Filomat 33:19 (2019), 6459–6468
https://doi.org/10.2298/FIL1919459W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Outer Inverse $f^{(2)}_{T,S}$ of a Homomorphism of Right $R-$Modules

Zhou Wang[a]

[a]*School of Mathematics, Southeast University, Nanjing 210096, P. R. China*

Abstract. In this paper, we introduce the definition of the generalized inverse $f^{(2)}_{T,S}$, which is an outer inverse of the homomorphism f of right $R-$modules with prescribed image T and kernel S. Some basic properties of the generalized inverse $f^{(2)}_{T,S}$ are presented. It is shown that the Drazin inverse, the group inverse and the Moore-Penrose inverse, if they exist, are all the generalized inverse $f^{(2)}_{T,S}$. In addition, we give necessary and sufficient conditions for the existence of the generalized inverse $f^{(1,2)}_{T,S}$.

1. Introduction

Let A be a matrix over the field of complex number. It is well known [3,12] that the group inverse, the Drazin inverse and the Moore-Penrose inverse of A are all the generalized inverse $A^{(2)}_{T,S}$, where $T,\ S$ are the range and null space of the outer inverse of A, respectively. In 1998, Wei presents an explicit expression for the generalized inverse $A^{(2)}_{T,S}$, and establishes the characterization and representation theorem (see [15]).

In 2005, Yu and Wang [13] introduce the definition of the generalized inverse $A^{(2)}_{T,S}$ of a matrix A over a commutative ring R. They also give an explicit expression for $A^{(2)}_{T,S}$ over integral domain. In addition, it is shown that over integral domain, the Drazin inverse, the group inverse and the Moore-Penrose inverse are all $A^{(2)}_{T,S}$. Furthermore, they extend the notion of the generalized inverse $A^{(2)}_{T,S}$ to the matrix A over an associative ring [14]. It is obtained that the Drazin inverse, the group inverse and the Moore-Penrose inverse, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$. They also give necessary and sufficient conditions for the existence of the generalized inverse $A^{(1,2)}_{T,S}$ and some explicit expressions for $A^{(1,2)}_{T,S}$.

From the view of homomorphisms, a matrix over the field of complex number can be regarded as a homomorphism (or a linear transformation) of finite dimensional vector spaces, and a matrix over a commutative (noncommutative) ring is corresponding exactly to a homomorphism of finitely generated

2010 *Mathematics Subject Classification*. Primary 16D10, 15A09; Secondary 16S50, 16U99
Keywords. the generalized inverse $f^{(2)}_{T,S}$; the $R-$homomorphism; Drazin inverse; group inverse; Moore-Penrose inverse
Received: 19 July 2019; Accepted: 25 December 2019
Communicated by Dijana Mosić
Research supported by the NNSF of China (No. 10971024)
Email address: zhouwang@seu.edu.cn (Zhou Wang)

free modules. Hence, one naturally wants to know whether the free modules could be generalized to arbitrary modules over an associative ring.

Throughout this paper, R denotes an associative ring with unity, and M, N denote right $R-$modules. If S is an R-submodule of M then we write $S \leq M$. We denote an $R-$homomorphism from M to N by $f \in \mathrm{Hom}_R(M, N)$. $\mathrm{Im}(f)$ and $\mathrm{Ker}(f)$ stand for the image and the kernel of f, respectively. Standard facts in ring and module theory used without mention in the text can be found in [1].

An $R-$homomorphism $f \in \mathrm{Hom}_R(M, N)$ is said to be *von Neumman regular* if there exists $g \in \mathrm{Hom}_R(N, M)$ such that $f = fgf$. In this case, g is called a $\{1\}$*-inverse* (or *inner inverse*) of f and denoted by $f^{(1)}$. Moreover, we recall that g is a $\{2\}$*-inverse* (or *outer inverse*) of f if $g = gfg$, and denoted by $f^{(2)}$. It is well known that $\{1\}$-invertible property implies $\{2\}$-invertible property, i.e., $\{1\}$-invertible property=$\{1, 2\}$-invertible property.

An endomorphism $f \in \mathrm{End}_R(M)$ is said to be *Drazin invertible* if for some positive integer k there exists an endomorphism g such that

$$(i)\ g = gfg,\ (ii)\ f^k = f^{k+1}g \text{ and } (iii)\ fg = gf.$$

If g exists then it is unique and is called the *Drazin inverse* of f and denoted by f^D. If k is the smallest positive integer such that g and f satisfy (i), (ii) and (iii), then it is called the *Drazin index* and denoted by $\mathrm{Ind}(f)$. If $k = 1$ then g is called the *group inverse* of f and denoted by $f^\sharp$.

Let $*$ be an involution on the $R-$homomorphisms. Recall that $f \in \mathrm{Hom}_R(M, N)$ is said to be *Moore-Penrose invertible* if there is a homomorphism $g \in \mathrm{Hom}_R(N, M)$ such that

$$f = fgf,\ g = gfg,\ (fg)^* = fg \text{ and } (gf)^* = gf.$$

Here g is called the *Moore-Penrose inverse* of f and denoted by $f^\dagger$.

