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An Ideal Version of the Star-C-Hurewicz Covering Property

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Abstract. A space X is said to have the star-C-$\mathcal{I}$-Hurewicz (SC$\mathcal{I}$H) property if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ of countably compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}$, where $\mathcal{I}$ is a proper admissible ideal of $\mathbb{N}$. We investigate the relationships among the SC$\mathcal{I}$H and related properties. We study the topological properties of the SC$\mathcal{I}$H property. This paper generalizes several results of [21, 24] to the larger class of spaces having the SC$\mathcal{I}$H property. The star-C-$\mathcal{I}$-Hurewicz game is introduced. It is shown that, in a paracompact Hausdorff space X, if TWO has a winning strategy in the SC$\mathcal{I}$H game on X, then TWO has a winning strategy in the $\mathcal{I}$-Hurewicz game on X.

1. Introduction

Scheepers, Just, Miller and Szeptycki in [13, 19] initiated the systematic study of selection principles in topology and their relations to game theory and Ramsey theory. Kočinac and Scheepers [16] studied the Hurewicz property in detail and found its relations with function spaces, game theory and Ramsey theory. Subsequently this topic became one of the most active area of set theoretic topology. The classical selection principles were used to define and characterize various topological properties [1, 13–15, 19]. Di Maio and Kočinac [9] introduced certain types of open covers and selection principles using the ideal of asymptotic density zero sets in $\mathbb{N}$. Now a natural question arises whether this investigation can be extended to arbitrary ideals instead of special type of ideal. Das, Chandra and Kočinac [2–5, 7] studied the open covers and selection principle using arbitrary ideals of $\mathbb{N}$. Further Das et al. [4, 6] defined the ideal analogues of some variants of the Hurewicz property such as the $\mathcal{I}$-Hurewicz ($\mathcal{I}$H), the star-$\mathcal{I}$-Hurewicz (S$\mathcal{I}$H) and the strongly star-$\mathcal{I}$-Hurewicz (SS$\mathcal{I}$H), where $\mathcal{I}$ is a proper admissible ideal of $\mathbb{N}$. Following this study, the authors [28] investigated the ideal analogue of the star-K-Hurewicz property called the star-K-$\mathcal{I}$-Hurewicz (SK$\mathcal{I}$H) (also see [29]).

This paper extends the use of arbitrary ideals of $\mathbb{N}$ to some covering properties. We defined ideal analogues of the star-C-Hurewicz (SCH) called the star-C-$\mathcal{I}$-Hurewicz (SC$\mathcal{I}$H). The SC$\mathcal{I}$H is the simultaneous generalization of the SK$\mathcal{I}$H, the star-C-Hurewicz and the $\mathcal{C}$-starcompact. We study the relationship among the SC$\mathcal{I}$H and related properties. The topological properties of the SC$\mathcal{I}$H are discussed.

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This paper is organized as follows. Section 2 contains the necessary preliminaries. In Section 3, we define the SC$\mathcal{I}$H property and study its relationship with other related properties. For example, it is shown that, in a paracompact Hausdorff space X, X has the $\mathcal{I}$H property if and only if X has the SS$\mathcal{I}$H property if and only if X has the SK$\mathcal{I}$H property if and only if X has the S$\mathcal{I}$H property if and only if X has the SC$\mathcal{I}$H property. In Section 4, we study the topological properties of the SC$\mathcal{I}$H property. Several results are proved in this section, for instance: (1) The SC$\mathcal{I}$H property is not preserved under regular closed G_δ-subsets, (2) The SC$\mathcal{I}$H property is inverse invariant under open perfect mappings. In Section 5 the star-C-$\mathcal{I}$-Hurewicz game is introduced and obtained some results related to game theory and open covers.

2. Preliminaries

Throughout the paper (X, τ) or X stands for a topological space.

Let A be a subset of X and $\mathcal{P}$ be a collection of subsets of X, then $St(A, \mathcal{P}) = \bigcup\{U \in \mathcal{P} : U \cap A \neq \emptyset\}$. We usually write $St(x, \mathcal{P}) = St(\{x\}, \mathcal{P})$.

A space X is said to have the star-K-Hurewicz [26] (in short, SKH) (resp., the star-C-Hurewicz [21] (in short, SCH), the star-L-Hurewicz (in short, SLH)) property if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ of compact (resp., countably compact, Lindelöf) subsets of X such that for each $x \in X$, $x \in St(K_n, \mathcal{U}_n)$ for all but finitely many n.

A space X is said to be starcompact [10] if for every open cover $\mathcal{U}$ of X there exists a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $St(\cup\mathcal{V}, \mathcal{U}) = X$.

A space X is said to be $\mathcal{C}$-starcompact [23] ($\mathcal{K}$-starcompact [22] if for every open cover $\mathcal{U}$ of X there exists a countably compact (resp., compact) subset F of X such that $St(F, \mathcal{U}) = X$.

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers.

A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$. An ideal $\mathcal{I}$ is said to be admissible ideal or free ideal of Y if $\{y\} \in \mathcal{I}$ for each $y \in Y$. An ideal $\mathcal{I}$ is proper in Y if $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$. If $\mathcal{I}$ is a proper ideal in Y, then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y whereas the coideal of $\mathcal{I}$ is $\mathcal{I}^+ = \{A \subset Y : A \notin \mathcal{I}\}$. Throughout the paper $\mathcal{I}$ will stand proper admissible ideal of $\mathbb{N}$. We denote the ideal of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_{fin}$.

