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Injective Edge Coloring of Graphs

Domingos M. Cardoso[a], J. Orestes Cerdeira[b], Charles Dominic[c], J. Pedro Cruz[a]

[a]*Center for Research and Development in Mathematics and Applications, Department of Mathematics, Universidade de Aveiro, 3810-193, Aveiro, Portugal*
[b]*Department of Mathematics and Center of Mathematics and Applications (CMA), Faculty of Sciences and Technology, New University of Lisbon, Quinta da Torre, 2829-516 Caparica, Portugal*
[c]*Department of Mathematics, CHRIST(Deemed to be University), Begaluru-560029, Karnataka, India*

Abstract. Three edges e_1, e_2 and e_3 in a graph G are consecutive if they form a path (in this order) or a cycle of lengths three. An injective edge coloring of a graph $G = (V, E)$ is a coloring c of the edges of G such that if e_1, e_2 and e_3 are consecutive edges in G, then $c(e_1) \neq c(e_3)$. The injective edge coloring number $\chi'_i(G)$ is the minimum number of colors permitted in such a coloring. In this paper, exact values of $\chi'_i(G)$ for several classes of graphs are obtained, upper and lower bounds for $\chi'_i(G)$ are introduced and it is proven that checking whether $\chi'_i(G) = k$ is NP-complete.

1. Introduction

Throughout this paper we deal with simple graphs G of *order* $n \geq 2$ (the number of vertices) and *size* $m \geq 1$ (the number of edges). The vertex set and edge set will be denoted by $V(G)$ and $E(G)$, respectively. A *proper vertex (edge) coloring* of a graph G is an assignment of colors to the vertices (edges) of G, that is, $c : V(G)(E(G)) \rightarrow \mathcal{C}$, where $\mathcal{C}$ is a set of colors, such that no two adjacent vertices (edges) have the same color, that is $c(x) \neq c(y)$ for every edge xy of G ($c(e) \neq c(e')$ for every pair of edges e, e' incident on the same vertex). The *(edge) chromatic number* ($\chi'(G)$) $\chi(G)$ of G is the minimum number of colors permitted in a such coloring.

Some variants of vertex and edge coloring have been considered.

An *injective vertex coloring* of G is a coloring of the vertices of G so that any two vertices with a common neighbor receive distinct colors. The *injective chromatic number* $\chi_i(G)$ of a graph G is the smallest number of colors in an injective coloring of G. Injective coloring of graphs was introduced by Hahn et. al in [6] and was originated from Complexity Theory on Random Access Machines, and can be applied in the theory of error correcting codes [6]. In [6] it was proved that, for $k \geq 3$, it is NP-complete to decide whether the

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Email addresses: dcardoso@ua.pt (Domingos M. Cardoso), jo.cerdeira@fct.unl.pt (J. Orestes Cerdeira), pedrocruz@ua.pt (J. Pedro Cruz)

injective chromatic number of a graph is at most k. Note that an injective coloring is not necessarily a proper coloring, and vice versa (see [2, 6]).

The following variant of edge coloring was proposed in [3]. In a graph G, three edges e_1, e_2 and e_3 (in this fixed order) are called *consecutive* if $e_1 = xy$, $e_2 = yz$ and $e_3 = zu$ for some vertices x, y, z, u (where $x = u$ is allowed). In other words, three edges are consecutive if they form a path or a cycle of lengths 3. A 3-consecutive edge coloring is a coloring of the edges such that for each three consecutive edges, e_1, e_2 and e_3, the color of e_2 is one of the colors of e_1 or e_3. The 3-*consecutive edge coloring number* of a graph G, $\psi'_{3c}(G)$, is the maximum number of colors of a 3-consecutive edge coloring of G. This concept was introduced and studied in some details in [3], where it is proven that the determination of the 3-consecutive edge coloring number for arbitrary graphs is NP-hard.

An *injective edge coloring* (i-edge coloring for short) of a graph G is a coloring $c : E(G) \to C$, such that if e_1, e_2 and e_3 are consecutive edges in G, then $c(e_1) \neq c(e_3)$. The *injective edge coloring number* or *injective edge chromatic index* of graph G, $\chi'_i(G)$, is the minimum number of colors permitted in an i-edge coloring. We say that graph G is *k edge i-colorable* if $\chi'_i(G) \leq k$. Note that an i-edge coloring is not necessarily a proper edge coloring, and vice versa. It is straightforward to see that for the edge chromatic number of G and the vertex chromatic number of its line graph $L(G)$, the equality $\chi'(G) = \chi(L(G))$ holds. However, it is not always true that $\chi'_i(G) = \chi_i(L(G))$. For instance, $\chi'_i(K_{1,n}) = 1$ and $\chi_i(L(K_{1,n})) = n$.

A motivation for the i-edge coloring is the following. We can model a Packet Radio Network (PRN) as an undirected graph $G = (V, E)$, where the vertices represent the set of stations and two vertices are joined by an edge if and only if the corresponding stations can hear each other transmissions, i.e, the set of edges E represents the common channel property between the pairs of stations (see [11, 12]). Assigning channels or frequencies to the edges of G we may define the secondary interference as the one obtained when two stations x and y that hear each other share the same frequency with one neighbor $x' \neq y$ of x and one neighbor $y' \neq x$ of y. An assignment of channels or frequencies to the edges between stations to avoid secondary interference corresponds to the i-edge coloring of the graph (where each color is a frequency or channel).

i-edge coloring is closely related with the concept of star arboricity recently introduced by Axenovich *et al.* [1]. The star arboricity of a graph G (isa(G)) is the smallest number of induced star-forests covering the edges of G. Ferdjallah *et al.* [5] prove that $\chi'_i(G) = \text{isa}(G)$.