More generally, an $R-$homomorphism of modules is regarded as a morphism in the category of modules, which is an additive category. The Moore-Penrose inverses and other generalized inverses of a morphism in an additive category are studied by many authors (see [4,6,9-11]).

Our goal in this paper is to extend the generalized inverse $A^{(2)}_{T,S}$ of a matrix A to $f^{(2)}_{T,S}$ of an R-homomorphism $f \in \mathrm{Hom}_R(M, N)$, which is $\{2\}$-inverse of f with prescribed image T and kernel S. In Section 2, we establish the definition of the generalized inverse $f^{(2)}_{T,S}$, and give some explicit expressions for $f^{(2)}_{T,S}$ by a projection or group inverses. In addition, we also show that the Drazin inverse f^D, the group inverse $f^\sharp$ and the Moore-Penrose inverse $f^\dagger$, if they exist, are all the generalized inverse $f^{(2)}_{T,S}$. In Section 3 we investigate necessary and sufficient conditions for the existence of the generalized inverse $f^{(1,2)}_{T,S}$. For any $h \in \mathrm{Hom}_R(N, M)$, we obtain some equivalent conditions for the existence of $f^{(1,2)}_{\mathrm{Im}(h),\mathrm{Ker}(h)}$. This paper is motivated by the interesting results of Yu and Wang [13,14], and some different methods are used in the proof of our main results.

2. The generalized inverse $f^{(2)}_{T,S}$ of a homomorphism of right $R-$modules

We begin this section from the following result.

Lemma 2.1. *Let $f \in Hom_R(M, N)$ and let $T \leqslant M$, $S \leqslant N$. Then the following are equivalent.*
(1) *There exists $g \in Hom_R(N, M)$ such that $gfg = g$, $Im(g) = T$ and $Ker(g) = S$.*
(2) *$f(T) \oplus S = N$ and $Ker(f) \cap T = \{0\}$.*

Proof. (1) $\Rightarrow$ (2). Let $s \in f(T) \cap S$. Then there exists $n \in N$ such that $s = fg(n) \in S$. From $S = \mathrm{Ker}(g)$, it follows that $g(n) = gfg(n) = g(s) = 0$. Then $s = fg(n) = 0$. This shows that $f(T) \cap S = \{0\}$. Take $n \in N$. Then $g(n) \in T$, and $(1 - fg)(n) \in S$ since $g = gfg$ and $\mathrm{Ker}(g) = S$. Thus, we get

$$n = fg(n) + (1 - fg)(n) \in f(T) + S.$$

This shows that $f(T) \oplus S = N$. Let $t \in \mathrm{Ker}(f) \cap T$. Then there exists $n \in N$ such that $t = g(n)$ and $f(t) = 0$. So we get $t = g(n) = gfg(n) = gf(t) = 0$, as required.

(2) $\Rightarrow$ (1). Define

$$g : N \to M,\ n = f(t) + s \mapsto t,\ \text{where } t \in T,\ s \in S.$$

We show first that g is well defined. In fact, assume that $f(t)+s = 0$. Since $N = f(T)\oplus S$, we have $s = f(t) = 0$. This implies that $t \in \mathrm{Ker}(f) \cap T = \{0\}$, i.e., $t = 0$. Next, it is sufficient to prove

$$\mathrm{Im}(g) = T,\ \mathrm{Ker}(g) = S \text{ and } gfg = g.$$

By the definition of g, we get $\mathrm{Im}(g) \subseteq T$. Let $t \in T$. Then $gf(t) = t$. This shows that $t \in \mathrm{Im}(g)$, and so $\mathrm{Im}(g) = T$. Let $n \in \mathrm{Ker}(g)$. From (2), we have $n = f(t) + s$ for some $t \in T$, $s \in S$. Then $t = g(n) = 0$. Thus, $n = s \in S$, i.e., $\mathrm{Ker}(g) \subseteq S$. Let $s \in S$. Then $g(s) = g(f(0) + s) = 0$, and so $s \in \mathrm{Ker}(g)$. This implies $\mathrm{Ker}(g) = S$. For any $n \in N$, we may calculate directly

$$gfg(n) = gfg(f(t) + s) = gf(t) = t = g(f(t) + s) = g(n).$$

Hence, $gfg = g$. $\square$

The following result should be well known, but we can not find it somewhere.

Lemma 2.2. *Let $f \in Hom_R(M, N)$. Then the following hold.*
(1) $P_H f = f$ if and only if $Im(f) \leqslant H$, where $N = H \oplus K$, $P_H : N \to N$, $h + k \mapsto h$.
(2) $fP_{H'} = f$ if and only if $K' \leqslant Ker(f)$, where $M = H' \oplus K'$, $P_{H'} : M \to M$, $h' + k' \mapsto h'$.

