Monotone Relations in Hadamard Spaces

Ali Moslemipour\(^a\), Mehdi Roohi\(^b\), Mohammad Reza Mardanbeigi\(^a\), Mahdi Azhini\(^a\)

\(^a\)Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
\(^b\)Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran

Abstract. In this paper, the notion of \(W\)-property for subsets of \(X \times X^0\) is introduced and investigated, where \(X\) is an Hadamard space and \(X^0\) is its linear dual space. It is shown that an Hadamard space \(X\) is flat if and only if \(X \times X^0\) has \(W\)-property. Moreover, the notion of monotone relation from an Hadamard space to its linear dual space is introduced. A characterization result for monotone relations with \(W\)-property (and hence in flat Hadamard spaces) is given. Finally, a type of Debrunner-Flor Lemma concerning extension of monotone relations in Hadamard spaces is proved.

1. Introduction and Preliminaries

Let \((X, d)\) be a metric space. We say that a mapping \(c : [0, 1] \rightarrow X\) is a geodesic path from \(x \in X\) to \(y \in X\) if \(c(0) = x, c(1) = y\) and \(d(c(t), c(s)) = |t - s|d(x, y)\), for each \(t, s \in [0, 1]\). The image of \(c\) is said to be a geodesic segment joining \(x\) and \(y\). A metric space \((X, d)\) is called a geodesic space if there is a geodesic path between every two points of \(X\). Also, a geodesic space \(X\) is called uniquely geodesic space if for each \(x, y \in X\) there exists a unique geodesic path from \(x\) to \(y\). From now on, in a uniquely geodesic space, we denote the set \(c([0, 1])\) by \([x, y]\) and for each \(z \in [x, y]\), we write \(z = (1 - t)x \oplus ty\), where \(t \in [0, 1]\). In this case, we say that \(z\) is a convex combination of \(x\) and \(y\). Hence, \([x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}\). More details can be found in [3, 5].

**Definition 1.1.** [9, Definition 2.2] Let \((X, d)\) be a geodesic space, \(v_1, v_2, v_3, \ldots, v_n\) be \(n\) points in \(X\) and \(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in (0, 1)\) be such that \(\sum_{i=1}^n \lambda_i = 1\). We define convex combination of \(\{v_1, v_2, v_3, \ldots, v_n\}\) inductively as following:

\[
\Theta_{i=1}^n \lambda_i v_i := (1 - \lambda_n) \left( \frac{\lambda_1}{1 - \lambda_n} v_1 \oplus \frac{\lambda_2}{1 - \lambda_n} v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} v_{n-1} \right) \oplus \lambda_n v_n. \tag{1}
\]

Note that for every \(x \in X\), we have \(d(x, \Theta_{i=1}^n \lambda_i v_i) \leq \sum_{i=1}^n \lambda_i d(x, v_i)\).

According to [3, Definition 1.2.1], a geodesic space \((X, d)\) is a CAT(0) space, if the following condition, so-called \(CN\)-inequality, holds:

\[
d(z, (1 - \lambda) x \oplus \lambda y)^2 \leq (1 - \lambda) d(z, x)^2 + \lambda d(z, y)^2 - \lambda(1 - \lambda) d(x, y)^2 \text{ for all } x, y, z \in X, \lambda \in [0, 1]. \tag{2}
\]
One can show that (for instance see [3, Theorem 1.3.3]) CAT(0) spaces are uniquely geodesic spaces. An Hadamard space is a complete CAT(0) space.

Let $X$ be an Hadamard space. For each $x, y \in X$, the ordered pair $(x, y)$ is called a bound vector and is denoted by $\vec{xy}$. Indeed, $X^2 = \{\vec{xy} : x, y \in X\}$. For each $x \in X$, we apply $0_x := \vec{xx}$ as zero bound vector at $x$ and $-\vec{xx}$ as the bound vector at $x$. The bound vectors $\vec{xy}$ and $\vec{yx}$ are called admissible if $y = u$. Therefore the sum of two admissible bound vectors $\vec{xy}$ and $\vec{yz}$ is defined by $\vec{xy} + \vec{yz} = \vec{xz}$. Ahmadi Kakavandi and Amini in [2] have introduced the dual space of an Hadamard space, by using the concept of quasilinearization of abstract metric spaces presented by Berg and Nikolaev in [4]. The quasilinearization map is defined as following:

\[
\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R}
\]

\[
\langle ab, cd \rangle := \frac{1}{2} \left[ d(a, b)^2 + d(b, c)^2 - d(a, c)^2 - d(b, d)^2 \right]; \quad a, b, c, d \in X.
\] (3)

Let $x, y \in X$, we define the mapping $\varphi_{\vec{xy}} : X \to \mathbb{R}$ by $\varphi_{\vec{xy}}(z) = \frac{1}{2}(d(x, z)^2 - d(y, z)^2)$; for each $z \in X$. We will see that $\varphi_{\vec{xy}}$ possess attractive properties that simplify some calculations. We observe that (3) can be rewritten as following:

\[
\langle ab, cd \rangle = \varphi_{\vec{ab}}(b) - \varphi_{\vec{ab}}(a) = \varphi_{\vec{ac}}(d) - \varphi_{\vec{ac}}(c).
\]

The metric space $(X, d)$ satisfies the Cauchy-Schwarz inequality if

\[
\langle ab, cd \rangle \leq d(a, b)d(c, d) \quad \text{for all} \ a, b, c, d \in X.
\]