In this paper we obtain exact values of $\chi'_i(G)$ for several classes of graphs, give upper and lower bounds for $\chi'_i(G)$, and we prove that checking whether $\chi'_i(G) = k$ is NP-complete.

For basic graph terminology we refer the reader to [7].

2. Exact values of $\chi'_i(G)$ for some classes of graphs

We start this section with a few basic results which are direct consequences of the definition of injective edge coloring number. As usually, the path, the cycle and the wheel with n vertices will be denoted by P_n, C_n and W_n, respectively. (The wheel W_n with n vertices is obtained by connecting a single vertex to all vertices of C_{n-1}. The wheel W_n is often a C_n with a universal vertex added.) The complete graph of order n is denoted by K_n and the complete bipartite graph with bipartite classes with p and q vertices is denoted by $K_{p,q}$. When $p = 1$, the complete bipartite graph $K_{1,q}$ is called the star of order $q + 1$. The star $K_{1,q}$ is often referred to as q-star. (In particular, P_2 is the star $K_{1,1}$ and P_3 is the star $K_{1,2}$.)

Considering the above notations and denoting the Petersen graph by $\mathcal{P}$, the following values for the injective edge coloring number can be easily derived.

Proposition 2.1.

1. $\chi'_i(P_n) = 2$, *for* $n \geq 4$.
2. $\chi'_i(C_n) = \begin{cases} 2, & \text{if } n \equiv 0 \ (mod\ 4), \\ 3, & \text{otherwise.} \end{cases}$
3. $\chi'_i(K_{p,q}) = \min\{p, q\}$.

4. $\chi'_i(\mathcal{P}) = 5$. *(A feasible 5 i-edge coloring of the Petersen graph is shown in Figure 1. Note that no pair of the edges labeled 1 to 5 can receive the same color.)*

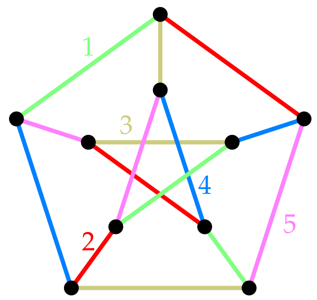


Figure 1: An injective edge coloring of Petersen graph with five colors.

Proposition 2.2. *Let G be a graph. Then $\chi'_i(G) = k$ if and only if k is the minimum positive integer for which the edge set of G, $E(G)$, can be partitioned into non-empty subsets $E_1, \ldots, E_k$, such that the end-vertices of the edges of each of these subsets E_j induces a subgraph G_j of G where each component is a star.*

*P*roof. Let us assume that $\chi'_i(G) = k$ and consider an injective edge coloring of the edges of G using k colors, $c_1, \ldots, c_k$. Then $E(G)$ can be partitioned into the subset of edges $E_1, \ldots, E_k$, where every edge in E_j has the color c_j, for $j = 1, \ldots, k$. Then, for each $j \in \{1, \ldots, k\}$ the end vertices of the edges of E_j must induce a graph without three consecutive edges (otherwise the color can not be the same for all the edges in E_j). Therefore, each component of the graph G_j induced by the end vertices of the edges in E_j are stars. Furthermore, the positive integer k is minimum (otherwise if there exists such partition of $E(G)$ into $k' < k$ subsets of edges $E'_1, \ldots, E'_k$ then $\chi'_i(G) \leq k' < k$).
Conversely, let us assume that k is the minimum positive integer for which $E(G)$ can be partitioned as described. Then, taking into account the first part of this proof, it is immediate that $\chi'_i(G) = k$. □

Applying Proposition 2.2, we may conclude that the injective edge coloring number of a wheel W_n, with $n \geq 4$ vertices, is:

$$\chi'_i(W_n) = \begin{cases} 6 & if \;\; n \;\; is \;\; even \\ 4 & if \;\; n \;\; is \;\; odd \;\; and \;\; n-1 \equiv 0 \pmod 4 \\ 5 & if \;\; n \;\; is \;\; odd \;\; and \;\; n-1 \not\equiv 0 \pmod 4 \end{cases}$$

From Proposition 2.2, it follows that whenever $\chi'_i(G) = k$, the adjacency matrix A_G of graph G can be given by

$$A_G = \sum_{j=1}^{k} A_{G_j}, \tag{1}$$

where each G_j is an induced subgraph of G, with at least one edge, and its components are stars or isolated vertices. Therefore, $\chi'_i(G_j) = 1$, for $j = 1, \ldots, k$, and $\chi'_i(G)$ is the minimum number of induced subgraphs G_j satisfying the conditions of Proposition 2.2.

Now, let us characterize the extremal graphs with largest and smallest injective chromatic index.

