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Some Characterizations of Partial Isometry Elements in Rings with Involutions

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Abstract. We give some sufficient and necessary conditions for an element in a ring with involution to be a partial isometry by using certain equations admitting solutions in a definite set.

1. Introduction

Let R be an associative ring with 1. An *involution* $a \mapsto a^*$ in R is an anti-isomorphism of degree 2 (see., [13]), that is,

$$(a^*)^* = a, \ (a+b)^* = a^* + b^*, \ (ab)^* = b^*a^*.$$

In this case R is called a $*-$*ring*.

An element $a \in R$ is said to be *Moore−Penrose invertible* (or *MP−invertible*) [14] if there exists some $b \in R$ such that the following Penrose equations hold:

$$(1)\ aba = a, \ (2)\ bab = b, \ (3)\ ab = (ab)^*, \ (4)\ ba = (ba)^*.$$

There is at most one b such that the above conditions hold (see., [3, 4, 7]). We call it the *Moore−Penrose inverse* (or *MP−inverse*) of a and denote it by $a^\dagger$. The set of all MP−invertible elements of R is denoted by $R^\dagger$.

An element $a \in R$ is said to be *group invertible* [13] if there is some $b \in R$ satisfying the following conditions:

$$aba = a, \ bab = b, \ ab = ba.$$

There is at most one b such that the above conditions hold. We call it the *group inverse* of a and denote it by $a^\#$. The set of all group invertible elements of R is denoted by $R^\#$.

An element $a \in R^\# \cap R^\dagger$ satisfying $a^\# = a^\dagger$ is said to be EP [5]. We denote the set of all EP elements of R by R^{EP}.

An element $a \in R^\dagger$ is called a *partial isometry* [11] if $a^* = a^\dagger$. We denote by R^{PI} the set of all the partial isometries of R. Partial isometries has been explored by many authors. In [1], using the representation of

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complex matrices provided in [6], O.M. Baksalary et al. investigated various classes of matrices, such as partial isometries, EP and star-dagger elements. In [9, 11], D. Mosić and D.S. Djordjević studied partial isometries by a purely algebraic technique, extending some already known results for complex matrices into the setting of the rings with involution. In addition, they presented a conjecture in [9] about an equivalent condition for a partial isometry a with $a \in R^{\dagger}$, which was negated by W. Chen [2] through a counter-example.

Motivated by these results, this paper is intended to provide, by using certain equations admitting solutions in a definite set, further equivalent conditions for an element in a ring with involution to be a partial isometry. Since there are close connections between partial isometries, EP elements and normal elements in rings with involution [9, 11], we present also several characterizations of the latter two kinds of elements.

2. Results

We give at first the following lemma, which follows by [9].

Lemma 2.1. *Let $a \in R^{\#} \cap R^{\dagger}$. If $a = a^2a^*$, then $a \in R^{PI}$.*

Remark 2.2. *The converse of Lemma 2.1 is not true. For instance, put $R = M_3(\mathbb{Z}_3)$ and, for any $A \in R$, define $A^* = A^T$, where A^T is the transpose of A. Thus R is a $*$-ring. Pick $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to check that $B \in R^{\#} \cap R^{\dagger}$, $B = B^{\#} = B^2$, and $B^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = B^*$. Therefore $B \in R^{PI}$, but $B^2B^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq B$.*

Let $a \in R^{PI}$. Then $aa^*a = a$ for $a^{\dagger} = a^*$ and consequently we can construct an equation as follows.

$$x = aa^*x. \tag{1}$$

Let $a \in R^{\#} \cap R^{\dagger}$ and write $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$. Then we have the following theorem.

Theorem 2.3. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{PI}$ if and only if equation (1) has at least one solution in χ_a.*

Proof. $\Rightarrow$ It is evident that $x = a$ is a solution of equation (1) in χ_a.

$\Leftarrow$ (1) If $x = a^{\#}$ is a solution of equation (1), then $a^{\#} = aa^*a^{\#}$, and so that $a \in R^{PI}$ in terms of [11, Theorem 2.1 (V)].

(2) If $x = a$ is a solution of equation (1), then $a = aa^*a$, which implies that $a \in R^{PI}$.

(3) If $x = a^{\dagger}$ is a solution of equation (1), then $a^{\dagger} = aa^*a^{\dagger}$, which gives $a \in R^{PI}$ by [9].

