$ is meromorphic.

Conversely, suppose that B^kA^k is generalized Drazin-meromorphic invertible. Then there exists $T' \in B(X)$ such that

$$T'B^kA^k = B^kA^kT', \quad T'B^kA^kT' = T' \quad \text{and} \quad B^kA^kT'B^kA^k - B^kA^k \text{ is meromorphic.}$$

Let $S' = A^kT'^{k+1}$. Then

$$S'A = A^kT'^{k+1}A = A^kT'^{k+2}B^kA^kA = A^kT'^{k+2}B^kA^{k+1} = A^kT'^{k+2}(B^kA^k)^2 = A^kT'^k$$

and

$$AS' = A^{k+1}T'^{k+1} = A^kT'^k.$$

Consider

$$\begin{aligned} AS' &= (A^kT'^{k+1}A)A^kT'^{k+1} = (A^kT'^k)A^kT'^{k+1} = A^kv^{k+1}B^kA^{2k}T'^{k+1} = A^kT'^{k+1}(B^kA^k)^{k+1} \\ &= S^{k+1} = A^kT'^{k+1} = S'. \end{aligned}$$

We claim that for all $n \in \mathbb{N}$ we have

$$(A - A^2S')^n = (A^n - A^{n+1}S').$$

We prove it by induction. Evidently, the result is true for $n = 1$. Assume it to be true for $n = p$. Consider

$$\begin{aligned} (A - A^2S')^{p+1} &= (A - A^2S')(A - A^2S')^p \\ &= (A - A^2S')(A^p - A^{p+1}S') \\ &= A^{P+1} - A^{P+2}S' - A^{P+2}S' + A^{P+3}S'^2 \\ &= A^{p+1} - A^{p+2}S'. \end{aligned}$$

Also,

$$\begin{aligned} B^k(A - A^2S')^k &= B^k(A^k - A^{k+1}S') = B^kA^k - B^kA^{k-1}A^2S' = B^kA^k - B^kA^{k-1}A^kT'^{k-1} \\ &= B^kA^k - B^kA^{2k-1}T'^{k-1} = B^kA^k - (B^kA^k)^kT'^{k-1} = B^kA^k - (B^kA^k)^2S'. \end{aligned}$$

Now consider

$$\begin{aligned} (A - A^2S')^kB^k(A - A^2S')^k &= (A^k - A^{k+1}S')B^k(A^k - A^{k+1}S') \\ &= A^kB^kA^k - A^{k+1}S'B^kA^k - A^kB^kA^kB^kA^kS' + A^{k+1}(B^kA^k)^2S'^2 \\ &= A^{k+1} - A^{k+2}S' = (A - A^2S')^{k+1}. \end{aligned}$$

Since $B^k(A - A^2S')^k = B^kA^k - (B^kA^k)^2T'$ is meromorphic, by Proposition 3.2 it follows that $A - A^2S'$ is meromorphic. □

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