This inequality characterizes CAT(0) spaces. Indeed, it follows from [4, Corollary 3] that a geodesic space $(X, d)$ is a CAT(0) space if and only if it satisfies in the Cauchy-Schwarz inequality. For an Hadamard space $(X, d)$, consider the mapping

\[
\Psi : \mathbb{R} \times X^2 \to C(X, \mathbb{R}),
\]

\[
(t, a, b) \mapsto \Psi(t, a, b) = t\vec{ab} = t(\vec{ab}, \overline{a}x); \quad a, b, x \in X, \ t \in \mathbb{R},
\]

where $C(X, \mathbb{R})$ denotes the space of all continuous real-valued functions on $X$. It follows from Cauchy-Schwarz inequality that $\Psi(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm

\[
L(\Psi(t, a, b)) = \|d(a, b)\| \quad \text{for all} \ a, b \in X \quad \text{and} \ t \in \mathbb{R},
\] (4)

where the Lipschitz semi-norm for any function $\varphi : (X, d) \to \mathbb{R}$ is defined by

\[
L(\varphi) = \sup \left\{ \frac{\varphi(x) - \varphi(y)}{d(x, y)} : x, y \in X, x \neq y \right\}.
\]

A pseudometric $D$ on $\mathbb{R} \times X^2$ induced by the Lipschitz semi-norm (4), is defined by

\[
D((t, a, b), (s, c, d)) = L(\Psi(t, a, b) - \Psi(s, c, d)); \quad a, b, c, d \in X, \ t, s \in \mathbb{R}.
\]

For an Hadamard space $(X, d)$, the pseudometric space $(\mathbb{R} \times X^2, D)$ can be considered as a subspace of the pseudometric space of all real-valued Lipschitz functions $\text{Lip}(X, \mathbb{R})$. Note that, in view of [2, Lemma 2.1], $D((t, a, b), (s, c, d)) = 0$ if and only if $t\vec{ab} = s\vec{cd}$, for all $x, y \in X$. Thus, $D$ induces an equivalence relation on $\mathbb{R} \times X^2$, where the equivalence class of $(t, a, b) \in \mathbb{R} \times X^2$ is

\[
[t\vec{ab}] = \{ s\vec{cd} : s \in \mathbb{R}, c, d \in X, D((t, a, b), (s, c, d)) = 0 \}.
\]

The dual space of an Hadamard space $(X, d)$, denoted by $X^*$, is the set of all equivalence classes $[t\vec{ab}]$ where $(t, a, b) \in \mathbb{R} \times X^2$, with the metric $D([t\vec{ab}], [s\vec{cd}]) := D((t, a, b), (s, c, d))$. Clearly, the definition of equivalence
classes implies that \( [\overrightarrow{ab}] = [\overrightarrow{bb}] \) for all \( a, b \in X \). The zero element of \( X^* \) is \( 0 := [\overrightarrow{0}] \), where \( a \in X \) and \( t \in \mathbb{R} \) are arbitrary. It is easy to see that the evaluation \( \langle 0, \cdot \rangle \) vanishes for any bound vectors in \( X^2 \). Note that in general \( X^* \) acts on \( X^2 \) by

\[
\langle x', \overrightarrow{xy} \rangle = t(\overrightarrow{ab}, \overrightarrow{xy}), \quad \text{where } x' = [\overrightarrow{ab}] \in X^* \text{ and } \overrightarrow{xy} \in X^2.
\]

The following notation will be used throughout this paper.

\[
\left\langle \sum_{i=1}^{n} \alpha_i x'_i, \overrightarrow{xy} \right\rangle := \sum_{i=1}^{n} \alpha_i (x'_i, \overrightarrow{xy}), \quad \alpha_i \in \mathbb{R}, \ x'_i \in X^*, \ n \in \mathbb{N}, \ x, y \in X.
\]

For an Hadamard space \((X, d)\), Chaipunya and Kumam in [7], defined the linear dual space of \( X \) by

\[
X^0 = \left\{ \sum_{i=1}^{n} \alpha_i x'_i : \alpha_i \in \mathbb{R}, \ x'_i \in X^*, \ n \in \mathbb{N} \right\}.
\]

Therefore, \( X^0 = \text{span} \ X^* \). It is easy to see that \( X^0 \) is a normed space with the norm \( ||x^0||_0 = L(x^0) \) for all \( x^0 \in X^0 \). Indeed:

**Lemma 1.2.** [14, Proposition 3.5] Let \( X \) be an Hadamard space with linear dual space \( X^0 \). Then

\[
||x^0||_0 := \sup \left\{ \frac{|\langle x^0, \overrightarrow{ab} \rangle - \langle x^0, \overrightarrow{cd} \rangle|}{d(a, b) + d(c, d)} : a, b, c, d \in X, (a, c) \neq (b, d) \right\},
\]

is a norm on \( X^0 \). In particular, \( ||[\overrightarrow{ab}]||_0 = ||\overrightarrow{ab}||_0 \).

2. Flat Hadamard Spaces and ‘W-property

Let \( M \) be a relation from \( X \) to \( X^0 \); i.e., \( M \subseteq X \times X^0 \). The domain and range of \( M \) are defined, respectively, by

\[
\text{Dom}(M) := \left\{ x \in X : \exists x^0 \in X^0 \text{ such that } (x, x^0) \in M \right\},
\]

and

\[
\text{Range}(M) := \left\{ x^0 \in X^0 : \exists x \in X \text{ such that } (x, x^0) \in M \right\}.
\]

**Definition 2.1.** Let \( X \) be an Hadamard space with linear dual space \( X^0 \). We say that \( M \subseteq X \times X^0 \) satisfies the \( \text{W-property} \) if there exists \( p \in X \) such that the following holds:

\[
\langle x^0, p((1 - \lambda)x_1 + \lambda x_2) \rangle \leq (1 - \lambda)\langle x^0, \overrightarrow{px_1} \rangle + \lambda \langle x^0, \overrightarrow{px_2} \rangle, \quad \forall \lambda \in [0, 1], \forall x^0 \in \text{Range}(M), \forall x_1, x_2 \in \text{Dom}(M).
\]

**Proposition 2.2.** Let \( X \) be an Hadamard space with linear dual space \( X^0 \) and let \( M \subseteq X \times X^0 \). Then the following statements are equivalent:

(i) \( M \subseteq X \times X^0 \) satisfies the \( \text{W-property} \) for some \( p \in X \).