A space X is said to have the $\mathcal{I}$-Hurewicz property [4, 6, 20] (in short, $\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \bigcup \mathcal{V}_n\} \in \mathcal{I}$.

A space X is said to have the star-$\mathcal{I}$-Hurewicz property [6] (in short, S$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)\} \in \mathcal{I}$.

A space X is said to have the strongly star-$\mathcal{I}$-Hurewicz property [6] (in short, SS$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(F_n, \mathcal{U}_n) \in \mathcal{I}$.

A space X is said to have the star-K-$\mathcal{I}$-Hurewicz property [28] (in short, SK$\mathcal{I}$H) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}$.

Throughout the paper it may be noted that a space is said to be *-space if it has *-property. For example, a space is said to be star-C-Hurewicz if it has the star-C-Hurewicz property and so on.

An open cover $\mathcal{U}$ of X is said to be large [13, 19] if for each point there are infinitely many sets in $\mathcal{U}$ containing that point. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be an $\mathcal{I}$-large cover [7] if for each $x \in X$ the set $\{n \in \mathbb{N} : x \in U_n\} \in \mathcal{I}^+$. The collection of all $\mathcal{I}$-large cover will be denoted by $\mathcal{I} - \Lambda$. An open cover $\mathcal{U}$ of X is said to be $\mathcal{I}$-groupable [4] if it can be expressed in the form $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, where $\mathcal{U}_n$'s are finite, pairwise disjoint and for each $x \in X$, $\{n \in \mathbb{N} : x \notin \cup\mathcal{U}_n\} \in \mathcal{I}$.

An open cover $\mathcal{U}$ of a X is said to be star-$\mathcal{I}$-groupable [8] if it can be expressed in the form $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, where $\mathcal{U}_n$'s are finite, pairwise disjoint and for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(\cup\mathcal{U}_n, \mathcal{U})\} \in \mathcal{I}$.

Throughout the paper, the extent $e(X)$ of a space X is the smallest cardinal number κ such that the cardinality of every discrete closed subset of X is not greater than κ. Let ω denote the first infinite cardinal,

ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of all real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For each pair of ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$, $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [11].

3. The Star-C-$\mathcal{I}$-Hurewicz Property and Related Properties

Definition 3.1. A space X is said to have the star-C-$\mathcal{I}$-Hurewicz (in short, SC$\mathcal{I}$H) property if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(K_n : n \in \mathbb{N})$ of countably compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}$.

Theorem 3.2. ([6]) *A paracompact Hausdorff space X has the S$\mathcal{I}$H property if and only if it has the $\mathcal{I}$H property.*

Theorem 3.3. *Let X be a paracompact Hausdorff space. Then the following statements are equivalent:*

1. *X has the $\mathcal{I}$H property;*
2. *X has the SS$\mathcal{I}$H property;*
3. *X has the SK$\mathcal{I}$H property;*
4. *X has the S$\mathcal{I}$H property;*
5. *X has the SC$\mathcal{I}$H property.*

Proof. The implications (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (4), (3) ⇒ (5) are obvious and (4) ⇒ (1) follows from Theorem 3.2. It only remains to show (5) ⇒ (1). This proof is similar to the proof of [21, Theorem 2.11] and we make only necessary modifications. Let X have the SC$\mathcal{I}$H property. Consider a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X. Since X is a paracompact space, for each $n \in \mathbb{N}$, there exists a locally finite open refinement $\mathcal{V}_n$ of $\mathcal{U}_n$. For each $n \in \mathbb{N}$ and for each $x \in X$, there exists an open neighborhood $V_{n,x}$ of x such that $V_{n,x} \subseteq V$ for some $V \in \mathcal{V}_n$ and $\{V \in \mathcal{V}_n : V_{n,x} \cap V \neq \phi\}$ is finite. Let

$$\mathcal{V}'_n = \{V_{n,x} : x \in X\}.$$

Then $\mathcal{V}'_n$ is an open refinement of $\mathcal{V}_n$. Thus there exists a locally finite open refinement $\mathcal{V}''_n$ of $\mathcal{V}'_n$. Applying the SC$\mathcal{I}$H property of X, there is a sequence $(K_n : n \in \mathbb{N})$ of countably compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{V}''_n)\} \in \mathcal{I}$. Consequently,

$$\{n \in \mathbb{N} : x \in St(K_n, \mathcal{V}''_n)\} \in \mathcal{F}(\mathcal{I}).$$

For each $n \in \mathbb{N}$, since K_n is a countably compact subset of X, then there exists a finite subset F_n of K_n such that $K_n \subseteq St(F_n, \mathcal{V}''_n)$. For each $n \in \mathbb{N}$, let

$$\mathcal{V}'''_n = \{V \in \mathcal{V}''_n : V \cap F_n \neq \phi\}.$$

Then $\mathcal{V}'''_n$ is a finite subset of $\mathcal{V}''_n$, since F_n is a finite and $\mathcal{V}''_n$ is locally finite. For each $n \in \mathbb{N}$, let

$$\mathcal{U}'_n = \{V \in \mathcal{V}_n : \text{there exists some } V' \in \mathcal{V}'''_n \text{ such that } V \cap V' \neq \phi\}.$$