Proof. (1). The implication follows from $\mathrm{Im}(f) = \mathrm{Im}(P_H f) \subseteq H$. For any $m \in M$, we have $f(m) = h + k$ for some $h \in H$, $k \in K$. Note that $\mathrm{Im}(f) \leqslant H$. Then $k = f(m) - h \in H \cap K = \{0\}$ since $N = H \oplus K$. This implies that $f(m) = h$, and so

$$P_H f(m) = P_H(h) = h = f(m).$$

Thus, $P_H f = f$.

(2). Let $k' \in K'$. Then $f(k') = fP_{H'}(k') = f(0) = 0$, as required. Conversely, for any $m \in M$, it follows that $m = h' + k'$ for some $h' \in H'$, $k' \in K'$ from $M = H' \oplus K'$. Then

$$fP_{H'}(m) = f(h') = f(h' + k') = f(m),$$

which shows $fP_{H'} = f$. $\square$

Let $M = H \oplus K$. Define $P_H : M \to M$; $h + k \mapsto h$. Then $P_H^2 = P_H$. Conversely, suppose $p^2 = p \in \mathrm{End}_R(M)$. Then $M = \mathrm{Im}(p) \oplus \mathrm{Im}(1 - p) := H \oplus K$, which implies $p = P_H$.

Proposition 2.3. *If the conditions of Lemma 2.1 are satisfied, then g is unique.*

Proof. Assume that g_1 and g_2 satisfy the conditions of Lemma 2.1. Then we have $\mathrm{Im}(g_1) = T = \mathrm{Im}(g_2) = \mathrm{Im}(g_2fg_2) \subseteq \mathrm{Im}(g_2f)$. Set $H = \mathrm{Im}(g_2f)$. Since $g_2fg_2 = g_2$, we get $\mathrm{Im}(g_1) \leqslant H$ and $M = H \oplus \mathrm{Im}(1 - g_2f)$. Note that $(g_2f)^2 = g_2f$. Then $P_H = g_2f$, and so we obtain that $g_1 = P_Hg_1 = (g_2f)g_1$ by Lemma 2.2(1). Since $g_1 = g_1fg_1$, we have $\mathrm{Im}(1 - fg_1) \subseteq \mathrm{Ker}(g_1) = S = \mathrm{Ker}(g_2)$. Take $H' = \mathrm{Im}(fg_1)$ and $K' = \mathrm{Im}(1 - fg_1)$. Then $K' \leqslant \mathrm{Ker}(g_2)$ and $M = H' \oplus K'$ with $P_{H'} = fg_1$. This implies $g_2 = g_2(fg_1)$ by Lemma 2.2(2). Thus, we get $g_1 = g_2$. $\square$

A homomorphism $g \in \mathrm{Hom}_R(N, M)$ is called **the generalized inverse**, which is an outer inverse of the homomorphism $f \in \mathrm{Hom}_R(M, N)$ with prescribed image T and kernel S if it satisfies the equivalent conditions in Lemma 2.1, and is denoted by $f^{(2)}_{T,S}$.

Proposition 2.4. *Let $f \in Hom_R(M, N)$ have the generalized inverse $f^{(2)}_{T,S}$ (say g). Set $N = f(T) \oplus S$, $T = Im(g)$ and $S = Ker(g)$. Define $f|_T : T \to f(T)$. Then $f|_T$ is an isomorphism, and*

$$g = (f|_T)^{-1} P_{f(T)},$$

where $P_{f(T)} : N \to N,\ f(t) + s \mapsto f(t)$.

Proof. It is clear that $f|_T$ is epimorphic. We show only that $f|_T$ is monomorphic. Let $f(t) = 0$ for $t \in T$. Then there exists $n \in N$ such that $t = g(n)$. Set $n = f(t') + s$, where $t' \in T,\ s \in S$. Then $t' = g(n')$ for some $n' \in N$ since $T = \mathrm{Im}(g)$. Thus, we have

$$0 = f(t) = f(g(n)) = fgf(t') = fg(n') = f(t').$$

This implies that $t = g(n) = g(f(t')+s) = g(s) = 0$, as required. Next, it is sufficient to prove $f^{(2)}_{T,S} = (f|_T)^{-1}P_{f(T)}$.

$$(f|_T)^{-1}P_{f(T)} f (f|_T)^{-1}P_{f(T)} = (f|_T)^{-1}P^2_{f(T)} = (f|_T)^{-1}P_{f(T)},$$

$$\mathrm{Im}((f|_T)^{-1}P_{f(T)}) = (f|_T)^{-1} f(T) = T,$$

$$\mathrm{Ker}((f|_T)^{-1}P_{f(T)}) = \mathrm{Ker}(P_{f(T)}) = S.$$

So the proof is completed. □

Corollary 2.5. *Let $f \in Hom_R(M, N)$. If the generalized inverse $f^{(2)}_{T,S}$ exists, then*
(1) $f^{(2)}_{T,S} fh = h$ if and only if $Im(h) \leqslant T$, where $h : X \to M$.
(2) $hff^{(2)}_{T,S} = h$ if and only if $S \leqslant Ker(h)$, where $h : N \to Y$.