(4) If $x = a^*$ is a solution of equation (1), then $a^* = aa^*a^*$. Applying the involution, we arrive at the result that $a = a^2a^*$. It is known by Lemma 2.1 that $a \in R^{PI}$.

(5) If $x = (a^{\#})^*$ is a solution of equation (1), then $(a^{\#})^* = aa^*(a^{\#})^*$. Using the involution, we obtain $a^{\#} = a^{\#}aa^*$, which yields $a = a^2a^{\#} = a^2a^*$. By Lemma 2.1, $a \in R^{PI}$.

(6) If $x = (a^{\dagger})^*$ is a solution of equation (1), then $(a^{\dagger})^* = aa^*(a^{\dagger})^* = aa^{\dagger}a = a$, from which the result $a \in R^{PI}$ follows. □

By the proof of Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are equivalent:*

(1) $a \in R^{PI}$;

(2) $a^{\dagger}a^{\dagger} = a^*a^{\dagger}$;

(3) $a^{\dagger}a^{\dagger} = a^{\dagger}a^*$;

(4) $a^{\#}(a^{\dagger})^* = a^{\#}a$;

(5) $(a^{\dagger})^*a^{\#} = aa^{\#}$.

Remark 2.2 illustrates that if $a \in R^{PI}$, we can not deduce that the equation (1) has solutions in $\{a^\dagger, a^*, (a^\#)^*\}$. Equation (1) yields by symmetricity the following equation.

$$x = xa^*a. \tag{2}$$

Theorem 2.5. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if equation (2) has at least one solution in χ_a.*

It is immediate that $a \in R^{PI}$ if and only if $a^* \in R^{PI}$, and it is not difficult to check that $\chi_a = \chi_{a^*}$. Applying the involution, we get the following equation.

$$x = xaa^*. \tag{3}$$

Theorem 2.6. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if equation (3) has at least one solution in χ_a.*

Let $a \in R^\# \cap R^\dagger$. We call a a strongly partial isometry element of R if $a^\# = a^* = a^\dagger$. The set of all strongly partial isometry elements of R is denoted by R^{SEP}. Certainly, $R^{SEP} = R^{EP} \cap R^{PI}$.

The following result follows by [9].

Lemma 2.7. *Let $a \in R^\# \cap R^\dagger$. If $a^\dagger = aa^\dagger a^*$, then $a \in R^{EP}$.*

Change equation (1) into the following one.

$$x = axa^*. \tag{4}$$

Theorem 2.8. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if equation (4) has at least one solution in χ_a.*

Proof. $\Rightarrow$ By $a \in R^{SEP}$, we conclude $a^2a^* = a^2a^\# = a$, which shows that $x = a$ is a solution of equation (4).

$\Leftarrow$ (1) If $x = a$ is a solution of equation (4), then $a = a^2a^*$, giving $a \in R^{SEP}$ by [9].

(2) If $x = a^\#$ is a solution of equation (4), then $a^\# = aa^\#a^*$. Multiplying this equality on the left by a^2, we arrive at the result that $a = a^2a^*$. According to (1), we have that $a \in R^{SEP}$.

(3) If $x = a^\dagger$ is a solution of equation (4), then $a^\dagger = aa^\dagger a^*$. It follows from Lemma 2.7 and Corollary 2.4 that $a \in R^{SEP}$.

(4) If $x = a^*$ is a solution of equation (4), then $a^* = aa^*a^*$. Applying the involution, we must get $a = a^2a^*$, yielding $a \in R^{SEP}$ by (1).

(5) If $x = (a^\#)^*$ is a solution of equation (4), then $(a^\#)^* = a(a^\#)^*a^*$. Using the involution, we obtain $a^\# = aa^\#a^*$. By (2), we know $a \in R^{SEP}$.

(6) If $x = (a^\dagger)^*$ is a solution of equation (4), then $(a^\dagger)^* = a(a^\dagger)^*a^*$. Applying the involution, we infer that $a^\dagger = aa^\dagger a^*$, forcing from (3) that $a \in R^{SEP}$. $\square$

Replacing a^* in equation (3) by $a^\dagger$, we get the following equation.

$$x = xaa^\dagger, \tag{5}$$

which together with equation (4) yields the following equation.

$$xaa^\dagger = axa^*. \tag{6}$$

Theorem 2.9. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if equation (6) has at least one solution in χ_a.*

Proof. $\Rightarrow$ Obviously $x = a$ is a solution of equation (6).