Then $\mathcal{U}'_n$ is finite, since $\mathcal{V}_n$ is locally finite. For each $V \in \mathcal{U}'_n$ choose $U_V \in \mathcal{U}_n$ such that $V \subseteq U_V$. Let $\mathcal{W}_n = \{U_V : V \in \mathcal{U}'_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$. Now let $x \in St(K_n, \mathcal{V}''_n)$ for some $n \in \mathbb{N}$ as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Choose $V'' \in \mathcal{V}''_n$ such that $x \in V''$ and $V'' \cap K_n \neq \phi$. For a finite subset F_n of K_n, $K_n \subseteq St(F_n, \mathcal{V}''_n)$, $\phi \neq V'' \cap K_n \cap F_n \subseteq V'' \cap F_n$, thus $V'' \in \mathcal{V}'''_n$. On the other hand, since $\mathcal{V}''_n$ is a open refinement of $\mathcal{V}'_n$ and $\mathcal{V}'_n$ is a open refinement of $\mathcal{V}_n$. For $V'' \in \mathcal{V}''_n$ choose $V' \in \mathcal{V}'_n$ and for $V' \in \mathcal{V}'_n$ choose $V \in \mathcal{V}_n$ such that $x \in V'' \subseteq V' \subseteq V$. Since $V'' \in \mathcal{V}'''_n$ and $V'' \cap V \neq \phi$, thus $V \in \mathcal{U}'_n$. Choose $U_V \in \mathcal{U}_n$ such that $x \in V \subseteq U_V \subset \bigcup \mathcal{W}_n$ for some n. Therefore

$$\{n \in \mathbb{N} : x \in St(K_n, \mathcal{V}''_n)\} \subset \{n \in \mathbb{N} : x \in \bigcup \mathcal{W}_n\}.$$

Hence X has the $\mathcal{I}$H property. □

4. Topological Properties of Star-C-$\mathcal{I}$-Hurewicz Spaces

In this section, we study the topological properties of star-C-$\mathcal{I}$-Hurewicz spaces.

Recall that a family $\mathcal{A} \subset 2^{\mathbb{N}}$ is said to be almost disjoint (in short *AD*) if every element of $\mathcal{A}$ is infinite and the set $A \cap B$ is finite for all distinct elements $A, B \in \mathcal{A}$. For an *AD* family $\mathcal{A}$, put $\psi(\mathcal{A}) = \mathcal{A} \cup \mathbb{N}$ and topologize $\psi(\mathcal{A})$ as follows: the natural numbers are isolated and for each element $A \in \mathcal{A}$ and each finite set $F \subset \mathbb{N}$, $\{A\} \cup (A \setminus F)$ is a basic open neighborhood of A. The spaces constructed in this manner are called Isbell-Mrówka ψ-spaces [18]. It is well known that $\mathcal{A}$ is maximal almost disjoint (in short *MAD*) family if and only if $\psi(\mathcal{A})$ is pseudocompact.

Let $\mathbb{N}^{\mathbb{N}}$ be the set of all functions from $\mathbb{N}$ to itself. For all $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq^* g$ means $f(n) \leq g(n)$ for all but finitely many n and ($f \leq g$ means $f(n) \leq g(n)$ for all n). A subset B of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ is bounded if there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for each $f \in B$. The unbounding number, denoted by $\mathfrak{b}$, is the smallest cardinality of an unbounded subset of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. The cardinality of a set is denoted by $|A|$. For an ideal $\mathcal{I}$ of $\mathbb{N}$, and ideal version of the unbounding number, denoted by $\mathfrak{b}(\mathcal{I})$ was introduced in [12] where $\mathfrak{b}(\mathcal{I}) = min\{|\mathcal{B}| : \mathcal{B} \subset (\mathbb{N}^{\mathbb{N}}, \leq^*)$ and for all $g \in \mathbb{N}^{\mathbb{N}}$ there is a $f \in \mathcal{B}$ such that $\{n \in \mathbb{N} : g(n) \leq f(n)\} \in \mathcal{I}^+\}$. It is not difficult to show that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{b}(\mathcal{I}) \leq \mathfrak{c}$ and it is known that $\omega_1 < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$, $\mathfrak{b} < \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{b}(\mathcal{I}) = \mathfrak{c}$ are all consistent with the axioms of *ZFC* (see [10, 12, 27]).

Lemma 4.1. ([6, Theorem 4.4]) *The following statements are equivalent under the assumptions $\neg CH$:*

1. $\psi(\mathcal{A})$ *has the SS$\mathcal{I}$H property;*
2. $|\mathcal{A}| < \mathfrak{b}(\mathcal{I})$.

The following lemma follows from [21, Lemma 2.4].

Lemma 4.2. *Let D be an infinite discrete space and $bD = D \cup \{d^*\}$ be the one point Lindelöfication, where $d^* \notin D$. Then $bD \times [0, \omega)$ has the SC$\mathcal{I}$H property.*

The following example is considered from [25, Example 2.2] with necessary modifications.

Example 4.3. Assuming $\omega_1 < \mathfrak{b}(\mathcal{I})$, there exists a Tychonoff space with the SC$\mathcal{I}$H property having regular closed G_δ-subspace which does not have the SC$\mathcal{I}$H property.