Proof. (1). Set $g = f^{(2)}_{T,S}$. Then the implication follows from

$$\mathrm{Im}(h) = \mathrm{Im}(gfh) \subseteq \mathrm{Im}(g) = T.$$

For any $x \in X$, say $t = h(x)$. Note that $\mathrm{Im}(h) \leqslant T$. Then there exists $n \in N$ such that $t = g(n)$. By $g = gfg$, we check easily that

$$gfh(x) = gf(t) = gfg(n) = g(n) = h(x).$$

So we get $gfh = h$.

(2). Let $s \in S = \mathrm{Ker}(g)$. Then $fh(s) = hfg(s) = 0$, i.e., $s \in \mathrm{Ker}(h)$, as required. Conversely, for any $n \in N$, say $n = f(t) + s$ for some $t \in T,\ s \in S$. From $\mathrm{Im}(g) = T$, there exists $n' \in N$ such that $t = g(n')$. Thus, we have

$$hfg(n) = hfgf(t) = hfgfg(n') = hfg(n') = hf(t).$$

On the other hand, $S \leqslant \mathrm{Ker}(h)$ implies $hf(t) = h(f(t) + s) = h(n)$, so one obtains $hfg(n) = h(n)$. The proof is completed. □

The following result is well known (also see [1, 3.6]).

Lemma 2.6. (The Factor Theorem) *Let $g,\ h : N \to T$ be two $R-$homomorphisms. If h is an epimorphism with $Ker(h) \leqslant Ker(g)$, then there exists unique homomorphism $\omega : T \to T$ such that $g = \omega h$.*

Theorem 2.7. *Let $f \in Hom_R(M, N)$ and $f^{(2)}_{T,S}$ exists (say g). If $h : N \to M$ satisfies $Im(h) = T$, $Ker(h) = S$, then there exists an isomorphism $\omega : M \to M$ such that $g = \omega h$.*

Proof. Note that $\mathrm{Im}(g) = T = \mathrm{Im}(h)$. Then g, h reduce two epimorphisms $\widetilde{g}$, $\widetilde{h}$ from N to T. Moreover, $\mathrm{Ker}(g) = S = \mathrm{Ker}(h)$ implies $\mathrm{Ker}(\widetilde{g}) = \mathrm{Ker}(\widetilde{h})$. By Lemma 2.6, there exist $\widetilde{\omega}$, $\widetilde{\upsilon} \in \mathrm{End}_R(T)$ such that $\widetilde{g} = \widetilde{\omega}\widetilde{h}$ and $\widetilde{h} = \widetilde{\upsilon}\widetilde{g}$. Thus, we have $\widetilde{g} = \widetilde{\omega}\widetilde{\upsilon}\widetilde{g}$ and $\widetilde{h} = \widetilde{\upsilon}\widetilde{\omega}\widetilde{h}$. Since both $\widetilde{g}$ and $\widetilde{h}$ are epimorphic, we get $\widetilde{\omega}\widetilde{\upsilon} = 1_T$, $\widetilde{\upsilon}\widetilde{\omega} = 1_T$, i.e., $\widetilde{\omega}$ is an isomorphism. Note that $T = \mathrm{Im}(g)$ is a direct summand of M since $gfg = g$, say $M = T \oplus X$. Define $\omega : M \to M$; $m = t + x \mapsto \widetilde{\omega}(t) + x$. It is easy to check that ω is an isomorphism and $g = \omega h$, as desired. □

Corollary 2.8. *Let $f \in Hom_R(M, N)$ and $f^{(2)}_{T,S}$ exists. If $h : N \to M$ satisfies $Im(h) = T$, $Ker(h) = S$, then there exists an isomorphism $\omega : M \to M$ such that*

$$\omega hfh = h \text{ and } hf\omega h = h.$$

Proof. Set $g = f^{(2)}_{T,S}$. By Corollary 2.5, we have $gfh = h$ and $hfg = h$. From Theorem 2.7, there exists an isomorphism $\omega \in \mathrm{End}_R(M)$ such that $g = \omega h$. Thus, we get

$$\omega hfh = gfh = h \text{ and } hf\omega h = hf\omega h = h.$$

The proof is completed. □

The following lemma is duo to Armendariz, Fisher and Snider [2, Proposition 2.3] (also see [7]).

Lemma 2.9. *Let α be an endomorphism of right $R-$module M. Then the following are equivalent.*
(1) *The endomorphism α is strongly regular.*
(2) *There exists a direct decomposition $M = Im(\alpha) \oplus Ker(\alpha)$.*
(2) *The endomorphism α is group invertible.*