(ii) \( M \subseteq X \times X^0 \) satisfies the \( \text{W-property} \) for any \( q \in X \).

(iii) For any \( q \in X \),

\[
\langle x^0, q(\sum_{i=1}^{n} \lambda_i x_i) \rangle \leq \sum_{i=1}^{n} \lambda_i \langle x^0, \overrightarrow{qx_i} \rangle, \quad \text{for all } x^0 \in \text{Range}(M), \{x_i\}_{i=1}^{n} \subseteq \text{Dom}(M), \{\lambda_i\}_{i=1}^{n} \subseteq [0, 1]. \quad (\text{W}_q(q))
\]
(iv) For some \( p \in X \), \( (\mathcal{W}_n(p)) \) holds.

Proof.

(i) \( \Rightarrow \) (ii): Let \( q \in X \) be any arbitrary element of \( X \), \( \lambda \in [0, 1] \), \( x^i \in \text{Range}(M) \), and \( x_1, x_2 \in \text{Dom}(M) \). Then

\[
\langle x^i, q((1 - \lambda)x_1 \oplus \lambda x_2) \rangle = \langle x^i, q\bar{\lambda} \rangle + \langle x^i, p((1 - \lambda)x_1 \oplus \lambda x_2) \rangle
\]

\[
= (1 - \lambda)\langle x^i, q\bar{\lambda} \rangle + \langle x^i, (1 - \lambda)x_1 \oplus \lambda x_2 \rangle
\]

\[
\leq (1 - \lambda)\langle x^i, q\bar{\lambda} \rangle + \lambda\langle x^i, \bar{\lambda} \rangle + \langle x^i, \bar{\lambda} \rangle
\]

\[
= (1 - \lambda)\langle x^i, q\bar{\lambda} \rangle + \lambda\langle x^i, \bar{\lambda} \rangle,
\]

as required.

(ii) \( \Rightarrow \) (iii): We proceed by induction on \( n \). By Definition 2.1 the claim is true for \( n = 2 \). Now assume that \( (\mathcal{W}_{n-1}(q)) \) is true. In view of equation (1),

\[
\langle x^i, q\oplus_{i=1}^{\lambda_i(x_i)} \rangle = \langle x^i, q((1 - \lambda_{n-1})x_1 \oplus \lambda_{n-1}x_2 \oplus \cdots \oplus (1 - \lambda_{n-1})x_{n-1} \oplus \lambda_n x_n) \rangle
\]

\[
\leq (1 - \lambda_{n-1})\langle x^i, q\rangle + \lambda_{n-1}\langle x^i, \bar{\lambda} \rangle
\]

\[
= \sum_{i=1}^{n-1} \lambda_i(\langle x^i, q\bar{\lambda} \rangle) + \lambda_n(\langle x^i, \bar{\lambda} \rangle)
\]

\[
= \sum_{i=1}^{n} \lambda_i(\langle x^i, q\bar{\lambda} \rangle).
\]

(iii) \( \Rightarrow \) (iv): Clear.

(iv) \( \Rightarrow \) (i): Take \( n = 2 \) in \( (\mathcal{W}_n(p)) \).

We are done. \( \Box \)

Remark 2.3. It should be noticed that Proposition 2.2 implies that \( \mathcal{W} \)-property is independent of the choice of the element \( p \in X \).

Definition 2.4. [11, Definition 3.1] An Hadamard space \( (X, d) \) is said to be flat if equality holds in the CN-inequality, i.e., for each \( x, y \in X \) and \( \lambda \in [0, 1] \), the following holds:

\[
d(z, (1 - \lambda)x \oplus \lambda y)^2 = (1 - \lambda)d(z, x)^2 + \lambda d(z, y)^2 - \lambda(1 - \lambda)d(x, y)^2, \text{ for all } z \in X.
\]

Proposition 2.5. Let \( X \) be an Hadamard space. The following statements are equivalent:

(i) \( X \) is a flat Hadamard space.

(ii) \( \langle x((1 - \lambda)x \oplus \lambda y), ab \rangle = \lambda(\bar{a}\bar{b}, ab) \), for all \( a, b, x, y \in X \), and all \( \lambda \in [0, 1] \).

(iii) \( X \times X^e \) has \( \mathcal{W} \)-property.
(iv) Any subset of $X \times X^\circ$ has $\mathcal{W}$-property.

(v) For each $p, z \in X$, the mapping $\varphi_{\mathcal{W}}$ is convex.

(vi) For each $p, z \in X$, the mapping $\varphi_{\mathcal{W}}$ is affine, in the sense that:

$$\varphi_{\mathcal{W}}((1 - \lambda)x \oplus \lambda y) = (1 - \lambda)\varphi_{\mathcal{W}}(x) + \lambda\varphi_{\mathcal{W}}(y), \quad \forall x, y \in X, \forall \lambda \in [0, 1].$$

Proof.

(i) $\Rightarrow$ (ii): [11, Theorem 3.2].