Assuming $\omega_1 < \mathfrak{b}(\mathcal{I})$, let

$$S_1 = \psi(\mathcal{A}) = \mathbb{N} \cup \mathcal{A}$$

be the Isbell-Mrówka ψ-space, where $\mathcal{A}$ is the *AD* family of infinite subsets of $\mathbb{N}$ with $|\mathcal{A}| = \omega_1$. Then by Lemma 4.1, S_1 has SC$\mathcal{I}$H property, since a space with the SS$\mathcal{I}$H property has the SC$\mathcal{I}$H property. Now let $D = \{d_\alpha : \alpha < \omega_1\}$ be a discrete space of cardinality ω_1 and let

$$bD = D \cup \{d^*\},$$

where $d^* \notin D$. Define topology on bD as follows: for each $\alpha < \omega_1$, $\{d_\alpha\}$ is isolated and set U containing d^* is open if and only if $bD \setminus U$ is countable. Then bD is Lindelöf and every countably compact subset of bD is finite. Let

$$S_2 = (bD \times [0, \omega]) \setminus \{\langle d^*, \omega\rangle\}$$

be the subspace of the product space $bD \times [0, \omega]$. In Example 2.2 of [25], Song shows that S_2 is not the star-C-Menger space. Hence it follows that S_2 is a Tychonoff space which does not have the SC$\mathcal{I}$H property, since a space having the SC$\mathcal{I}$H property must be star-C-Menger. Let $\pi : \mathcal{A} \to D \times \{\omega\}$ be a bijection and let X be the quotient image of the disjoint sum $S_1 \oplus S_2$ by identifying A of S_1 with $\pi(A)$ of S_2 for every $A \in \mathcal{A}$. Let

$$\phi : S_1 \oplus S_2 \to X$$

be the quotient map. Then $\phi(S_2)$ is a regular closed subspace of X. For each $n \in \mathbb{N}$, let

$$F_n = \{m \in \mathbb{N} : m \leq n\}.$$

For each $n \in \mathbb{N}$, let

$$U_n = \phi(\{\{A\} \cup (A \setminus F_n) : A \in \mathcal{A}\}) \cup \phi(bD \times [0, \omega)).$$

Then U_n is open in X and

$$\phi(S_2) = \bigcap_{n \in \mathbb{N}} U_n.$$

Thus $\phi(S_2)$ is a regular closed G_δ-subspace of X. However $\phi(S_2)$ does not have the SC$\mathcal{I}$H property, since it is homeomorphic to S_2.

Finally we show that X has the SC$\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since $\phi(S_1)$ has the SC$\mathcal{I}$H property, since $\phi(S_1)$ homeomorphic to S_1. Thus there exists a sequence $(K_n : n \in \mathbb{N})$ of countably compact subsets of $\phi(S_1)$ such that for each

$$x_1 \in \phi(S_1), \{n \in \mathbb{N} : x_1 \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}.$$

On the other hand, the set $\phi(bD \times [0,\omega))$ is homeomorphic to $(bD \times [0,\omega)$. Then by Lemma 4.2, $\phi(bD \times [0,\omega))$ has the SC$\mathcal{I}$H property. Thus there exist a sequence $(K'_n : n \in \mathbb{N})$ of countably compact subsets of $\phi(bD \times [0,\omega))$ such that for each

$$x_2 \in \phi(bD \times [0,\omega)), \{n \in \mathbb{N} : x_2 \notin St(K'_n, \mathcal{U}_n)\} \in \mathcal{I}.$$

For each $n \in \mathbb{N}$, let $K_n = K_n \cup K'_n$. Then $(K_n : n \in \mathbb{N})$ is a sequence of countably compact subsets of X such that for each $x \in X$,

$$\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}.$$

Hence X has the SC$\mathcal{I}$H property.

Now we give several results on the SC$\mathcal{I}$H property.

Theorem 4.4. *The SC$\mathcal{I}$H property is preserved under open F_σ-subsets.*

Proof. Let X be a space having the SC$\mathcal{I}$H property and let $Y = \cup\{H_n : n \in \mathbb{N}\}$ be an open F_σ-subset of X, where the H_n is closed in X for each $n \in \mathbb{N}$. Without loss of generality, we can assume that $H_n \subseteq H_{n+1}$ for each $n \in \mathbb{N}$. To show that Y has SC$\mathcal{I}$H property. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. We have to find a sequence $(K_n : n \in \mathbb{N})$ of countably compact subsets of Y such that for each $y \in Y$, $\{n \in \mathbb{N} : y \notin St(K_n, \mathcal{U}_n)\} \in \mathcal{I}$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \mathcal{U}_n \bigcup\{X \setminus H_n\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X and by the SC$\mathcal{I}$H property of X, there exists a sequence $(K'_n : n \in \mathbb{N})$ of countably compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K'_n, \mathcal{V}_n)\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : x \in St(K'_n, \mathcal{V}_n)\} \in \mathcal{F}(\mathcal{I})$. For each $n \in \mathbb{N}$, let $K_n = K'_n \cap Y$. Thus $(K_n : n \in \mathbb{N})$ is a sequence of countably compact subsets of Y. Let $y \in X \cap Y$. If $y \in St(K'_n, \mathcal{V}_n)$ for some n. By the construction of $\mathcal{V}_n$ choose $U \in \mathcal{U}_n$ such that $y \in U$ and $U \cap K'_n \cap Y \neq \phi$, which implies $U \cap K_n \neq \phi$. Therefore $\{n \in \mathbb{N} : x \in St(K'_n, \mathcal{V}_n)\} \subset \{n \in \mathbb{N} : y \in St(K_n, \mathcal{U}_n)\}$. This completes the proof. □

Corollary 4.5. *The SC$\mathcal{I}$H property is preserved under clopen subsets.*

A cozero-set in a space X is a set of form $f^{-1}(\mathbb{R} \setminus \{0\})$ for some real-valued continuous function f on X. Since a cozero-set is an open F_σ-subset, the following corollary follows.