Theorem 2.10. *Let $f \in Hom_R(M, N)$, $T \leqslant M$, $S \leqslant N$. Suppose that $f^{(2)}_{T,S}$ exists. If there is $h \in Hom_R(N, M)$ such that $Im(h) = T$ and $Ker(h) = S$, then both fh and hf are group invertible. Furthermore,*

$$f^{(2)}_{T,S} = h(fh)^\sharp = (hf)^\sharp h.$$

Proof. We prove firstly that fh is group invertible. By Lemma 2.9, it is sufficient to show that

$$N = \mathrm{Im}(fh) \oplus \mathrm{Ker}(fh).$$

Note that $\mathrm{Im}(fh) = f\mathrm{Im}(h) = f(T)$ and $S = \mathrm{Ker}(h) \subseteq \mathrm{Ker}(fh)$. For any $n \in \mathrm{Ker}(fh)$, by Lemma 2.1(2), we have

$$h(n) \in \mathrm{Ker}(f) \cap \mathrm{Im}(h) = \mathrm{Ker}(f) \cap T = \{0\}.$$

This shows that $n \in \mathrm{Ker}(h)$, and so

$$\mathrm{Ker}(fh) = \mathrm{Ker}(h) = S.$$

Thus, by Lemma 2.1(1), we have

$$N = f(T) \oplus S = \mathrm{Im}(fh) \oplus \mathrm{Ker}(fh).$$

Next, for any $m \in \mathrm{Im}(h)$, there exists $n \in N$ such that $m = h(n)$. Then

$$f(m) = fh(n) = (fh)(fh)^\sharp(fh)(n) \in \mathrm{Im}((fh)(fh)^\sharp),$$

i.e., $f(m) = (fh)(fh)^{\sharp}(n')$ for some $n' \in N$. Thus, we get

$$m - h(fh)^{\sharp}(n') \in \mathrm{Ker}(f) \cap T = \{0\},$$

and so

$$m = h(fh)^{\sharp}(n') \in \mathrm{Im}(h(fh)^{\sharp}).$$

This shows that

$$\mathrm{Im}(h(fh)^{\sharp}) = \mathrm{Im}(h) = T.$$

Note that $\mathrm{Ker}(fh) = \mathrm{Ker}(h) = S$. Then it is necessary to check that

$$\mathrm{Ker}(h(fh)^{\sharp}) = \mathrm{Ker}(fh).$$

Let $fh(n) = 0$. Then

$$fh(fh)^{\sharp}(n) = (fh)^{\sharp}fh(n) = 0.$$

This implies that

$$h(fh)^{\sharp}(n) \in \mathrm{Ker}(f) \cap T = \{0\},$$

and so $n \in \mathrm{Ker}(h(fh)^{\sharp})$. Thus, we have $\mathrm{Ker}(fh) \subseteq \mathrm{Ker}(h(fh)^{\sharp})$. Note that

$$fh = (fh)^2(fh)^{\sharp} = (fhf)(h(fh)^{\sharp}).$$

Then

$$\mathrm{Ker}(h(fh)^{\sharp}) \subseteq \mathrm{Ker}(fh).$$

This shows that $\mathrm{Ker}(h(fh)^{\sharp}) = \mathrm{Ker}(fh)$, and so $\mathrm{Ker}(h(fh)^{\sharp}) = S$. Note that

$$(h(fh)^{\sharp})f(h(fh)^{\sharp}) = h(fh)^{\sharp}(fh)(fh)^{\sharp} = h(fh)^{\sharp}.$$

Thus, this shows that $f^{(2)}_{T,S} = h(fh)^{\sharp}$.
Set $g = f^{(2)}_{T,S}$. By Theorem 2.7, we have $g = \omega h$ for some automorphism of M. It follows that

$$\mathrm{Im}(hf) \subseteq \mathrm{Im}(h) = \mathrm{Im}(hfg) \subseteq \mathrm{Im}(hf)$$

from Corollary 2.5(2). This implies that

$$\mathrm{Im}(hf) = \mathrm{Im}(h) = T = \mathrm{Im}(g) = \mathrm{Im}(gf)$$

since $gfg = g$. Note that

$$\mathrm{Ker}(hf) = \mathrm{Ker}(\omega hf) = \mathrm{Ker}(gf) = \mathrm{Im}(1 - gf),$$

and so we have

$$M = \mathrm{Im}(gf) \oplus \mathrm{Im}(1 - gf) = \mathrm{Im}(hf) \oplus \mathrm{Ker}(hf).$$

It follows that hf is group invertible from Lemma 2.9. Moreover, we can check that $f^{(2)}_{T,S} = (hf)^{\sharp}h$. □

In the next result, we will show that for an arbitrary homomorphism f of right R-modules, Drazin inverse f^D, group inverse $f^{\sharp}$ and Moore-Penrose inverse $f^{\dagger}$, if they exist, are all the generalized inverse $f^{(2)}_{T,S}$.