Proposition 2.3. *For any graph G of order $n \geq 2$, $\chi'_i(G) = 1$ if and only if G is the disjoint union of $k \geq 1$ stars, i.e., $G = \cup_{j=1}^{k} K_{1,l_j}$, with $\sum_{j=1}^{k} l_j = n - k$ and $V(K_{1,l_j}) \cap V(K_{1,l_{j'}}) = \emptyset$, for $j \neq j'$.*

*P*roof. The proof is a direct consequence of Proposition 2.2. □

A trivial upper bound on the injective edge chromatic number of a graph G is its size, that is, $\chi_i'(G) \leq |E(G)|$. The Proposition 2.4 characterizes the graphs for which this upper bound is attained.

Proposition 2.4. *Consider a graph G of order n and size m, with no isolated vertices. Then $\chi_i'(G) = m$ if and only if G is complete.*

*P*roof. Assume that G is the complete graph K_n, and consider two arbitrary edges e_i and e_j of K_n. Then either e_i is adjacent to e_j and thus they are both included in a triangle or there exists an edge e_k such that e_i, e_k and e_j are three consecutive edges. In any of these cases e_i and e_j must have different colors. Therefore, we have $\chi_i'(G) = n(n-1)/2 = m$.

Conversely, let us assume that $\chi_i'(G) = m$. Clearly G has to be connected, since otherwise the same color could be used on edges from different components. If G has size one then it is complete. Let us suppose that the size of G is greater than one and G is not complete. Since G is connected, there are two adjacent edges in G not lying in the same triangle. Coloring these two edges by the color c_1 and all the remaining edges differently, we produce an injective edge coloring with less than m colors, which is a contradiction. Therefore G is complete. □

3. ω' edge injective colorable graphs

The clique number of a graph G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. Denoting the number of edges in a maximum clique of G by $\omega'(G)$, it is immediate that $\omega'(G) = \frac{\omega(G)(\omega(G)-1)}{2}$.

Proposition 3.1. *For any connected graph G of order $n \geq 2$, $\chi_i'(G) \geq \omega'(G)$.*

*P*roof. Let K_r be a maximum clique in G. From Proposition 2.4, $\chi_i'(K_r) = r(r-1)/2 = \omega'(G)$. Therefore, we need at least $r(r-1)/2$ colors to color the edges of G, i.e., $\chi_i'(G) \geq r(r-1)/2 = \omega'(G)$. □

Before to proceed let us recall the Turan's theorem.

Theorem 3.2 (Theorem of Turan [14]). *Let G be a graph of order n and size m, without a q-clique, with $q > 1$. Then,*

$$m \leq \frac{(q-2)n^2}{2(q-1)}. \tag{2}$$

As a consequence we have the following result.

Corollary 3.3. *Let G be a graph of order n and size m, and consider a positive integer $q > 1$. Then*

$$m > \frac{(q-2)n^2}{2(q-1)} \Rightarrow \chi_i'(G) \geq \frac{q(q-1)}{2}.$$

*P*roof. From Theorem 3.2, $m > \frac{(q-2)n^2}{2(q-1)}$ implies that G as a complete subgraph K_q, that is, $\omega(G) \geq q$. Therefore, $\omega'(G) \geq \frac{q(q-1)}{2}$ and from Proposition 3.1 we obtain $\chi_i'(G) \geq \frac{q(q-1)}{2}$. □

We say that G is an ω' *edge injective colorable* (ω'EIC-)graph if $\chi_i'(G) = \omega'(G)$.

Example 3.4. *The following graphs are examples of ω'EIC-graphs.*

1. *The complete graph, K_n.*
2. *The star, $K_{1,q}$.*

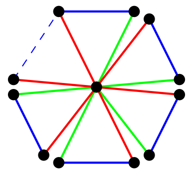


Figure 2: The friendship graph.

3. *The friendship graph, i.e., the graph with $n = 2p + 1$ vertices formed by $p \geq 1$ triangles all attached to a common vertex (see Figure 2).*

Proposition 3.5. *For any positive integer $p \geq 3$, consider the complete graph K_p, with $V(K_p) = \{v_1, \dots, v_p\}$, and a family of stars $K_{1,q_1}, \dots, K_{1,q_p}$, with $q_j \geq 1$. Let G be the graph obtained coalescing a maximum degree vertex of the star K_{1,q_j} with the vertex v_j of K_p, for $j = 1, \dots, p$. Then G is an ω'EIC-graph.*

*P*roof. Consider the Hamiltonian cycle of K_p, $C_p = v_1, v_2 \dots v_p, v_1$. Color each edge $e_i = v_i v_{i+1}$, for $i = 1, \dots, p-1$ of C_p with color c_i, $1 \leq i \leq p-1$ and the edge $e_p = v_p v_1$ with color c_p. For each $j \in \{1, \dots, p\}$, color all the edges of the star K_{1,q_j} with color c_j. Now color all the remaining edges of K_p differently. Since this coloring produces an injective edge coloring, we have

$$\chi_i'(G) \leq p(p-1)/2 = \omega'(K_p) = \omega'(G). \tag{3}$$

The result now follows from Proposition 3.1 □

Notice that the *corona* $K_p \circ K_1$, that is, the graph obtained from K_p by adding a pendant edge to each of its vertices, is a particular case of the graphs G considered in Proposition 3.5, which is obtained setting $K_{1,q_j} = K_{1,1}$ for $j = 1, \dots, p$.