$\Leftarrow$ (1) If $x = a$ is a solution of equation (6), then $a^2a^\dagger = a^2a^*$. It follows that $a \in R^{PI}$ by [11, Theorem 2.1 (i)].

(2) If $x = a^\#$ is a solution of equation (6), then $a^\#aa^\dagger = aa^\#a^*$. Multiplying this equality on the left by a^2, we deduce $a^2a^\dagger = a^2a^*$. According to (1), we see that $a \in R^{PI}$.

(3) If $x = a^\dagger$ is a solution of equation (6), then $a^\dagger = a^\dagger aa^\dagger = aa^\dagger a^*$, meaning $a^\dagger a^\dagger = a^\dagger a^*$. It follows from Corollary 2.4 that $a \in R^{PI}$.

(4) If $x = a^*$ is a solution of equation (6), then $a^* = a^*aa^\dagger = aa^*a^*$. Using the involution, we conclude then that $a = a^2a^*$, yielding $a \in R^{PI}$ by Lemma 2.1.

(5) If $x = (a^\#)^*$ is a solution of equation (6), then $(a^\#)^*aa^\dagger = a(a^\#)^*a^*$. Using the involution, we arrive at the result that $a^\# = aa^\dagger a^\# = aa^\#a^*$. Thus

$$a^\#(a^\dagger)^* = aa^\#a^*(a^\dagger)^* = a^\#aa^\dagger a = a^\#a,$$

which implies from Corollary 2.4 that $a \in R^{PI}$.

(6) If $x = (a^\dagger)^*$ is a solution of equation (6), then $(a^\dagger)^*aa^\dagger = a(a^\dagger)^*a^*$. Using the involution, we obtain that $aa^\dagger a^\dagger = aa^\dagger a^*$, and furthermore

$$a^\dagger a^\dagger = a^\dagger(aa^\dagger a^\dagger) = a^\dagger(aa^\dagger a^*) = a^\dagger a^*.$$

So it is the case that $a \in R^{PI}$. □

Applying the involution on equation (6), we obtain the following equation.

$$aa^\dagger x = axa^*, \tag{7}$$

which gives the following theorem.

Theorem 2.10. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if equation (7) has at least one solution in χ_a.*

Combining equations (6) and (7), we get the following equation.

$$aa^\dagger x = xaa^\dagger. \tag{8}$$

Theorem 2.11. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if equation (8) has at least one solution in χ_a.*

Proof. $\Rightarrow$ Since $a \in R^{EP}$, we have $a^2a^\dagger = a^2a^\# = a = aa^\dagger a$. Therefore $x = a$ is a solution of equation (8).

$\Leftarrow$ (1) If $x = a$ is a solution of equation (8), then $a = aa^\dagger a = a^2a^\dagger$, which implies $a \in R^{EP}$ by [12].

(2) If $x = a^\#$ is a solution of equation (8), then $aa^\dagger a^\# = a^\#aa^\dagger$. That is, $a^\# = a^\#aa^\dagger$, stating that $a \in R^{EP}$.

(3) If $x = a^\dagger$ is a solution of equation (8), then $aa^\dagger a^\dagger = a^\dagger aa^\dagger = a^\dagger$. So, $a^\dagger a = aa^\dagger a^\dagger a$ and $a^\dagger a = (a^\dagger a)^* = a^\dagger a^2a^\dagger$. Accordingly $a = aa^\dagger a = a(a^\dagger a^2a^\dagger) = a^2a^\dagger$, which indicates $a \in R^{EP}$ by (1).

(4) If $x = a^*$ is a solution of equation (8), then $aa^\dagger a^* = a^*aa^\dagger = a^*$. It may be concluded that $a = a^2a^\dagger$, proving $a \in R^{EP}$ by (1).

(5) If $x = (a^\#)^*$ is a solution of equation (8), then $aa^\dagger(a^\#)^* = (a^\#)^*aa^\dagger$. Applying the involution, we get $a^\#aa^\dagger = aa^\dagger a^\# = a^\#$. Hence $a \in R^{EP}$ according to (2).