Theorem 2.11. *Let M, N be right $R-$modules.*
(1) *Let $f \in End_R(M)$. If f^D exists with $Ind(f) = k$, then $f^D = f^{(2)}_{Im(f^k),Ker(f^k)}$.*
(2) *Let $f \in End_R(M)$. If $f^{\sharp}$ exists, then $f^{\sharp} = f^{(2)}_{Im(f),Ker(f)}$.*
(3) *Let $f \in Hom_R(M, N)$. If $f^{\dagger}$ exists with an involution $*$ on homomorphisms of modules, then $f^{\dagger} = f^{(2)}_{Im(f^*),Ker(f)^*}$.*

Proof. (1). Since $f^D f f^D = f^D$, by Lemma 2.1(1), it is sufficient to show that

$$\mathrm{Im}(f^D) = \mathrm{Im}(f^k) \text{ and } \mathrm{Ker}(f^D) = \mathrm{Ker}(f^k).$$

Note that $ff^D = f^D f$ and $f^k = f^D f^{k+1}$. Then

$$\mathrm{Im}(f^D) = \mathrm{Im}(f^D f f^D) = \mathrm{Im}((f^D f)^k f^D) = \mathrm{Im}(f^k (f^D)^{k+1}) \subseteq \mathrm{Im}(f^k),$$

and

$$\mathrm{Im}(f^k) = \mathrm{Im}(f^D f^{k+1}) \subseteq \mathrm{Im}(f^D).$$

It follows that $\mathrm{Im}(f^D) = \mathrm{Im}(f^k)$. Since $f^k = f^{k+1} f^D$ and $f^D = f^D f f^D = f^D (f f^D)^k = (f^D)^{k+1} f^k$, this implies that $\mathrm{Ker}(f^D) = \mathrm{Ker}(f^k)$. Thus, we have $f^D = f^{(2)}_{\mathrm{Im}(f^k),\mathrm{Ker}(f^k)}$.

(2). Take $k = 1$ in (1).

(3). Note that $f^\dagger f f^\dagger = f^\dagger$. By Lemma 2.1(1), it is only necessary to check that

$$\mathrm{Im}(f^\dagger) = \mathrm{Im}(f^*) \text{ and } \mathrm{Ker}(f^\dagger) = \mathrm{Ker}(f^*).$$

Since $f \in f^{\{1,2\}}$ and $f^* \in (f^*)^{\{1,2\}}$, we can get easily that

$$\mathrm{Im}(f^\dagger) = \mathrm{Im}(f^\dagger f) = \mathrm{Im}((f^\dagger f)^*) = \mathrm{Im}(f^*(f^\dagger)^*) = \mathrm{Im}(f^*),$$

and

$$\mathrm{Ker}(f^\dagger) = \mathrm{Ker}(ff^\dagger) = \mathrm{Ker}((ff^\dagger)^*) = \mathrm{Ker}((f^\dagger)^* f^*) = \mathrm{Ker}(f^*).$$

The proof is completed. □

3. The generalized inverse $f^{(1,2)}_{T,S}$ of a homomorphism of right R−modules

If the generalized inverse $f^{(2)}_{T,S}$ satisfies $ff^{(2)}_{T,S}f = f$, then it is called the generalized inverse which is a $\{1,2\}$-inverse of a homomorphism f of modules with prescribed image T and kernel S, and is denoted by $f^{(1,2)}_{T,S}$.

Theorem 3.1. *Let $f \in Hom_R(M, N)$ and let $T \leqslant M$, $S \leqslant N$. Then the following are equivalent.*
(1) *$f(T) \oplus S = N$, $Im(f) \cap S = 0$ and $Ker(f) \cap T = \{0\}$.*
(2) *There exists some $g \in Hom_R(N, M)$ such that*

$$fgf = f,\ gfg = g,\ Im(g) = T \text{ and } Ker(g) = S.$$

(3) *$Im(f) \oplus S = N$ and $Ker(f) \oplus T = M$.*

Proof. (1) ⇒ (2). From $f(T) \oplus S = N$ and $\mathrm{Ker}(f) \cap T = 0$, we get that $g = f^{(2)}_{T,S}$ exists and that $\mathrm{Im}(g) = T$, $\mathrm{Ker}(g) = S$ by Lemma 2.1. We only need to show that $fgf = f$. Note that $gfg = g$. Then we have $gfgf = gf$, which implies

$$\mathrm{Im}(fgf - f) \subseteq \mathrm{Im}(f) \cap \mathrm{Ker}(g) = \mathrm{Im}(f) \cap S = \{0\}.$$

So $fgf = f$, as required.

(2) ⇒ (3). From $\mathrm{Im}(g) = T$, we have $f(T) = \mathrm{Im}(fg)$. Note that $fgf = f$ implying $\mathrm{Im}(fg) = \mathrm{Im}(f)$. Then $f(T) = \mathrm{Im}(f)$. By (2), we know $g = f^{(2)}_{T,S}$. Hence, $N = f(T) \oplus S = \mathrm{Im}(f) \oplus S$. Next, $f = fgf$ implies that $\mathrm{Ker}(f) = \mathrm{Im}(I_M - gf)$. From $\mathrm{Im}(g) = T$, we have

$$M = \mathrm{Im}(I_M - gf) + \mathrm{Im}(g) = \mathrm{Ker}(f) + T.$$

Hence, it follows from $\mathrm{Ker}(f) \cap T = \{0\}$.