(ii) $\Rightarrow$ (iii): Let $x, y \in X, \lambda \in [0, 1]$ and $(x, x') \in X \times X^\circ$. Then $x' = \sum_{i=1}^n a_i[a_i, b_i] \in X^\circ$, and hence by using (ii) we get:

$$\langle x', p((1 - \lambda)x \oplus \lambda y) \rangle = \sum_{i=1}^n a_i t_i \langle a_i b_i, p^x \rangle + x((1 - \lambda)x \oplus \lambda y)$$

$$= \sum_{i=1}^n a_i t_i \langle a_i b_i, p^x \rangle + \lambda \langle a_i b_i, xy \rangle$$

$$= \sum_{i=1}^n a_i t_i \langle a_i b_i, p^x \rangle + \lambda \langle a_i b_i, p^y - p^x \rangle$$

$$= \sum_{i=1}^n a_i t_i \langle 1 - \lambda \rangle \langle a_i b_i, p^x \rangle + \lambda \langle a_i b_i, p^y \rangle$$

$$= (1 - \lambda) \sum_{i=1}^n a_i t_i \langle a_i b_i, p^x \rangle + \lambda \sum_{i=1}^n a_i t_i \langle a_i b_i, p^y \rangle$$

$$= (1 - \lambda) \langle \sum_{i=1}^n a_i t_i b_i, p^x \rangle + \lambda \langle \sum_{i=1}^n a_i t_i b_i, p^y \rangle$$

$$= (1 - \lambda) \langle x', p^x \rangle + \lambda \langle x', p^y \rangle.$$

Therefore $X \times X^\circ$ has $\mathcal{W}$-property.

(iii) $\Leftrightarrow$ (iv): Straightforward.

(iv) $\Rightarrow$ (v): Let $x, y \in X$ and $\lambda \in [0, 1]$, then

$$(1 - \lambda)\varphi_{\mathcal{W}}(x) + \lambda\varphi_{\mathcal{W}}(y) = \lambda(\varphi_{\mathcal{W}}(y) - \varphi_{\mathcal{W}}(x)) + \varphi_{\mathcal{W}}(x) - \varphi_{\mathcal{W}}((1 - \lambda)x \oplus \lambda y)$$

$$= \lambda \langle p^y, y \rangle + \langle p^x, ((1 - \lambda)x \oplus \lambda y) \rangle$$

$$= \lambda \langle p^y, p^y - p^x \rangle + \langle p^x, p((1 - \lambda)x \oplus \lambda y) \rangle$$

$$= \lambda(\langle p^y, p^y \rangle - \langle p^x, p((1 - \lambda)x \oplus \lambda y) \rangle) \geq 0.$$

Therefore, $\varphi_{\mathcal{W}}$ is convex.

(v) $\Rightarrow$ (vi): It is easy.
(vi) ⇒ (iii): Let \( x, y, p \in X, \lambda \in [0, 1] \) and \( x^t = \sum_{i=1}^{n} \alpha_i [t, p, z_i] \in X^t \) be given. Then

\[
\langle x^t, p((1 - \lambda)x \oplus \lambda y) \rangle = \left\langle \sum_{i=1}^{n} \alpha_i x^*, p((1 - \lambda)x \oplus \lambda y) \right\rangle
\]

\[
= \sum_{i=1}^{n} \alpha_i \langle t, p(z_i^*, p((1 - \lambda)x \oplus \lambda y) \rangle
\]

\[
= \sum_{i=1}^{n} \alpha_i \langle t, q_{p, z_i}((1 - \lambda)x \oplus \lambda y) - q_{p, z_i}(p) \rangle
\]

\[
= \sum_{i=1}^{n} \alpha_i \langle t, (1 - \lambda)q_{p, z_i}(x) + \lambda q_{p, z_i}(y) - q_{p, z_i}(p) \rangle
\]

\[
= \sum_{i=1}^{n} \alpha_i \langle t, (1 - \lambda)(q_{p, z_i}(x) - q_{p, z_i}(p)) + \lambda(q_{p, z_i}(y) - q_{p, z_i}(p)) \rangle
\]

\[
= \sum_{i=1}^{n} \alpha_i \langle t, (1 - \lambda)(z_i^*, p^X) + \lambda(z_i^*, p^Y) \rangle
\]

\[
= (1 - \lambda) \sum_{i=1}^{n} \alpha_i \langle t, p(z_i^*, p^X) \rangle + \lambda \sum_{i=1}^{n} \alpha_i \langle t, p(z_i^*, p^Y) \rangle
\]

\[
= (1 - \lambda) \langle x^*, p^X \rangle + \lambda \langle x^*, p^Y \rangle;
\]

i.e., \( X \times X^t \) has \( W \)-property.

(iii)⇒ (ii): For \( a, b, x, y \in X \) and \( \lambda \in [0, 1] \), we have:

\[
\lambda\langle ab, xy \rangle - \langle ab, x((1 - \lambda)x \oplus \lambda y) \rangle = \lambda(\langle ab, p^Y - p^X \rangle) - \langle ab, p((1 - \lambda)x \oplus \lambda y) \rangle
\]

\[
= \lambda(\langle ab, p^Y \rangle - \langle ab, p^X \rangle) - \langle ab, p((1 - \lambda)x \oplus \lambda y) \rangle + \langle ab, p^X \rangle
\]

\[
= (1 - \lambda)(\langle ab, p^X \rangle + \lambda(\langle ab, p^Y \rangle - \langle ab, p((1 - \lambda)x \oplus \lambda y) \rangle)
\]

\[
= (1 - \lambda)\langle x^*, p^X \rangle + \lambda(\langle x^*, p^Y \rangle - \langle x^*, p((1 - \lambda)x \oplus \lambda y) \rangle)
\]

where \( x^t = [ab] \in X^t \). Since \( X \times X^t \) has \( W \)-property, one can deduce that:

\[
\lambda(\langle ab, x((1 - \lambda)x \oplus \lambda y) \rangle) \geq \langle ab, x((1 - \lambda)x \oplus \lambda y) \rangle.
\]