In [15] a construction of ω'EIC-graphs (therein called perfect ω'EIC-graphs) is obtained from a friendship graph F_n, with n triangles $T_1, \dots, T_n$, replacing each triangle T_i by an arbitrary complete graph K_{q_i} (in fact, in [15, Lem. 2.2] by mistake it is written K_i instead of K_{q_i}).

Proposition 3.6. *If G is a unicyclic graph with K_3, then G is an ω'EIC-graph.*

*P*roof. Let the vertices of the cycle K_3 be v_1, v_2, v_3, and the edges $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_1$. Color the edge e_i with color c_i, for $i = 1, 2, 3$. Let T_1, T_2 and T_3 be the trees which are incident to v_1, v_2 and v_3, respectively, and color the edges of these trees as follows.

- Color the edges in T_1 which are incident to v_1 with color c_1, and call C_1^1 the set of these edges. Consider the edges in T_1 which are adjacent to C_1^1 edges, and color all these edges with color c_2. Call C_2^1 the set of these edges. Now consider the edges in T_1 which are adjacent to edges of $C_2^1 \setminus C_1^1$, color these edges with color c_3, and call the set of these edges C_3^1. Again, consider the edges in T_1 which are adjacent to edges of $C_3^1 \setminus C_2^1$, color these edges with color c_1, and call the set of these edges C_1^2 edges. Continue this procedure until all edges in T_1 have been colored.

- Color the edges in T_2 which are incident to v_2 with color c_2, and call the set of these edges C_2^1. Consider the edges in T_2 which are adjacent to C_2^1 edges, color all these edges with color c_3, and call C_3^1 the set of these edges. Now consider the edges in T_2 which are adjacent to $C_3^1 \setminus C_2^1$, color these edges with color c_1, and call C_1^1 the set of these edges. Again, consider the edges in T_2 which are adjacent to $C_1^1 \setminus C_3^1$ edges, color these edges with color c_2, and call the set of these edges C_2^2. Continue this procedure until all edges in T_2 have been colored.

- Color the edges in T_3 which are incident to v_3 with color c_3 and call the set of these edges C_3^1. Consider the edges in T_3 which are adjacent to C_3^1 edges, color these edges with color c_1, and call the set of these edges C_1^1. Now consider the edges in T_3 which are adjacent to $C_1^1 \setminus C_3^1$ edges, color these edges with color c_2, and call C_2^1 the set of these edges. Again, consider the edges in T_3 which are adjacent to $C_2^1 \setminus C_1^1$, color these edges with color c_3 and call the set of these edges C_3^2. Continue this procedure until all edges in T_3 have been colored.

This clearly produces a feasible 3 i-edge coloring of G, and since 3 colors are needed for coloring the triangle K_3, we can conclude that $\chi_i'(G) = 3 = \omega'(G)$, and the result follows. □

4. Bounds on the injective chromatic index

Now we consider the injective edge coloring number of bipartite graphs.

Proposition 4.1. *If G is a bipartite graph with bipartition $V(G) = V_1 \cup V_2$, and G has no isolated vertices, then* $\chi_i'(G) \leq \min\{| V_1 |, | V_2 |\}$.

*P*roof. The proof follows directly from Proposition 2.1 - item 3. □

Note that the above bound is attained for every complete bipartite graph $K_{p,q}$.

We now combine Proposition 4.1 with results from [3] on the 3-consecutive edge coloring of graphs to obtain bounds on the injective edge chromatic index for bipartite graphs.

Bujtás et. al [3] proved the following results.

Proposition 4.2. *[3] If G is a bipartite graph with bipartition $V(G) = V_1 \cup V_2$, and G has no isolated vertices, then* $\max\{|V_1|, |V_2|\} \leq \psi_{3c}'(G) \leq \alpha(G)$, *where $\alpha(G)$ is the independence number of G.*

Proposition 4.3. *[3] Let G be a graph of order n.*

- *If G is connected, then $\psi_{3c}'(G) \leq n - \frac{n-1}{\Delta(G)}$, where $\Delta(G)$ denotes the maximum degree of G;*
- *$\psi_{3c}'(G) \leq n - i(G)$, where $i(G)$ is the independence domination number of G, i.e., the minimum cardinality among all maximal independent sets of G.*

From Propositions 4.1, 4.2 and 4.3 we can directly conclude the following.

Corollary 4.4. *Let G be a connected bipartite graph of order $n \geq 2$. Then*

$$\begin{aligned} \chi_i'(G) &\leq n - \frac{n-1}{\Delta(G)}, \\ \chi_i'(G) &\leq n - i(G), \\ \chi_i'(G) &\leq \alpha(G). \end{aligned}$$

Now we introduce an upper bound on the edge injective coloring number of a graph G in terms of its size and *diameter*, which we denote by diam(G).

Proposition 4.5. *For any connected graph G of size $m \geq 3$, $\chi_i'(G) \leq m - diam(G) + 2$. This upper bound is attained if and only if G is P_{m+1}.*

*P*roof. Let P_d be a diametral path of G. We can color the path P_d with 2 colors. Coloring all the other edges differently with $m -$ diam(G) colors, we produce an injective edge coloring of G, and then $\chi_i'(G) \leq m -$ diam$(G) + 2$.