(3) $\Rightarrow$ (1). It is clear that $\mathrm{Im}(f) \cap S = \{0\}$ and $\mathrm{Ker}(f) \cap T = \{0\}$. To obtain $f(T) \oplus S = N$, it is sufficient to show $f(T) = \mathrm{Im}(f)$. For any $n \in \mathrm{Im}(f)$, we have $n = f(m)$ for some $m \in M$. Since $\mathrm{Ker}(f) \oplus T = M$, we can say $m = m_1 + m_2$ where $m_1 \in \mathrm{Ker}(f)$, $m_2 \in T$. Thus, we get

$$n = f(m) = f(m_1) + f(m_2) = f(m_2) \in f(T),$$

and so $\mathrm{Im}(f) \subseteq f(T)$. Clearly, $f(T) \subseteq \mathrm{Im}(f)$. Hence, $f(T) = \mathrm{Im}(f)$. □

Theorem 3.2. *Let $f \in Hom_R(M, N)$ and let $T \leqslant M$, $S \leqslant N$.*
(1) If $Ker(f) + T = M$, then $f(T) = Im(f)$.
(2) If $f(T) \oplus S = N$, then

$$f(T) = Im(f) \text{ if and only if } Im(f) \cap S = \{0\}.$$

Proof. (1) follows easily from the observation that

$$f(M) = f(\mathrm{Ker}(f) + T) \subseteq f(\mathrm{Ker}(f)) + f(T) = f(T).$$

(2). Suppose that $\mathrm{Im}(f) \cap S = \{0\}$. Obviously, we have $f(T) \subseteq \mathrm{Im}(f)$. For any $x \in \mathrm{Im}(f)$, $x = x_1 + x_2$, where $x_1 \in f(T)$, $x_2 \in S$. From $f(T) \subseteq \mathrm{Im}(f)$, $x_1 \in \mathrm{Im}(f)$. Thus, $x_2 = x - x_1 \in \mathrm{Im}(f) \cap S = \{0\}$. Therefore, we get $x_2 = 0$ and then $x = x_1 \in f(T)$. Hence $\mathrm{Im}(f) \subseteq f(T)$. Conversely, assume that $f(T) = \mathrm{Im}(f)$. From $f(T) \oplus S = N$, we have $\mathrm{Im}(f) \cap S = f(T) \cap S = \{0\}$. □

Lemma 3.3. (Jacobson Lemma) *Let a, $b \in R$. Then $1 - ab$ is invertible if and only if $1 - ba$ is invertible.*

Let $S = \mathrm{End}(_R N)$ and $T = \mathrm{End}(_R M)$. The following lemma is duo to Puystjens and Hartwig [8, Corollary 1.]. We will give a proof for the sake of completeness.

Lemma 3.4. *Suppose $f \in Hom_R(M, N)$ is regular, and let $f = ff^{(1)}f$. Then the following are equivalent for any $h \in Hom_R(N, M)$.*
(1) $u = fhff^{(1)} + I_N - ff^{(1)}$ is invertible in S.
(2) $v = f^{(1)}fhf + I_M - f^{(1)}f$ is invertible in T.
(3) $Sfhf = Sf$ and $fhfT = fT$.

Proof. (1) $\Leftrightarrow$ (2). Note that $u = I_N - (f - fhf)f^{(1)}$ and $v = I_M - f^{(1)}(f - fhf)$. Then, by Lemma 3.3, u is invertible in S if and only if v is invertible in T.

(1) (and (2)) $\Rightarrow$ (3). It follows that $uf = fhf = fv$ from (1) and (2). Note that u and v are both invertible. Then it implies that $Sfhf = Sf$ and $fhfT = fT$.

(3) $\Rightarrow$ (1). Suppose that $xfhf = f = fhfy$ for some $x \in S$, $y \in T$. Take $\alpha = fyf^{(1)} + I_N - ff^{(1)}$ and $\beta = xff^{(1)} + I_N - ff^{(1)}$. Then we can directly calculate that $u\alpha = \beta u = I_N$, as required. □

Theorem 3.5. *Let $f \in Hom_R(M, N)$, $h \in Hom_R(N, M)$. Then the following are equivalent.*
(1) f is regular, $u = fhff^{(1)} + I_N - ff^{(1)}$ is invertible in S and $Ker(f) \cap Im(h) = \{0\}$.
(2) f is regular, $v = f^{(1)}fhf + I_M - f^{(1)}f$ is invertible in T and $Ker(f) \cap Im(h) = \{0\}$.
(3) $f^{(1,2)}_{Im(h),Ker(h)}$ exists.

Proof. (1) $\Leftrightarrow$ (2) is clear from Lemma 3.4.