Hence, by interchanging the role of \( a \) and \( b \) in (5), we obtain:

\[
\langle ab, x((1 - \lambda)x \oplus \lambda y) \rangle \geq \lambda(\langle ab, xy \rangle).
\]

Finally, (5) and (6) yield:

\[
\langle ab, p((1 - \lambda)x \oplus \lambda y) \rangle = \lambda(\langle ab, xy \rangle).
\]

We are done. \( \square \)

The next example shows that there exists a relation \( M \subseteq X \times X^t \) in the non-flat Hadamard spaces which doesn't have the \( W \)-property.
Example 2.6. Consider the following equivalence relation on $\mathbb{N} \times [0, 1]$

$$(n, t) \sim (m, s) \iff t = s = 0 \text{ or } (n, t) = (m, s).$$

Set $X := \mathbb{N} \times [0, 1]$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d([(n, t)], [(m, s)]) = \begin{cases} |n - s| & n = m, \\ t + s & n \neq m. \end{cases}$$

The geodesic joining $x = [(n, t)]$ to $y = [(m, s)]$ is defined as follows:

$$(1 - \lambda)x \oplus \lambda y := \begin{cases} [(n, (1 - \lambda)t - \lambda s)] & 0 \leq \lambda \leq \frac{1}{10}, \\ [(m, (\lambda - 1)t + \lambda s)] & \frac{1}{10} \leq \lambda \leq 1, \end{cases}$$

whenever $x \neq y$ and vacuously $(1 - \lambda)x \oplus \lambda x := x$. It is known that (see [1, Example 4.7]) $(X, d)$ is an $\mathbb{R}$-tree space.

It follows from [3, Example 1.2.10], that any $\mathbb{R}$-tree space is an Hadamard space. Let $x = [(2, \frac{1}{2})], y = [(1, \frac{1}{2})], a = [(3, \frac{1}{2})], b = [(2, \frac{1}{2})]$ and $\lambda = \frac{1}{2}$. Then $\frac{1}{5}x \oplus \frac{1}{5}y = [(2, \frac{1}{5})]$ and

$$\langle x(\frac{4}{5}x \oplus \frac{1}{5}y), ab \rangle = \frac{-1}{6} \neq \frac{-1}{10} = \frac{1}{5} \langle y, ab \rangle.$$ 

Now, Proposition 2.5(ii) implies that $(X, d)$ is not a flat Hadamard space. For each $n \in \mathbb{N}$, set $x_n := [(n, \frac{1}{2})]$ and $y_n := [(n, \frac{1}{n})]$. Now, we define

$$M := \left\{ (x_n, [\overline{y_n y_n}]) : n \in \mathbb{N} \right\} \subseteq X \times X^\circ.$$ 

Take $p = [(1, 1)] \in X, [\overline{y_5 y_4}] \in \text{Range}(M)$ and $\lambda = \frac{1}{4}$. Clearly, $\hat{x} := (1 - \lambda)x_1 \oplus \lambda x_3 = [(1, \frac{1}{4})]$ and $\langle [\overline{y_5 y_4}], \overline{p x} \rangle = \frac{1}{24},$ while

$$\frac{2}{3} \langle [\overline{y_5 y_4}], \overline{p x} \rangle + \frac{1}{3} \langle [\overline{y_5 y_4}], \overline{p x} \rangle = \frac{1}{40}.$$ 

Therefore, $M$ doesn’t have the $\mathcal{W}$-property.

3. Monotone Relations

Ahmadi Kakavandi and Amini [2] introduced the notion of monotone operators in Hadamard spaces. In [10], Khatibzadeh and Ranjbar, investigated some properties of monotone operators and their resolvents and also proximal point algorithm in Hadamard spaces. Chaipunya and Kumam [7] studied the general proximal point method for finding a zero point of a maximal monotone set-valued vector field defined on Hadamard spaces. They proved the relation between the maximality and Minty’s surjectivity condition. Zamani Eskandani and Raeisi [14], by using products of finitely many resolvents of monotone operators, proposed an iterative algorithm for finding a common zero of a finite family of monotone operators and a common fixed point of an infinitely countable family of non-expansive mappings in Hadamard spaces. In this section, we will characterize the notion of monotone relations in Hadamard spaces based on characterization of monotone sets in Banach spaces [8, 12, 13].

Definition 3.1. Let $X$ be an Hadamard space with linear dual space $X^\circ$. The set $M \subseteq X \times X^\circ$ is called monotone if $\langle x^\circ - y^\circ, \overline{y^\circ} \rangle \geq 0$, for all $(x, x^\circ), (y, y^\circ) \in M.$
Example 3.2. Let $x_n$, $y_n$ and $M$ be the same as in Example 2.6. Let $(u, u'), (v, v') \in M$. There exists $m, n \in \mathbb{N}$ such that $u = x_n$, $u' := [y_{m+1}y_n]$, $v = x_m$ and $v' := [y_{m+1}y_m]$. Then

$$
\langle u' - v', \overline{v'u} \rangle = \langle u', \overline{v'u} \rangle - \langle v', \overline{v'u} \rangle = \frac{(m+1, \frac{1}{m+1})}{\left[\left(\frac{1}{n} \right) \left(\frac{1}{m} \right) \left(\frac{1}{m+1} \right)\right]} - \frac{(m+1, \frac{1}{m+1})}{\left[\left(\frac{1}{n} \right) \left(\frac{1}{m} \right) \left(\frac{1}{m+1} \right)\right]}
$$

Therefore, $\langle u' - v', \overline{v'u} \rangle \geq 0$ which shows that $M$ is a monotone relation.