The proof of the last part of this proposition can be divided in two cases:

1. If G is a path, with $n \geq 4$, it follows from Proposition 2.1- item 1 that $\chi_i'(P_n) = 2$ and, since $\mathrm{diam}(G) = m$, the result holds.
2. Let us assume that G is not a path.
   - If $\mathrm{diam}(G) \leq 2$, Proposition 2.4 implies that
   $$\chi_i'(G) < m - \mathrm{diam}(G) + 2.$$
   In fact, if $\mathrm{diam}(G) = 1$, then G is complete and thus $m - \mathrm{diam}(G) + 2 > m = \chi_i'(G)$. If $\mathrm{diam}(G) = 2$, then G is not complete and thus $m - \mathrm{diam}(G) + 2 = m > \chi_i'(G)$.
   - If $\mathrm{diam}(G) > 2$, consider a diametral path $P_d = x_1, \ldots, x_{d+1}$. Since G is connected and is not a path, then there exists a vertex $u \notin V(P_d)$ which has (i) one, (ii) two or at most (iii) three neighbors in P_d, otherwise P_d is not diametral.
     (i) Suppose u has a unique neighbor, say x_i, in P_d. As P_d is a diametral path, x_i has to be an interior vertex of P_d, i.e., $i \neq 1, d+1$, and the edges of P_d can be colored in a way such that $x_{i-1}x_i$ and x_ix_{i+1} have the same color c and this color c can also be used for coloring the edge ux_i.The remaining m − diam(G) − 1 edges can be colored with no more than m − diam(G) − 1 colors, and thus producing an i-edge coloring with at most $2 + m - \mathrm{diam}(G) - 1$ colors, and therefore $\chi_i'(G) < m - \mathrm{diam}(G) + 2$.
     (ii) If u has two neighbors in P_d, say x_i and x_j, then they must have at most one vertex between them, i.e., $j = i + 1$ or $j = i + 2$. If $j = i + 2$, the two edges ux_i and ux_j can be colored with the same color, different from each of the two colors used for the edges of P_d, and using a different color for each of the m − diam(G) − 2 other edges. Thus, $\chi_i'(G) \leq 2 + 1 + m - \mathrm{diam(G)} - 2 < m - \mathrm{diam}(G) + 2$. If $j = i+1$, then use two colors to color the edges on path $(P_d \setminus x_ix_j) \cup ux_i \cup ux_j$, a new color for edge x_ix_j and a different color for each of the remaining m − diam(G) − 2 edges. Again, $\chi_i'(G) \leq 2 + 1 + m - \mathrm{diam(G)} - 2 < m - \mathrm{diam}(G) + 2$.
     (iii) If u has three neighbors in P_d, then they must be consecutive (otherwise P_d is not diametral), say x_i, x_{i+1}, x_{i+2}. Coloring again the edges of P_d using two colors, say c_1 and c_2, edges ux_i and ux_{i+2} can be colored with an additional color c_3, and edge $x_{i+1}u$ with a different color c_4. Using a different color for each of the m − diam(G) − 3, we can conclude that $\chi_i'(G) \leq 2 + 1 + 1 + m - \mathrm{diam}(G) - 3 < m - \mathrm{diam}(G) + 2$.

□

Proposition 4.6. *For any tree T of order $n \geq 2$, $1 \leq \chi_i'(T) \leq 3$.*

*P*roof. If $n = 2$, $\chi_i'(T) = 1$. If $n \geq 3$, an edge can be added to T in such a way that the resulting graph H includes a triangle. We then have, $\chi_i'(T) \leq \chi_i'(H)$, and using Proposition 3.6, $\chi_i'(H) = 3$. □

These lower and upper bounds are sharp. According to Proposition 2.3, the stars are the only connected graphs G such that $\chi_i'(G) = 1$. Regarding the upper bound, the tree T' of Figure 3 is a minimum size tree with $\chi_i'(T') = 3$.

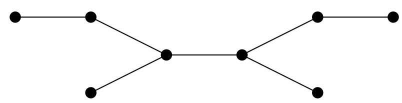


Figure 3: a minimum size tree T' with $\chi_i'(T') = 3$.

Since for a subgraph H of a graph G, any 3-consecutive edges of H are also 3-consecutive edges of G, we have the following result.

Proposition 4.7. *If H is a subgraph of a connected graph G, then $\chi_i'(H) \leq \chi_i'(G)$.*

As an immediate consequence, we have the corollary below.

Corollary 4.8. *Let G be a connected graph of order n.*

1. *If x is an edge of G, then* $\chi'_i(G) \leq \chi'_i(G + x)$.
2. *If G includes a cycle* C_p *and* $4 \leq p \not\equiv 0$ $(mod\ 4)$, *then* $\chi'_i(G) \geq 3$.
3. *If G includes a complete graph* K_p, *then* $\chi'_i(G) \geq p(p-1)/2$.
4. *If G includes the tree depicted in Figure 3, then* $\chi'_i(G) \geq 3$.
5. *If G is a tree T, which includes the subtree T′ depicted in Figure 3, then* $\chi'_i(T) = 3$.

In computer science a *perfect binary tree* is a tree data structure with exactly one vertex of degree two and where each of the other vertices has degree one or three. Now we have another corollary.

Corollary 4.9. *Let T be a perfect binary tree with diam(T)* ≥ 7. *Then* $\chi'_i(T) = 3$.

*P*roof. Note that every perfect binary tree T with $diam(T) \geq 7$ has to include the tree depicted in Figure 3 as induced subgraph. Therefore, from Corollary 4.8 - item 5, the result follows. □

We proceed deriving a characterization of the injective edge chromatic index, which is checkable in polynomial time for trees.