Corollary 4.6. *The SC$\mathcal{I}$H property is preserved under cozero-subsets.*

Theorem 4.7. *The SC$\mathcal{I}$H property is preserved under continuous mappings.*

Proof. Let $f : X \to Y$ be a continuous mapping and X be a space having the SC$\mathcal{I}$H property. Suppose that $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of $f(X) = Y$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X. Apply the SC$\mathcal{I}$H property of X to $(\mathcal{V}_n : n \in \mathbb{N})$ to obtain a sequence $(K_n : n \in \mathbb{N})$ of countably compact subsets of X such that for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{V}_n)\} \in \mathcal{I}$. Consequently $\{n \in \mathbb{N} : x \in St(K_n, \mathcal{V}_n)\} \in \mathcal{F}(\mathcal{I})$. For each $n \in \mathbb{N}$, let $K'_n = f(K_n)$. Then $(K'_n : n \in \mathbb{N})$ is a sequence of countably compact subsets of Y. Let $y \in Y$. Then there is a $x \in X$ such that $f(x) = y$. Let $x \in St(K_n, \mathcal{V}_n)$ for some n. Thus $y = f(x) \in St(f(K_n), \{f(U) : U \in \mathcal{V}_n\}) = St(K'_n, \mathcal{U}_n)$. Hence the result follows from the fact that $\{n \in \mathbb{N} : x \in St(K_n, \mathcal{V}_n)\} \subset \{n \in \mathbb{N} : x \in St(K'_n, \mathcal{U}_n)\}$. □

Consider the Alexandorff duplicate $A(X) = X \times \{0,1\}$ of a space X. The basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \bigcup (U \times \{1\} \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X and each points $\langle x, 1\rangle \in X \times \{1\}$ are isolated points.

Example 4.8. There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is a Tychonoff space having the SC$\mathcal{I}$H property, but $A(X)$ does not have the SC$\mathcal{I}$H property.

Let X be the space considered by Song in [21, Example 2.3]. Then X is a Tychonoff space having the SCH property and hence the SC$\mathcal{I}$H. Now to show that $A(X)$ does not have the SC$\mathcal{I}$H property. Let $A = \{\langle\langle d_\alpha, \mathfrak{c}\rangle, 1\rangle : \alpha < \mathfrak{c}\}$. Then A is an open and closed subset of $A(X)$ with $|A| = \mathfrak{c}$, and each point $\langle\langle d_\alpha, \mathfrak{c}\rangle, 1\rangle$ is isolated. Hence $A(X)$ does not have the SC$\mathcal{I}$H property, since by Corollary 4.5, the SC$\mathcal{I}$H property preserved under clopen subsets, but A does not have the SC$\mathcal{I}$H property. Let $f : A(X) \to X$ be the projection. Then f is a closed 2-to-1 continuous map.

The proof of the following theorem is quite similar to the proof of Theorem 4.9 in [21] and will be omitted.

Theorem 4.9. *The SC$\mathcal{I}$H property is inverse invariant under open perfect mappings.*

Example 4.10. In [24], Song and Yin gave Example 3.4 of spaces in which product of two countably compact spaces is not the star-C-Menger, hence not the SC$\mathcal{I}$H, since the SC$\mathcal{I}$H property implies the star-C-Menger property. They also gave Example 3.5, of spaces in which product of countably compact and Lindelöf spaces is not the star-C-Menger, hence not the SC$\mathcal{I}$H.

By Theorem 4.9, the following corollary follows.

Corollary 4.11. *Let X have the SC$\mathcal{I}$H property and Y be a compact space. Then $X \times Y$ has the SC$\mathcal{I}$H property.*

The proof of the following theorem is an easy modification of the proof of Lemma 2.16 in [21].

Theorem 4.12. *The SC$\mathcal{I}$H property is closed under countable unions.*

The following corollary follows from Corollary 4.11 and Theorem 4.12.

Corollary 4.13. *If X has SC$\mathcal{I}$H property and Y is σ-compact, then $X \times Y$ has the SC$\mathcal{I}$H property.*

Theorem 4.14. *If a space X has a countably compact dense subset, then X has SC$\mathcal{I}$H property.*

Proof. Let D be a countably compact dense subspace of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $n \in \mathbb{N}$, let $K_n = D$. Hence the sequence $(K_n : n \in \mathbb{N})$ of countably compact dense subsets of X. Thus for each $x \in X$, $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U}_n)\} = \phi \in \mathcal{I}$. Therefore X has the SC$\mathcal{I}$H property. □

The following problem remains open. Does the space $A(X)$ of a space X having the SC$\mathcal{I}$H property with $e(X) < \omega_1$ also have the SC$\mathcal{I}$H property?

