A Fixed Point Theorem for Mappings Satisfying a New Common Range Property

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Abstract. In this paper a general fixed point theorem for two pairs of mappings satisfying a new type of common range property without limit of sequences in metric spaces are proved.

1. Introduction and Preliminaries

Let X be a non empty set and $A, S : X \to X$ two self mapping on X. A point $x \in X$ is a coincidence point of A and S if $w = Ax = Sx$ for some $x \in X$.

The set of all coincidence points of A and S is denoted by $C(A, S)$, and w is said to be a point of coincidence of A and S.

Definition 1.1. \cite{7} Let X be a nonempty set and A and S be two self mappings on X. A and S are weakly compatible if $ASu = SAu$ for all $u \in C(A, S)$.

In 2011, Sintunavarat and Kumam \cite{12} introduced the notion of common limit range property in metric spaces.

Definition 1.2. \cite{12} A pair of self mappings A and S on a metric space $(X, d)$ is said to satisfy common limit range property with respect to S, denoted $\text{CLR}(S)$ property if there exists a sequence $x_n \in X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = t \in S(X).$$

Recently, Imdad et all. \cite{3} extend this notion of common limit range property for two pairs of mappings.

Definition 1.3. \cite{3}. Two pairs $(A, S)$ and $(B, T)$ of self mappings on a metric space $(X, d)$ satisfy common limit range property with respect to $(ST)$, denoted $\text{CLR}(S, T)$ property if there exist two sequences $x_n$ and $y_n \in X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = u \in S(X) \cap T(X).$$

Some fixed point results for two pairs of mappings with theorems with $\text{CLR}(S)$ and $\text{CLR}(S, T)$ - properties are obtained in \cite{4},\cite{5},\cite{6} and other papers. Quite recently, a new type of common limit range property is introduced in \cite{11}.

Definition 1.4. \cite{11} Let $A, S$ and $T$ be self mappings of a metric space $(X, d)$. The pair $(A, S)$ is said to satisfy a common limit range property with respect to $T$, denoted by $\text{CLR}(A, S, T)$ - property if there exist a sequence $x_n$ such that

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\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = u \in S(X) \cap T(X) \]

**Remark 1.1.** In all definitions 1.2 – 1.4 there exists some convergent sequences in \( X \). We introduce a new type of common range property without limits of sequences.

**Definition 1.5.** \((A, S)\) and \(T\) satisfy \(\text{CRP}_{(A,S,T)}\) - coincidence range property with respect to \(T\), if there exists \( u \in C(A, S) \), with \( z = u \in T(X) \).

**Example 1.1.** Let \( X = [1, \infty) \) with the usual metric, and \( Ax = (x^2 + 1)/2 \), \( Sx = (x + 1)/2 \) and \( Tx = x \), then \( T(X) = [1, \infty) \), and \( Sx = Ax \) implies \( x = 1 \). As a consequence, \( A1 = S1 = z = 1 \in T(X) = [1, \infty) \).

2. Implicit relations

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition, by an implicit function in [9] and [10] and other papers. In 2008, Ali and Imdad [2] had introduced a new class of implicit relations. We will introduce a new class of implicit relations, similarly with [2].

**Definition 2.1.** Let \( F_{C} \) be a family of functions \( F(t_{1}, \ldots, t_{k}) : R^{k}_{+} \to R \) satisfying: (F1): \( F(t_{0}, 0, t, t, 0) > 0 \), for all \( t > 0 \), (F2): \( F(t, t, 0, t, t, t) > 0 \), for all \( t > 0 \). The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying \( \text{CRP}_{(A,S,T)} \) - property and an implicit relation.

**Example 2.2.** \( F(t_{1}, \ldots, t_{k}) = t_{1} - k \max \{ t_{2}, \ldots, t_{k} \} \), where \( k \in (0,1) \).

**Example 2.3.** \( F(t_{1}, \ldots, t_{k}) = k \max \{ t_{2}, t_{3}, t_{4}, \frac{t_{5} + t_{6}}{2} \} \), where \( k \in (0,1) \).

**Example 2.4.** \( F(t_{1}, \ldots, t_{k}) = t_{1} - a \max \{ t_{1}, t_{4} \} - c \max \{ t_{2}, t_{3} \} \), where \( a, b, c \geq 0 \) and \( a + b + c < 1 \).

**Example 2.5.** \( F(t_{1}, \ldots, t_{k}) = t_{1} - a \max \{ t_{3}, t_{5} \} - (1-a)(a.t_{5} + b.t_{6}) \), where \( a, b, c \geq 0 \) and \( a + b < 1 \).

**Example 2.6.** \( F(t_{1}, \ldots, t_{k}) = t_{1} - a \frac{t_{3}}{1 + t_{5}} + b.t_{6} \), where \( a, b \geq 0 \) and \( a + 2b < 1 \).

**Example 2.7.** \( F(t_{1}, \ldots, t_{k}) = t_{1} - t_{1}(a.t_{2} + b.t_{3} + c.t_{4}) - d.t_{5}.t_{6} \), where \( a, b, c, d \geq 0 \) and \( a + b + c + d < 1 \).