Let G be a graph of size $m \geq 1$, $\bar{G}$ the graph with m vertices corresponding to the edges of G and where, for every pair of vertices $x, y \in V(\bar{G})$, $xy \in E(\bar{G})$ if and only if there is an edge $e \in E(G)$ such that x, e, y are consecutive edges of G. We obviously have the following.

Lemma 4.10. *If G is a graph of size* $m \geq 1$, $\chi'_i(G) = \chi(\bar{G})$.

From Lemma 4.10 we can conclude the following.

Proposition 4.11. *If G is a graph of size* $m \geq 1$, *then* $\chi'_i(G) \leq 2$ *if and only if* $\bar{G}$ *is bipartite.*

*P*roof. Note that the chromatic number of a graph with no edges is 1, and is equal to 2 if and only if it has at least one edge and is bipartite. Lemma 4.10 completes the proof. □

We therefore have the following characterization of graphs having injective edge chromatic index equal to 2.

Proposition 4.12. *Let G be a graph of size* $m \geq 1$. *Then,* $\chi'_i(G) = 2$ *if and only if G is not a disjoint union of stars and* $\bar{G}$ *has no odd cycle.*

*P*roof. Proposition 2.3 states that $\chi'_i(G) = 1$ if and only if G is a disjoint union of stars. Proposition 4.11 states that if $\bar{G}$ is bipartite then $\chi'_i(G) \leq 2$. □

Taking into account Propositions 2.3 and 4.6, Proposition 4.12 reads for trees as follows.

Proposition 4.13. *Let T be a tree. Then, either*

- $\chi'_i(T) = 1$ *if T is a star, or*
- $\chi'_i(T) = 3$ *if* $\bar{T}$ *includes an odd cycle, or*
- $\chi'_i(T) = 2$, *in any other case.*

For example, the graph $\bar{T}'$ that is obtained from the tree T' of Figure 3, which has $\chi'_i(T') = 3$, includes cycles C_5 and C_7.

Note that Proposition 4.13 gives a polynomial time algorithm to determine the injective edge chromatic index for trees.

The next result relates the injective edge chromatic index of a graph and of its square.

Let us denote the *distance* between the vertices u and v in G by $d_G(u, v)$. The *square* of a simple graph G is the simple graph G^2, where $e = uv$ is an edge in G^2 if and only if $d_G(u, v) \leq 2$. Using this concept and this notation we have the corollary.

Corollary 4.14. *For any connected graph G, $\chi_i'(G) \leq \chi_i'(G^2)$.*

*P*roof. Notice that G is a subgraph of G^2. Therefore, applying Proposition 4.7, the result follows. □

Previously we have considered the unicyclic graphs which include a triangle and we proved that those are ω'EIC-graphs. Now, the following proposition states a lower and upper bounds on the injective chromatic index of more general unicyclic graphs.

Proposition 4.15. *Let G be a unicyclic graph and C_p the cycle in G. If $p \geq 4$, then $2 \leq \chi_i'(G) \leq 4$.*

*P*roof. The left inequality follows directly from Proposition 2.1, item 2.

To prove that $\chi_i'(G) \leq 4$, let v be an arbitrary vertex of the cycle C_p and consider $G - v$ (the graph obtained from G deleting v and every edge of G incident to v). As $G - v$ is a forest we can properly i-coloring its edges with three colors, say colors c_1, c_2, c_3. Now use a different color, say c_4, to color all edges of G incident to vertex v. This is clearly a feasible i-edge coloring of G using 4 color, showing that the result holds. □

Notice that the upper bound on the injective edge coloring number obtained in Proposition 4.15 is attained for the unicyclic graph $\aleph$ depicted in Figure 5. Regarding the lower bound, it is attained for a graph G if and only if $\bar{G}$ is bipartite.

5. The injective chromatic index of some mesh graphs and cartesian products

Herein we call mesh graphs the graphs considered in [13]. Among these graphs we pay particular attention to the cartesian products $P_n \square K_2$ and $P_r \square P_s$ and also to the honeycomb graph. The *Cartesian product $G \square H$* of two graphs G and H is the graph with vertex set equal to the Cartesian product $V(G) \times V(H)$ and where two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$ if and only if either $g_1 = g_2$ and h_1 is adjacent to h_2 or $h_1 = h_2$ and g_1 is adjacent to g_2.

Proposition 5.1. *Let P_n be a path of order $n \geq 3$. Then $\chi_i'(P_n \square K_2) = 3$*

*P*roof. Mark all the vertices in $P_n \square K_2$ from the left to right as follows: mark the first upper vertex in the ladder by 1, the second lower vertex by 2, the third upper vertex by 3 and so on, as it is in Figure 4. Now color the edges with one end vertex labeled 1 by the color c_1, the edges with one end vertex labeled 2 by the color c_2, the edges with one end vertex labeled 3 by the color c_3 and so on. This coloring yields an injective edge coloring of $P_n \square K_2$. Therefore, $\chi_i'(P_n \square K_2) \leq 3$. On the other hand, it is easy to find C_6 as a subgraph of $P_n \square K_2$ and, from Proposition 2.1-2, $\chi_i'(C_6) = 3$. Then, applying Proposition 4.7, it follows that $\chi_i'(P_n \square K_2) \geq 3$. □