In the sequel, we need the following notations. Let $X$ be an Hadamard space and $Y \subseteq X$. Put

$$
\varsigma_Y := \left\{ \eta : Y \to [0, +\infty[ \quad \text{supp } \eta \text{ is finite and } \sum_{x \in Y} \eta(x) = 1 \right\}
$$

where $\text{supp } \eta = \{ y \in Y : \eta(y) \neq 0 \}$. Clearly, for each $\emptyset \neq A \subset Y$, $\varsigma_A = \{ \eta \in \varsigma_Y : \text{supp } \eta \subseteq A \}$. It is obvious that $\varsigma_A$ is a convex subset of $\mathbb{R}^1$. Moreover, if $\emptyset \neq A \subset B$, then $\varsigma_A \subseteq \varsigma_B$. Suppose $u \in Y$ be fixed. Define $\delta_u \in \varsigma_Y$ by

$$
\delta_u(x) = \begin{cases} 1 & x = u, \\ 0 & x \neq u. \end{cases}
$$

Let $M \subseteq X \times X^*$ and $\eta \in \varsigma_A$. Then $\text{supp } \eta = \{ \lambda_1, \ldots, \lambda_n \}$ where $\lambda_i = \eta(x_i, x_i')$, for each $1 \leq i \leq n$. Let $p \in X$ be fixed. Define $\alpha : \varsigma_{X \times X^*} \to X$ (resp. $\beta : \varsigma_{X \times X^*} \to X^*$ and $\theta_p : \varsigma_{X \times X^*} \to \mathbb{R}$) by

$$
\alpha(\eta) = \sum_{i=1}^n \lambda_i x_i, \quad \beta(\eta) = \sum_{i=1}^n \lambda_i x_i^* \quad \text{and } \theta_p(\eta) = \sum_{i=1}^n \lambda_i (x_i', \overline{p x_i'}). \tag{7}
$$

Proposition 3.3. Let $X$ be an Hadamard space, $M \subseteq X \times X^*$ and $p \in X$. Set

$$
\Theta_{p, M} := \left\{ \eta \in \varsigma_M : \theta_p(\eta) \geq \langle \beta(\eta), \overline{p x(\eta)} \rangle \right\}.
$$

Then $\Theta_{p, M} = \Theta_{q, M}$ for any $q \in X$.

Proof. It is enough to show that $\Theta_{p, M} \subseteq \Theta_{q, M}$. Let $\eta \in \Theta_{p, M}$ be such that $\text{supp } \eta = \{ \lambda_1, \ldots, \lambda_n \}$ where $\lambda_i = \eta(x_i, x_i')$, for each $1 \leq i \leq n$. Then

$$
\theta_q(\eta) = \sum_{i=1}^n \lambda_i (x_i', \overline{q x_i'}) = \sum_{i=1}^n \lambda_i (x_i', \overline{q p x_i'}) + \sum_{i=1}^n \lambda_i (x_i', \overline{p x_i'})
$$

$$
= \langle \sum_{i=1}^n \lambda_i x_i', \overline{q p} \rangle + \theta_p(\eta) = \langle \beta(\eta), \overline{q p} \rangle + \theta_p(\eta)
$$

$$
\geq \langle \beta(\eta), \overline{p x(\eta)} \rangle.
$$

Therefore, $\eta \in \Theta_{q, M}$, i.e., $\Theta_{p, M} \subseteq \Theta_{q, M}$. \qed
According to Proposition 3.3, for each \( M \subseteq X \times X^* \), the set \( \Theta_{pM} \) is independent of the choice of the element \( p \in X \) and hence we denote the set \( \Theta_{pM} \) by \( \Theta_M \).

**Theorem 3.4.** Let \( X \) be an Hadamard space and \( M \subseteq X \times X^* \) satisfies the \( \mathcal{W} \)-property. Then \( M \) is a monotone set if and only if \( \Theta_M = \varsigma_M \).

**Proof.** Let \( M \) be a monotone set. In view of (7), it is enough to show that \( \varsigma_M \subseteq \Theta_M \). Let \( \eta \in \varsigma_M \) be such that \( \text{supp}\eta = \{ \lambda_1, \ldots, \lambda_n \} \) where \( \lambda_i = \eta(x_i, x^*_i) \), for each \( 1 \leq i \leq n \). By using Proposition 2.2, we obtain:

\[
\theta_p(\eta) - \langle \beta(\eta), \overline{px(\eta)} \rangle = \sum_{i=1}^{n} \lambda_i(x^*_i, \overline{px_i}) - \left\langle \sum_{i=1}^{n} \lambda_i(x^*_i, \overline{px_i}) \right\rangle \\
= \sum_{i=1}^{n} \lambda_i(x^*_i, \overline{px_i}) - \sum_{i=1}^{n} \lambda_i(x^*_i, \overline{px_i}) \\
\geq \sum_{i=1}^{n} \lambda_i(x^*_i, \overline{px_i}) - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (x^*_i, \overline{px_i}) \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (x^*_i - x^*_j, \overline{px_i}) \\
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (x^*_i - x^*_j, \overline{px_i}) \\
\geq \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (x^*_i - x^*_j, \overline{px_i}) \geq 0.
\]