Theorem 4.15. *If X is a T_1-space and $A(X)$ has the SC$\mathcal{I}$H property, then $e(X) < \omega_1$.*

Proof. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset B of X such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point. By Theorem 4.5, $A(X)$ does not have the SC$\mathcal{I}$H property, since $B \times \{1\}$ does not have the SC$\mathcal{I}$H property. □

The following diagram follows from the definitions and the above results.

$$\begin{tikzcd}
\text{Starcompact} & \mathcal{K}\text{-starcompact} \arrow[l] \arrow[r] \arrow[d] & \text{SK}\mathcal{I}\text{H} \arrow[d] & \text{SKH} \arrow[l] \arrow[d] \\
& \mathcal{C}\text{-starcompact} \arrow[r] & \text{SC}\mathcal{I}\text{H} & \text{SCH} \arrow[l]
\end{tikzcd}$$

A simple modification of Example 2.3 in [23] will demonstrate that the SC$\mathcal{I}$H does not imply the SK$\mathcal{I}$H property. Also Example 2.3 in [24] will demonstrate that the SC$\mathcal{I}$H property does not imply the C-starcompactness. However, the authors could not find a space with property SC$\mathcal{I}$H but not having the SCH property. For the remaining counterexample see [21–24, 29].

5. The Star-C-$\mathcal{I}$-Hurewicz Game

Recall that the *$\mathcal{I}$-Hurewicz game* [7] on a topological space X is defined as follows. Players ONE and TWO play an inning per positive integer n. In the n-th inning ONE chooses an open cover $\mathcal{O}_n$ of X and TWO responds by choosing a finite set $\mathcal{T}_n \subset \mathcal{O}_n$. A play $(\mathcal{O}_1, \mathcal{T}_1, ..., \mathcal{O}_n, \mathcal{T}_n, ...)$ is won by TWO if for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin \cup\mathcal{T}_n\} \in \mathcal{I}$.

The *star-$\mathcal{I}$-Hurewicz game* [8] on a topological space X is as follows. For each positive integer n, the player ONE chooses an open cover $\mathcal{O}_n$ of X and TWO responds by choosing a finite set $\mathcal{T}_n \subset \mathcal{O}_n$. A play $(\mathcal{O}_1, \mathcal{T}_1, ..., \mathcal{O}_n, \mathcal{T}_n, ...)$ is won by TWO if for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin St(\cup\mathcal{T}_n, \mathcal{O}_n)\} \in \mathcal{I}$.

Now we define the *star-C-$\mathcal{I}$-Hurewicz game* (in short SC$\mathcal{I}$H game) on a topological space X as follows. For each positive integer n, the player ONE chooses an open cover $\mathcal{O}_n$ of X and TWO responds by choosing a countably compact subset K_n of X. A play $(\mathcal{O}_1, K_1, ..., \mathcal{O}_n, K_n, ...)$ is won by TWO if for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{O}_n)\} \in \mathcal{I}$.

From the following example, it follows that either the class of the SC$\mathcal{I}$H spaces will contain the class of the S$\mathcal{I}$H spaces or these classes are independent of each other. Thus our results in this section for the SC$\mathcal{I}$H spaces are justified.

Example 5.1. There exists a Tychonoff SC$\mathcal{I}$H space X which is not the S$\mathcal{I}$H.

Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let
$$X = (\beta D \times [0, \mathfrak{c})) \cup (D \times \{\mathfrak{c}\})$$
be the subspace of the product space $\beta D \times [0, \mathfrak{c}]$. By [21, Example 2.3], X is a SCH space, hence SC$\mathcal{I}$H. Now we show that X is not the S$\mathcal{I}$H space. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times (\alpha, \mathfrak{c}]$. Then U_α is open in X and $U_\alpha \cap U'_\alpha = \phi$ for each $\alpha \neq \alpha'$. For each $n \in \mathbb{N}$, let
$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\beta D \times [0, \mathfrak{c})\}.$$
Then $\mathcal{U}_n$ is an open cover of X. It suffices to show that there exists $x \in X$ such that
$$\{n \in \mathbb{N} : x \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)\} = \mathbb{N} \notin \mathcal{I},$$
for any sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. Since for each each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite, thus $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is countable, so there exists $\alpha' < \mathfrak{c}$ such that $U_{\alpha'} \notin \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, that is for each $n \in \mathbb{N}$, $U_{\alpha'} \cap (\cup\mathcal{V}_n) = \emptyset$, since $U_{\alpha'}$ is the only element of $\mathcal{U}_n$ containing the point $(d_{\alpha'}, \mathfrak{c})$. Thus $(d_{\alpha'}, \mathfrak{c}) \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, that is $\{n \in \mathbb{N} : (d_{\alpha'}, \mathfrak{c}) \notin St(\cup\mathcal{V}_n, \mathcal{U}_n)\} = \mathbb{N} \notin \mathcal{I}$.