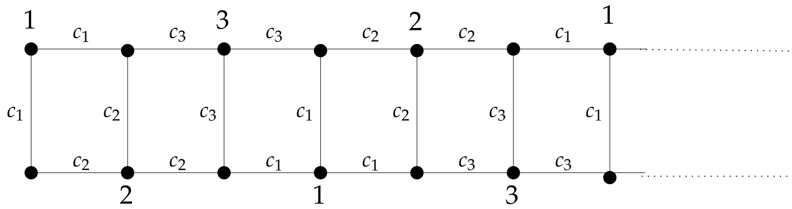
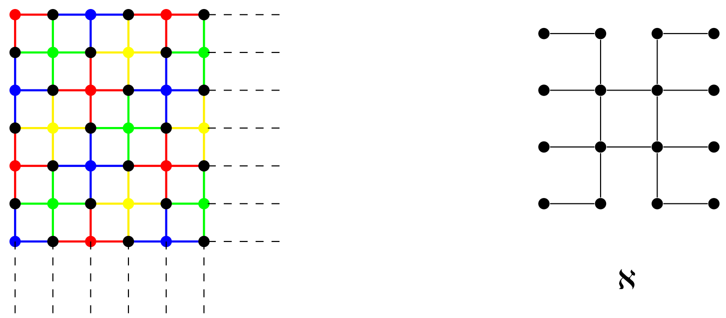


Figure 4: The Cartesian product $P_n \square K_2$ with $\chi_i'(G) = 3$.

Before the next proposition it is worth to recall that a *two dimensional grid* graph is the graph obtained by the cartesian product $P_r \square P_s$, where r and s are integers.

Proposition 5.2. *If $r, s \geq 4$, then $\chi_i'(P_r \square P_s) = 4$.*

Proof. We start to choose a color, say red, for the edges having the upper left corner vertex v of $G = P_r \square P_s$ as end vertex. Then, we color the edges having as end vertex the vertices which form a diagonal (starting at v) of the grid G alternating between red and another color, say green. The parallel diagonals are colored in the same way, using two different colors, say blue and yellow (see Figure 5). It is easy to check that this coloring produces an injective edge coloring of G and therefore, $\chi'_i(G) \leq 4$. Since the graph $\aleph$ depicted in Figure 5 is a subgraph of G such that $\chi'_i(\aleph) = 4$, applying Proposition 4.7, it follows that $4 = \chi'_i(\aleph) \leq \chi'_i(G) \leq 4$. □



$P_r \square P_s$ $\aleph$

Figure 5: Injective edge coloring of $G = P_r \square P_s$ which has $\chi'_i(G) = 4$ and the unicyclic graph $\aleph$, where $\chi'_i(\aleph) = 4$.

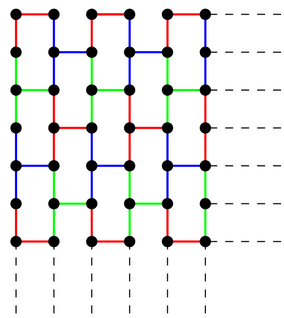


Figure 6: Injective edge coloring of a honey comb graph using three colors.

Honeycomb graphs are hexagonal tessellations which appear in the literature as models of many applications. Among several examples presented in [13] we may emphasize the applications to cellular phone station placement, representation of benzenoid hydrocarbons, computer graphics and image processing, etc.

Proposition 5.3. *If G is a honeycomb graph, then $\chi'_i(G) = 3$.*

Proof. Since the honeycomb graph G has a hexagonal tessellation, then C_6 is a subgraph G and, by Proposition 2.1-2, $\chi'_i(C_6) = 3$. Therefore, considering the coloring of the honeycomb graph G presented in Figure 6 which (as can be easily checked) is an injective edge coloring, it follows that $3 = \chi'_i(C_6) \leq \chi'_i(G) \leq 3$. □

Proposition 5.4. *For any connected graph G of order $n \geq 2$, $\chi'_i(G \square K_2) \leq n^2 - n$, and this bound is sharp. Furthermore, for any complete graph K_n with at $n \geq 2$ $\chi'_i(K_n \square K_2) = n^2 - n$.*

Proof. Let G' and G'' be the two copies of G in $G\square K_2$. Give different colors to all edges in G' and G'' say $1, 2, \ldots, m, m+1, \ldots, 2m$,where $m \leq \frac{n(n-1)}{2}$. Now consider the colorless edges in G', all colorless edges of G' have one end vertex in G' and the other in G''. We can color these edges by the colors $1, 2, \ldots, m+1$. □

The n-cube Q_n is defined repeatedly by $Q_1 = K_2$ and $Q_n = Q_{n-1}\square K_2$. Thus we have the following

Corollary 5.5. *For the n-cube* Q_n, $\chi_i'(Q_n) \leq 2^{(n-1)}(n-1)$.

6. Computational complexity of injective edge coloring

To establish the complexity of i-edge coloring we use a graph, denoted by B_k, which can properly be i-edge colored with k colors. Graph B_4 is represented in Figure 7 from which it should be clear how to construct graph B_k, for arbitrary $k \geq 1$. Note that if we remove from B_k all edges incident with b_k and vertices u_k, b_k, v_k, we obtain B_{k-1}.