Then \( \varsigma_M \subseteq \Theta_M \) and hence \( \varsigma_M = \Theta_M \). For the converse, let \( (x, x^*), (y, y^*) \in M \) and set \( \eta := \frac{1}{2} \delta_{(x, x^*)} + \frac{1}{2} \delta_{(y, y^*)} \in \varsigma_M \). By using \( \mathcal{W} \)-property, we get:

\[
\frac{1}{4} (x^* - y^*, \overline{px^*} - \overline{py^*}) = \frac{1}{4} (x^* - y^*, \overline{px^*} - \overline{py^*}) \\
= \frac{1}{4} ((x^* - y^*, \overline{px^*}) - (x^* - y^*, \overline{py^*})) \\
= \frac{1}{4} (x^*, \overline{px^*}) + \frac{1}{4} (y^*, \overline{py^*}) - \frac{1}{4} (x^*, \overline{py^*}) - \frac{1}{4} (y^*, \overline{py^*}) \\
= \frac{1}{2} (x^*, \overline{px^*}) + \frac{1}{2} (y^*, \overline{py^*}) - \frac{1}{2} (x^*, \overline{px^*}) - \frac{1}{2} (y^*, \overline{py^*}) \\
\geq \frac{1}{2} (x^*, \overline{px^*}) + \frac{1}{2} (y^*, \overline{py^*}) - \left( \frac{1}{2} (x^* + y^*, p(\frac{1}{2} x \oplus \frac{1}{2} y)) \right) \\
= \frac{1}{2} (x^*, \overline{px^*}) + \frac{1}{2} (y^*, \overline{py^*}) - \frac{1}{2} (x^*, p(\frac{1}{2} x \oplus \frac{1}{2} y)) - \frac{1}{2} (y^*, p(\frac{1}{2} x \oplus \frac{1}{2} y)) \\
= \theta_p(\eta) - \langle \beta(\eta), \overline{px(\eta)} \rangle \geq 0.
\]

Therefore, \( M \) is monotone. \( \square \)
Corollary 3.5. Let $X$ be a flat Hadamard space and $M \subseteq X \times X^\circ$. Then $M$ is a monotone set if and only if $\Theta_M = \varsigma_M$.

Proof. Since $X$ is flat, Proposition 2.5 implies that $M \subseteq X \times X^\circ$ satisfies the $\mathcal{W}$-property. Then the conclusion follows immediately from Theorem 3.4.  

A fundamental result concerning monotone operators is the extension theorem of Debrunner-Flor (for a proof see [6, Theorem 4.3.1] or [15, Proposition 2.17]). In the sequel, we prove a type of this result for monotone relations from an Hadamard space to its linear dual space. First, we recall some notions and results.

Definition 3.6. [2, Definition 2.4] Let $\{x_n\}$ be a sequence in an Hadamard space $X$. The sequence $\{x_n\}$ is said to be weakly convergent to $x \in X$, denoted by $x_n \rightharpoonup x$, if $\lim_{n \to \infty} \langle x_n, x \rangle = 0$, for all $y \in X$.

One can easily see that convergence in the metric implies weak convergence.

Lemma 3.7. [14, Proposition 3.6] Let $\{x_n\}$ be a bounded sequence in an Hadamard space $(X, d)$ with linear dual space $X^\circ$ and let $\{x^*_n\}$ be a sequence in $X^\circ$. If $\{x_n\}$ is weakly convergent to $x$ and $x^*_n \rightharpoonup_{X^\circ} x^*$, then $\langle x^*_n, x_n \rangle \to \langle x^*, x \rangle$, for all $z \in X$.

Theorem 3.8. Let $X$ be an Hadamard space and $M \subseteq X \times X^\circ$ be a monotone relation satisfies the $\mathcal{W}$-property. Let $C \subseteq X^\circ$ be a compact and convex set, and $\varphi : C \to X$ be a continuous function. Then there exists $z^* \in C$ such that $\{(\varphi(z^*)), z^*\} \cup M$ is monotone.

Proof. Let $x \in X$, $u^i, v^i \in X^\circ$ be arbitrary and fixed element. Consider the function $\tau : C \to \mathbb{R}$ defined by

$$\tau(x^i) = \langle x^i - v^i, \overline{x\varphi(u^i)} \rangle, \quad x^i \in C.$$ 

Let $\{x^*_n\} \subseteq C$ be such that $x^*_n \rightharpoonup_{X^\circ} x^i$, for some $x^i \in C$. By Lemma 3.7,

$$\langle x^*_n - v^i, \overline{x\varphi(u^i)} \rangle \to \langle x^i - v^i, \overline{x\varphi(u^i)} \rangle.$$ 

Thus $\tau(x^*_n) \to \tau(x^i)$. Hence $\tau$ is continuous. For every $(y, y^i) \in M$, set

$$U(y, y^i) := \{u^i \in C : \langle u^i - y^i, \overline{y\varphi(u^i)} \rangle < 0\}.$$ 

Continuity of $\tau$ implies that $U(y, y^i)$ is an open subset of $C$. Suppose that the conclusion fails. Then for each $u^i \in C$ there exists $(y, y^i) \in M$ such that $u^i \in U(y, y^i)$. This means that the family of open sets $\{U(y, y^i)\}_{(y, y^i) \in M}$ is an open cover of $C$. Using the compactness of $C$, we obtain that $C = \bigcup_{i=1}^n U(y_i, y^i_i)$. In addition, [15, Page 756] implies that there exists a partition of unity associated with this finite subcover. Hence, there are continuous functions $\psi_i : X^\circ \to \mathbb{R}$ ($1 \leq i \leq n$) satisfying

(i) $\sum_{i=1}^n \psi_i(x^i) = 1$, for all $x^i \in C$.
(ii) $\psi_i(x^i) \geq 0$, for all $x^i \in C$ and all $i \in \{1, \ldots, n\}$.
(iii) $\{x^i \in C : \psi_i(x^i) > 0\} \subseteq U_i := U(y_i, y^i_i)$ for all $i \in \{1, \ldots, n\}$.