Lemma 5.2. *If X is a topological space such that TWO has a winning strategy in the SC$\mathcal{I}$H game on X, then each $\mathcal{I}$-large cover of X is star-$\mathcal{I}$-groupable.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an $\mathcal{I}$-large cover of X. Consider the following strategy σ of ONE in the SC$\mathcal{I}$H game on X. In the first inning ONE plays $\sigma(\emptyset) = \mathcal{U}$. Suppose TWO responds a countably compact subset K_1 of X, then there is a finite set $\mathcal{T}_1 \subset \mathcal{U}$. Then ONE plays $\sigma(K_1) = \mathcal{U} \setminus \mathcal{T}_1$. Suppose TWO responds a countably compact subset K_2 of X, then there is a finite set $\mathcal{T}_2 \subset \mathcal{U} \setminus \mathcal{T}_1$. Then ONE plays $\sigma(K_1, K_2) = \mathcal{U} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$ and so on. Note that σ is a winning strategy for TWO in SC$\mathcal{I}$H game on X. Consider a σ play $(\sigma(\emptyset), K_1, \sigma(K_1), K_2, \sigma(K_1, K_2), K_3, ...)$, in which TWO wins. Since the sequence $(\mathcal{T}_n : n \in \mathbb{N})$ are disjoint from each other and for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin St(K_n, \mathcal{U} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2 \cup ... \cup \mathcal{T}_{n-1}))\} \in \mathcal{I}$. Now $\{n \in \mathbb{N} : x \in St(K_n, \mathcal{U} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2 \cup ... \cup \mathcal{T}_{n-1}))\} \subset \{n \in \mathbb{N} : x \in St(K_n, \mathcal{U})\} \subset \{n \in \mathbb{N} : x \in St(\cup\mathcal{T}_n, \mathcal{U})\}$. If any element of $\mathcal{U}$ is not present in the sequence $(\mathcal{T}_n : n \in \mathbb{N})$, then after the construction of the play the remaining sets of $\mathcal{U}$ can be distributed among $\mathcal{T}_n$'s so that $\mathcal{T}_n$'s remain pairwise disjoint finite subsets of $\mathcal{U}$ and $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{T}_n$. Hence $\mathcal{U}$ is star-$\mathcal{I}$-groupable. □

Theorem 5.3. *For a paracompact Hausdorff space X, if TWO has a winning strategy in the SC$\mathcal{I}$H game on X, then TWO has a winning strategy in the $\mathcal{I}$-Hurewicz game on X.*

Proof. First, let ONE choose an open cover $\mathcal{U}_1$ of X in the $\mathcal{I}$-Hurewicz game on X. Since X is paracompact, $\mathcal{U}_1$ has a locally finite open refinement $\mathcal{V}_1$, that is for each $x \in X$, there exists an open neighborhood $V_{n,x}$ of x such that $V_{n,x} \subseteq V$ for some $V \in \mathcal{V}_1$ and $\{V \in \mathcal{V}_1 : V_{n,x} \cap V \neq \emptyset\}$ is finite. Then $\mathcal{V}'_1 = \{V_{n,x} : x \in X\}$ is an open refinement of $\mathcal{V}_1$ and $\mathcal{V}'_1$ has a locally finite open refinement $\mathcal{V}''_1$.

Let in the first inning of SC$\mathcal{I}$H game, ONE chooses open cover $\mathcal{V}_1''$. Suppose TWO responds by choosing a countably compact subset K_1 of X. Since K_1 is countably compact subset of X, then there is a finite subset of F_1 of K_1 such that $K_1 \subseteq St(F_1, \mathcal{V}_1'')$. For each $n \in \mathbb{N}$, let $\mathcal{V}_1''' = \{V \in \mathcal{V}_1'' : V \cap F_1 \neq \phi\}$. Then $\mathcal{V}_1'''$ is a finite subset of $\mathcal{V}_1''$, since F_1 is a finite and $\mathcal{V}_1''$ is locally finite. For each $n \in \mathbb{N}$, let $\mathcal{U}_1' = \{V \in \mathcal{V}_1$: there exists some $V' \in \mathcal{V}_1'''$ such that $V \cap V' \neq \phi\}$. Then $\mathcal{U}_1'$ is finite, since $\mathcal{V}_1$ is locally finite. For each $V \in \mathcal{U}_1'$ choose $U_V \in \mathcal{U}_1$ such that $V \subseteq U_V$. Let $\mathcal{W}_1 = \{U_V \in \mathcal{U}_1 : V \in \mathcal{U}_1'\}$. Suppose TWO responds by choosing $\mathcal{W}_1$ in the $\mathcal{I}$-Hurewicz game.

Next, in the 2nd inning of the $\mathcal{I}$-Hurewicz game, let ONE chooses an open cover $\mathcal{U}_2$ of X. In the 2nd inning of SC$\mathcal{I}$H game, ONE chooses open cover $\mathcal{V}_2''$ (as $\mathcal{V}_1''$ constructed above) and TWO responds by choosing a countably compact subset K_2 of X. Similarly construct $\mathcal{W}_2 = \{U_V \in \mathcal{U}_2 : V \in \mathcal{U}_2'\}$ as above. Suppose TWO responds by choosing $\mathcal{W}_2$ in the $\mathcal{I}$-Hurewicz game.

In this way in the n-th inning, ONE choose an open cover $\mathcal{U}_n$ of X. In the 2nd inning of the SC$\mathcal{I}$H game, ONE chooses open cover $\mathcal{V}_n''$ (as $\mathcal{V}_1''$ constructed above) and TWO responds by choosing a countably compact subset K_n of X. Construct similarly $\mathcal{W}_n = \{U_V \in \mathcal{U}_n : V \in \mathcal{U}_n'\}$ as in 1st inning and the game has been played in this way.

Now for each $n \in \mathbb{N}$, we have $St(k_n, \mathcal{V}_n'') \subseteq \cup\mathcal{U}_n' \subset \cup\mathcal{W}_n$.

As in the SC$\mathcal{I}$H game $(\mathcal{V}_1'', K_1, \mathcal{V}_2'', K_2, .., \mathcal{V}_n'', K_n ...)$ TWO has a winning strategy so for each $x \in X$, the set $\{n \in \mathbb{N} : x \in St(K_n, \mathcal{V}_n'')\} \in \mathcal{F}(\mathcal{I})$. Therefore for each $x \in X$, the set $\{n \in \mathbb{N} : x \in \cup\mathcal{W}_n\} \in \mathcal{F}(\mathcal{I})$ and so TWO has a winning strategy in the $\mathcal{I}$-Hurewicz game on X. $\square$

Our next result gives certain conditions under which $\mathcal{I}$-groupabable open cover coincides with large open covers. For this we need the following theorem from Das, Kočinac and Chandra [7].