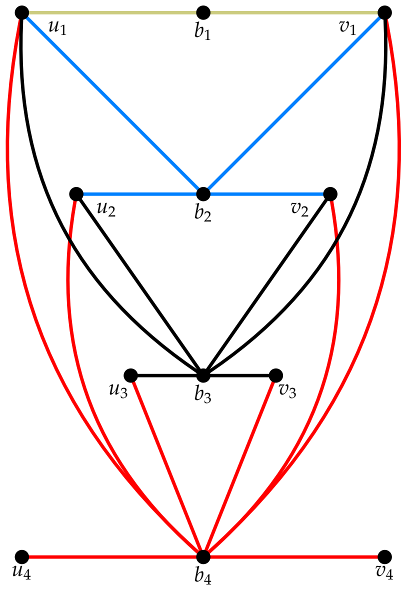


Figure 7: Graph B_4

As shown for B_4 in Figure 7, a feasible k i-edge coloring of B_k is obtained giving the same color to all the edges incident with each vertex b_i, and using different colors for edges incident with different b_i.

It can be easily verified that

Lemma 6.1. $\chi_i'(B_k) = k$. *Moreover, in every feasible k i-edge coloring of* B_k *all edges incident with vertex* b_i *receive the same color,* $i = 1, \ldots, k$, *and pairs of edges, one incident with* b_i *and the other with* b_j, $i \neq j = 1, \ldots, k$, *receive different colors.*

Proof. In any feasible i-edge coloring none of the colors of edges u_k, b_k and b_k, v_k can be used to color the edges incident with b_i, for $i = 1, \ldots, k-1$. Taking into account that the edges of B_{k-1} are the edges of B_k not incident with b_k, this shows that B_k cannot be properly i-edge colored with less than k colors, thus implying $\chi_i'(B_k) = k$, and additionally that edges u_i, b_i and b_i, v_i must have the same color in any feasible i-edge coloring.

Since the color assigned to edge u_i, b_k (b_k, v_i) cannot have the same color of edges b_i, v_i (u_i, b_i), with $i = 1, \ldots, k-1$, we can conclude that all edges incident with b_k have the same color.

The remark above, noting that B_{k-1} is obtained removing from B_k all edges incident with b_k and vertices u_k, b_k, v_k, completes the proof. □

We now use Lemma 6.1 and the NP-completeness of deciding whether $\chi'(G) = k$, for $k \geq 3$ ([8, 10]), to establish the computational complexity of i-edge coloring.

Theorem 6.2. *It is NP-complete to recognize graphs having edge injective chromatic number equal to positive integer* $k \geq 3$.

*P*roof. Given an arbitrary graph G with maximum degree $\Delta(G) = k$, we construct graph $G(B_k)$ replacing every edge uv of G by graph B_k, such that each edge uv of G is now the edges u_kb_k and b_kv_k of B_k. Therefore, in the modified graph $G(B_k)$ we have a graph B_k for each edge of G. Given a $k \geq 3$ edge coloring of G we obtain a k i-edge coloring of $G(B_k)$ by (i) assigning the color used on each edge uv of G to every edge incident with vertex b_k of subgraph B_k of $G(B_k)$ corresponding to edge uv; and (ii) using the remaining $k-1$ colors to feasibly i-edge color the other edges of $G(B_k)$. Clearly, the resulting k i-edge coloring of $G(B_k)$ is feasible if and only if the k edge coloring of G is feasible.

Conversely, given a feasible $k \geq 3$ i-edge coloring of $G(B_k)$, where each subgraph B_k corresponding to each edge of G has exactly k colors, we obtain a feasible k edge coloring of G assigning to every edge uv of G the color used on the edges incident with b_k of the subgraph B_k of $G(B_k)$ corresponding to uv.

We thus have $\chi'(G) = \chi'_i(G(B))$. Finally, since recognizing graphs G with edge injective chromatic number, $\chi'_i(G)$, equal k is obviously in NP, the result follows. □

7. Conclusions and open problems

In this paper we have characterized graphs having injective chromatic index equal to one (Proposition 2.3) and two (Proposition 4.12), and graphs with injective chromatic index equal to their sizes (Proposition 2.4). These graphs are recognized in polynomial time. We showed that trees have injective chromatic index equal to 1, 2 or 3 (Proposition 4.13), and identified the trees T with $\chi'_i(T) = i$, for $i = 1, 2, 3$ (Proposition 4.13).

In Section 3, we have introduced the notion of $\omega' EIC$-graphs (for which $\chi'_i(G) = \omega'(G)$) and presented a few examples of these graphs. Following the results published in [4], a few additional families of these type of graphs (therein called perfect $\omega' EIC$-graphs) were constructed in [15]. However, the characterizations of $\omega' EIC$-graphs remains open.

Some lower and upper bounds on the injective chromatic index of a graph were obtained in Section 4.

Regarding mesh graphs, in Section 5, the injective chromatic index of the cartesian products $P_n \square K_2$ and $P_r \square P_s$ as well as the honey comb graphs were determined. However, it is not known the injective chromatic index of several other mesh graphs as it is the case of hexagonal mesh graphs (see [13, Fig. 2]). It is also open to compute the injective chromatic index for planar graphs.

Finally, in Section 6, we have proved that determining the injective chromatic index of graphs is NP-hard.

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