Set $K := \text{co}((y^i_1, \ldots, y^i_n)) \subseteq C$ and define

$$\iota : K \to K$$

$$u^i \mapsto \sum_{i=1}^n \psi_i(u^i) y^i_i.$$ 

Let $\{w^i_n\} \subseteq K$ be such that $w^i_n \to u^i$,

$$\|\iota(w^i_n) - \iota(u^i)\|_v = \left\| \sum_{i=1}^n \psi_i(u^i_n) y^i_i - \sum_{i=1}^n \psi_i(u^i_n) y^i_i \right\|_v$$
\[
\begin{align*}
&= \left\| \sum_{i=1}^{n} (\psi_i(u_i^n) - \psi_i(u^\circ)) y_i^\circ \right\|_p \\
&\leq \sum_{i=1}^{n} \left\| (\psi_i(u_i^n) - \psi_i(u^\circ)) y_i^\circ \right\|_p \\
&\leq \sum_{i=1}^{n} |\psi_i(u_i^n) - \psi_i(u^\circ)| \left\| y_i^\circ \right\|_p.
\end{align*}
\]

By continuity of \(\psi_i (1 \leq i \leq n)\), letting \(m \to +\infty\), then \(\psi_i(u_i^n) \to \psi_i(u^\circ)\) and this implies that \(\iota(u_i^n) \to \iota(u^\circ)\) and so \(\iota\) is continuous. One can identify \(K\) with a finite-dimensional convex and compact set. By using Brouwer fixed point theorem [15, Proposition 2.6], there exists \(\bar{w} \in K\) such that \(\iota(\bar{w}) = \bar{w}\). Moreover, by using Proposition 2.2 we get:

\[
0 = \left\{ \iota(\bar{w}) - \bar{w}, \overline{\phi(\bar{w})} (\oplus_i \psi_i(\bar{w}) y_i) \right\} \\
= \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} (\oplus_i \psi_i(\bar{w}) y_i) \right\} \\
= \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} (\oplus_i \psi_i(\bar{w}) y_i) \right\} - \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} \\
\leq \sum_i \psi_i(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} - \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} \\
= \sum_i \psi_i(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} - \sum_j \psi_j(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} \\
= \sum_i \psi_i(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} - \sum_j \psi_j(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} \\
= \sum_i \psi_i(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} - \sum_j \psi_j(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\} \\
= \sum_i \psi_i(\bar{w}) \left\{ \sum_i \psi_i(\bar{w}) (y_i^\circ - \bar{w}), \overline{\phi(\bar{w})} \right\}.
\]

Set \(a_{ij} = (y_i^\circ - \bar{w}, \overline{\phi(\bar{w})} y_j)\). It follows from monotonicity of \(M\) that

\[
\begin{align*}
a_{ii} + a_{ij} - a_{ji} - a_{ii} &= (y_i^\circ - \bar{w}, \overline{\phi(\bar{w})} y_i) + (y_i^\circ - \bar{w}, \overline{\phi(\bar{w})} y_j) \\
&\quad - (y_i^\circ - \bar{w}, \overline{\phi(\bar{w})} y_j) - (y_j^\circ - \bar{w}, \overline{\phi(\bar{w})} y_i) \\
&= (y_i^\circ - y_j^\circ, \overline{\phi(\bar{w})} y_j - \overline{\phi(\bar{w})} y_i) \\
&= (y_i^\circ - y_j^\circ, \bar{y}_j y_i) \geq 0;
\end{align*}
\]

i.e.,

\[
a_{ii} + a_{ij} \geq a_{ij} + a_{ii}.
\]
Applying (8) and (9), we obtain:

\[
0 \leq \sum_{i,j} \psi_i(w^o)\psi_j(w^o)a_{ij}
\]

\[
= \sum_{i\neq j} \psi_i(w^o)\psi_j(w^o)a_{ij} + \sum_{i=1}^{n} \psi_i(w^o)^2a_{ii} + \sum_{i>j} \psi_i(w^o)\psi_j(w^o)a_{ij}
\]

\[
= \sum_{i=1}^{n} \psi_i(w^o)^2a_{ii} + \sum_{i<j} \psi_i(w^o)\psi_j(w^o)(a_{ii} + a_{jj})
\]

\[
\leq \sum_{i=1}^{n} \psi_i(w^o)^2a_{ii} + \sum_{i<j} \psi_i(w^o)\psi_j(w^o)(a_{ii} + a_{jj}).
\]

(10)

Set \(I(w^o) := \{ i \in [1, \ldots, n] : w^o \in U_i \} \). Applying property (iii) of the partition of unity in (11) we get:

\[
0 \leq \sum_{i \in I(w^o)} \psi_i(w^o)^2a_{ii} + \sum_{i,j \in I(w^o)} \psi_i(w^o)\psi_j(w^o)(a_{ii} + a_{jj}).
\]

(12)

By using property (iii) of the partition of unity and the definition of \( U_i \), one deduce that all terms in the right-hand side of (12) are nonpositive. So all of \( \psi_i(w^o) \)'s must be vanish, which contradicts with (i).

**Corollary 3.9.** Let \( X \) be a flat Hadamard space and \( M \subseteq X \times X^o \) be a monotone set. Let \( C \subseteq X^o \) be a compact and convex set, and \( \varphi : C \to X \) be a continuous function. Then there exists \( z^o \in C \) such that \( \{(\varphi(z^o), z^o)\} \cup M \) is monotone.

**Proof.** Since \( X \) is flat, it follows from Proposition 2.5 that \( M \subseteq X \times X^o \) has \( \mathcal{W} \)-property. The inclusion follows from Theorem 3.8.

---

**References**