Theorem 5.4. ([7]) *If X is a topological space such that TWO has a winning strategy in the $\mathcal{I}$-Hurewicz game on X, then each $\mathcal{I}$-large cover of X is $\mathcal{I}$-groupable.*

Combining Theorem 5.1, Theorem 5.2 and Theorem 5.3, we have the following.

Theorem 5.5. *For a paracompact Hausdorff space X, if TWO has a winning strategy in the SC$\mathcal{I}$H game on X, then each $\mathcal{I}$-large cover is star-$\mathcal{I}$-groupable as well as $\mathcal{I}$-groupable open cover of X.*

Corollary 5.6. *For a paracompact Hausdorff space X, if TWO has a winning strategy in the SC$\mathcal{I}$H game on X and $\mathcal{U}$ is an open cover of X, then following statements are equivalent:*

1. *$\mathcal{U}$ is a large cover;*
2. *$\mathcal{U}$ is a $\mathcal{I}$-large cover;*
3. *$\mathcal{U}$ is a $\mathcal{I}$-groupable.*

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References

[1] M. Bonanzinga, F. Cammaroto, Lj.D.R. Kočinac, Star-Hurewicz and related properties, Appl. Gen. Topol. 5 (2004) 79–89.
[2] D. Chandra, P. Das, Some further investigation of open covers and selection principles using ideals , Topology Proc. 39 (2012) 281–291.
[3] P. Das, Some further results on ideal convergence in topology space, Topology Appl. 159 (2012) 2621–2625.
[4] P. Das, Certain types of open covers and selection principles using ideals, Houston J. Math. 39 (2013) 637–650.
[5] P. Das, D. Chandra, Some further results on $\mathcal{I} - \gamma$ and $\mathcal{I} - \gamma_k$-covers, Topology Appl. 16 (2013) 2401–2410.
[6] P. Das, D. Chandra, U. Samanta, On certain variations $\mathcal{I}$-Hurewicz property, Topology Appl. 241 (2018) 363–376.
[7] P. Das, Lj.D.R. Kočinac, D. Chandra, Some remarks on open covers and selection principles using ideals, Topology Appl. 202 (2016) 183–193.
[8] P. Das, U. Samanta, D. Chandra, On certain generalized versions of groupability, Topology Appl. 258 (2019) 47–57.

[9] G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in topology, Topology Appl. 156 (2008) 28–45.
[10] E.K. van Douwen, G.K. Reed, A.W. Roscoe, I.J. Tree, Star covering properties, Topology Appl. 39 (1991) 71–103.
[11] R. Engelking, General Topology, PWN, Warszawa (1977).
[12] B. Farkas, L. Soukup, More on cardinal invariants of analytic P-ideals, Comment. Math. Univ. Carolin. 50(2) (2009) 281–295.
[13] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, The combinatorics of open covers (II), Topology Appl. 73 (1996) 241–266.
[14] Lj.D.R. Kočinac, Star-Menger and realted spaces, Publ. Math. Debrecen 55 (1999) 421–431.
[15] Lj.D.R. Kočinac, Star selection principles: A survey, Khayyam J. Math. 1 (2015) 82–106.
[16] Lj.D.R. Kočinac, M. Scheepers, Combinatorics of open covers (VII): Groupability, Fund. Math. 179 (2003) 131–155.
[17] M.V. Matveev, A survey on star-covering properties, Topology Atlas, preprint No 330, 1998.
[18] S. Mrówka, On completely regular spaces, Fund. Math. 41 (1954) 105–106.
[19] M. Scheepers, The combinatorics of open covers (I): Ramsey theory, Topol. Appl. 69 (1996) 31–62.
[20] S.G. Silva, The $\mathcal{I}$-Hurewicz property and bounded families modulo an ideal, Quest. Ans. Gen. Topol. 36 (2018) 31–38.
[21] Y.-K. Song, On star-C-Hurewicz spaces, Stud. Sci. Math. Hung. 54 (2017) 411–425.
[22] Y.-K. Song, On $\mathcal{K}$-starcompact spaces, Bull. Malays. Math. Sci. Soc. 30 (2007) 59–64.
[23] Y.-K. Song, On $\mathcal{C}$-starcompact spaces, Math. Bohemica 133 (2008) 259–266.
[24] Y.-K. Song, Z. Yin, On star-C-Menger spaces, Quaest. Math. 37 (2014) 337–347.
[25] Y.-K. Song, Remarks on star-C-Menger spaces, Quaest. Math. 39 (2016) 479–486.
[26] Y.-K. Song, On star-K-Hurewicz spaces, Filomat 31 (2017) 1129–1285.
[27] M. Staniszewski, On ideal equal convergence II, J. Math. Anal. Appl. 451 (2017) 1179–1197.
[28] B.K. Tyagi, S. Singh, M. Bhardwaj, Ideal analogues of some variants of Hurewicz property, Filomat 33:9 (2019), 2725–2735.
[29] S. Singh, B.K. Tyagi, M. Bhardwaj, On star-K-$\mathcal{I}$-Hurewicz property, communicated.
[30] R.C. Walker, The Stone-Čech Compactification, Springer, Berlin, 1974.