**Example 2.8.** \( F(t_{1}, \ldots, t_{k}) = t_{1} - \max \{ c.t_{2}, c.t_{3}, c.t_{4}, a.t_{5} + b.t_{6} \} \), where \( a, b, c \geq 0 \) and \( \max \{ c, a + b \} < 1 \).

The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying \( \text{CRP}_{(A,S,T)} \) - properties without the use of limits of mappings.

3. Main result:

**Lemma 3.1** [1]. Let \( f, g \) be two weakly compatible mappings of a non empty set \( X \). If \( f \) and \( g \) have a unique point \( w \) of coincidence where \( w = fx = gx \), for that \( xeX \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

**Theorem 3.2** Let \( A, B, S, T \) be self mappings of a metric space such that: (3.1)\( F(d(Ax, By)), d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ax, Ty) \leq 0 \) for all \( x, y \in X \) and some \( F \in F_{C} \).

If \((A, S)\) and \(T\) satisfy \( \text{CRP}_{(A,S,T)} \) property then \( C(B, T) \neq \emptyset \). Moreover, if \((A, S)\) and \( (B, T)\) are two pairs of weakly compatible mappings, then \( A, B, S \), and \( T \) have a unique common fixed point.

**Proof:** Since \((A, S)\) and \(T\) satisfy \( \text{CRP}_{(A,S,T)} \) property, there exist \( u \in X \) such that \( z = Av = Sv \) with \( z \in T(X) \). Hence, there exists \( u \in X \) such that \( z = T(u) \).

By 3.1. for \( x = u \) and \( y = u \) we obtain: \( F(d(Av, Bu)), d(Sv, Tu), d(Sv, Av), d(Tu, Bu), d(Sv, Bu), d(Av, Tu) \leq 0 \), \( F(d(z, Bu), 0, 0, d(z, Bu), d(z, Bu), 0) \leq 0 \). A contradiction with (F1) if \( d(z, Bu) > 0 \), hence \( d(z, Bu) = 0 \). Which implies \( z = Bu = Tu \). Therefore \( z = Av = Sv = Tu = Bu \). Therefore, \( z \) is a common point of coincidence of \((A, S)\) and \((B, T)\).

We prove that \( z \) is the unique point of coincidence for \( A \) and \( S \). Suppose that \( t = Aw = Bw \) for some \( w \in X \). By 3.1 we obtain for \( x = w \) and \( y = u \) that \( F(d(Av, Bu)), d(Sv, Tu), d(Sv, Av), d(Tu, Bu), d(Sv, Bu), d(Av, Tu) \leq 0 \), \( F(d(t, z), d(t, z), 0, 0, d(z, t), d(t, z)) \leq 0 \). A contradiction of (F2) if \( d(z, t) > 0 \). Which implies \( d(z, t) = 0 \), i.e. \( z = t \). And \( z \) is the unique point of coincidence of \( A \) and \( S \). Similarly \( z \) is the unique point of coincidence, moreover, if \((A, S)\) and \((B, T)\) are weakly compatible, by Lemma 3.1, \( z \) is the unique common fixed point of \( A, B, S, T \).
Remark 3.3: For the proof of this theorem we have to do the following steps:

Step 1. Solve the equation $Sx = Ax$ on $X$ and establish $C(A, S) = \{z | xX \text{ and } Sx = Ax, z = Ax\}$. If $C(A, S) = \Phi$ the theorem is not applicable.

Step 2. If $C(A, S) \neq \Phi$ we have to select $z$ from $C(A, S)$ such that $z$ exists an $x \in X$ such that $T(x) = z$. As a consequence, $A, S, T$ satisfy the CRP$_{(A, S, T)}$ property.

Step 3. Verify if the pairs $(A, S)$ and $(B, T)$ are weakly compatible. I.e. solve the $Az = Sz, z \in C(A, S)$ and similarly, for $(B, T)$: solve the $Bq = Tq, q \in C(B, T)$. If one of those pairs are not weakly compatible, the theorem cannot be applied. Stop.

Step 4. If the relation 3.1 is satisfied then, by Theorem 3.1, $A, S, B, T$ have a unique fixed point: $z$.

Example 3.4 Let $x = [0,1]$ be a metric space with $d$, the usual metric and $Ax = 0, Sx = \frac{x}{3}, Bx = \frac{x}{2}, Tx = x$. If $Ax = Sx$ then $x = 0$ and $C(A, S) = \{0\}$. Then, $z = 0, zT(x) = x$. Hence, $(A, S)$ and $T$ satisfy CRP$_{(A, S, T)}$ property.

Moreover, $AS0 = SA0 = 0$, and $BT0 = TB0 = 0$, hence $(A, S)$ and $(B, T)$ are weakly compatible. Otherhand, $d(Ax, By) = \frac{x}{4}, d(Ty, By) = \frac{x}{2}$, which implies, $d(Ax, By) \leq k d(Ty, By)$, where $k \in (\frac{1}{2}, 1)$. Then $d(Ax, By) \leq k \max[d(Sx, Ty), d(Sx, Ax)], d(Ty, By), d(Sx, By), d(Ax, Ty)$, with $ke(\frac{1}{2}, 1)$.

By Theorem 3.2, and Example 2.1, $A, B, S$ and $T$ have a unique common fixed point $z = 0$.

Remark 3.4 By Theorem 3.2 and example 2.2-2.8 we can obtain new particular results.

References