(6) If $x = (a^\dagger)^*$ is a solution of equation (8), then $aa^\dagger(a^\dagger)^* = (a^\dagger)^*aa^\dagger$. Using the involution, we infer that $a^\dagger = a^\dagger aa^\dagger = aa^\dagger a^\dagger$, so that $a \in R^{EP}$ by (3). □

Corollary 2.12. *Let $a \in R^\# \cap R^\dagger$. Then the following conditions are equivalent:*

(1) $a \in R^{EP}$;
(2) $a = a^2a^\dagger$;
(3) $a = a^\dagger a^2$;
(4) $a^\dagger = aa^\dagger a^\dagger$;
(5) $a^\dagger = a^\dagger a^\dagger a$;
(6) $a^\# = a^\#aa^\dagger$;
(7) $a^\# = a^\dagger aa^\#$.

We remark that the (2) and (3) of the above corollary appeared in [12], and the (6) of that appeared in [13].

Replacing $a^\dagger$ in equation (8) by a^*, we obtain an equation as follows.

$$aa^*x = xaa^*, \tag{9}$$

Theorem 2.13. *Let $a \in R^\# \cap R^\dagger$. Then a is a normal element if and only if equation (9) has at least one solution in χ_a.*

Proof. $\Rightarrow$ Let a be a normal element. Then $aa^* = a^*a$ and evidently $x = a$ is a solution of equation (9).

$\Leftarrow$ (1) If $x = a$ is a solution of equation (9), then $aa^*a = a^2a^*$, which implies by [10] that a is normal.

(2) and (3) follow also from [10].

(4) If $x = a^*$ is a solution of equation (9), then $aa^*a^* = a^*aa^*$. Applying involution on it, the rest follows by (1).

(5) If $x = (a^\#)^*$ is a solution of equation (9), then $aa^*(a^\#)^* = (a^\#)^*aa^*$. Multiplying this equality on the right by $(a^\dagger)^*$, we deduce $a(a^\dagger a^\# a)^* = (a^\#)^*a$. Using the involution, we conclude $a^\dagger a^\# aa^* = a^*a^\#$. Multiplying this equality on the left by a, we arrive at $aa^\#a^* = aa^*a^\#$. By (2), we know that a is normal.

(6) If $x = (a^\dagger)^*$ is a solution of equation (9), then $aa^*(a^\dagger)^* = (a^\dagger)^*aa^*$. Applying the involution, we have that $a^\dagger aa^* = aa^*a^\dagger$, i.e., $a^* = aa^*a^\dagger$, which gives that a is normal by (3). □

References

[1] O. M. Baksalary, G. P. H. Styan, G. Trenkler, On a matrix decomposition of Hartwig and Spindelböck, Linear Algebra Appl. 430 (10) (2009) 2798-2812.

[2] W. Chen, On EP elements, normal elements and partial isometries in rings with involution, Electron. J. Linear Algebra 23 (1) (2012) 553-561.

[3] J.J. Koliha, D. Djordjević, D. Cvetković, Moore-Penrose inverse in rings with involution, Linear Algebra Appl. 426 (2-3) (2007) 371-381.

[4] R. E. Harte, M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math. 103 (1992) 71-77.

[5] R. E. Hartwig, Block generalized inverses, Arch. Rational Mech. Anal. 61 (3) (1976) 197-251.

[6] R. E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ commute, Linear Multilinear Algebra 14 (1984) 241-256.

[7] J. J. Koliha, The Drazin and Moore-Penrose inverse in C^*-algebras, Math. Proc. Royal Irish Acad. 99A (1999) 17-27.

[8] J. J. Koliha, P. Patrício, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc. 72 (1) (2002) 137-152.

[9] D. Mosić, D. S. Djordjević, Partial isometries and EP elements in rings with involution, Electron. J. Linear Algebra 18 (2009) 761-772.

[10] D. Mosić, D. S. Djordjević, Moore-Penrose-invertible normal and Hermitian elements in rings, Linear Algebra Appl. 431 (5-7) (2009) 732-745.

[11] D. Mosić, D. S. Djordjević, Further results on partial isometries and EP elements in rings with involution, Math. Comput. Modelling 54 (1) (2011) 460-465.

[12] D. Mosić, D.S. Djordjević, New characterizations of EP, generalized normal and generalized Hermitian elements in rings, Appl. Math. Comput. 218 (12) (2012) 6702-6710.

[13] D. Mosić, D. S. Djordjević, J. J. Koliha, EP elements in rings. Linear Algebra Appl. 431 (2009) 527-535.

[14] R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc. 51 (1955) 406-413.