(1) (and (2)) $\Rightarrow$ (3). From (1) and (2), we can check easily that $uf = fhf = fv$ and $fv^{-1} = u^{-1}f$. Set $\varphi = fv^{-2}h$. Then we have

$$\varphi(fh) = fv^{-2}hfh = u^{-2}fhfh = u^{-1}fh = fv^{-1}h = fhfv^{-2}h = (fh)\varphi,$$

$$\varphi(fh)\varphi = u^{-1}fhfv^{-2}h = fv^{-2}h = \varphi,$$

and

$$(fh)\varphi(fh) = fhfv^{-1}h = fh.$$

This shows that fh is group invertible and $\varphi = (fh)^\sharp$. Set $g = h(fh)^\sharp$. It is easy to check that

$$gfg = h(fh)^\sharp fh(fh)^\sharp = h(fh)^\sharp = g,$$

and

$$fgf = fhfv^{-2}hf = u^{-1}fhf = f.$$

Next, it is sufficient to show that $\mathrm{Im}(g) = \mathrm{Im}(h)$ and $\mathrm{Ker}(g) = \mathrm{Ker}(h)$. Since $fh = (fh)^2(fh)^\sharp = fhfg$, we get $f(h - hfg) = 0$. This implies that

$$\mathrm{Im}(h - hfg) \subseteq \mathrm{Ker}(f) \cap \mathrm{Im}(h) = \{0\},$$

and so

$$h = hfg = hfh(fh)^\sharp = h(fh)^\sharp fh = gfh.$$

Note that

$$g = h(fh)^\sharp = h((fh)^\sharp)^2 fh.$$

Then we can obtain that $\mathrm{Im}(g) = \mathrm{Im}(h)$ and $\mathrm{Ker}(g) = \mathrm{Ker}(h)$. Thus, it follows that $f^{(1,2)}_{Im(h),Ker(h)}$ exists and $g = f^{(1,2)}_{Im(h),Ker(h)}$.

(3) $\Rightarrow$ (1). Suppose $f^{(1,2)}_{Im(h),Ker(h)}$ exists and say $g = f^{(1,2)}_{Im(h),Ker(h)}$. By Theorem 2.10, we have $g = h(fh)^\sharp$. Take $S = \mathrm{End}(_RN)$ and $T = \mathrm{End}(_RM)$. Note that

$$Sfhf \subseteq Sf = Sfgf = Sfh(fh)^\sharp f = S(fh)^\sharp fhf \subseteq Sfhf.$$

Then $Sfhf = Sf$. It is easy to see

$$fhfT \subseteq fT \subseteq fgfgfT = fh(fh)^\sharp fh(fh)^\sharp fT = (fhf)h((fh)^\sharp)^2 fT \subseteq fhfT,$$

so we get $fhfT = fT$. By Lemma 3.4, u is invertible in S. $\square$

Theorem 3.6. *Let M, N be right $R-$modules.*
(1) If $f \in End_R(M)$, then $f^{(1,2)}_{Im(f),Ker(f)}$ exists if and only if $f^\sharp$ exists. Moreover, $f^\sharp = f^{(1,2)}_{Im(f),Ker(f)}$.
(2) If $f \in Hom_R(M, N)$ and $*$ is an involution on the homomorphisms of modules, then $f^{(1,2)}_{Im(f^*),Ker(f^*)}$ exists if and only if $f^\dagger$ exists. Moreover, $f^\dagger = f^{(1,2)}_{Im(f^*),Ker(f^*)}$

Proof. (1). By Theorem 2.11, it is sufficient to show that the existence of $f^{(1,2)}_{Im(f),Ker(f)}$ implies existence of $f^\sharp$. Then, by Theorem 2.10, $f^{(1,2)}_{Im(f),Ker(f)} = f(f^2)^\sharp = (f^2)^\sharp f$, and so $ff^{(1,2)}_{Im(f),Ker(f)} = f^{(1,2)}_{Im(f),Ker(f)}f$. Hence, $f^{(1,2)}_{Im(f),Ker(f)}$ is the group inverse of f.

(2). To show that existence of $f^{(1,2)}_{Im(f^*),Ker(f^*)}$ implies existence of $f^\dagger$, take $h = f^*$ as in Theorem 2.10. Then $f^{(1,2)}_{Im(f^*),Ker(f^*)} = f^*(ff^*)^\sharp = (ff^*)^\sharp f^*$. This implies that

$$(ff^{(1,2)}_{Im(f^*),Ker(f^*)})^* = ff^{(1,2)}_{Im(f^*),Ker(f^*)} \text{ and } (f^{(1,2)}_{Im(f^*),Ker(f^*)}f)^* = f^{(1,2)}_{Im(f^*),Ker(f^*)}f.$$

Hence $f^{(1,2)}_{Im(f^*),Ker(f^*)}$ is the Moore-Penrose inverse of f. Conversely, it follows from Theorem 2.11. $\square$

Acknowledgments. The authors are grateful to the referees and Shen Guan for their very useful and detailed comments and suggestions which greatly improve the presentation.

References

[1] F. W. Anderson, K. R. Fuller, Rings and categories of modules (2nd edition), Springer-Verlag, Berlin, New York, Heidelberg, 2004.
[2] E. P. Armendariz, J. W. Fisher, R. L. Snider, On injective and surjective endomorphisms of finitely generated modules, Comm. Algebra 6 (1978) 659-672.
[3] A. Ben-Israel, T. N. E. Greville, Generalized inverses: Theory and applications (2nd edition), Springer-Verlag, New York Heidelberg Berlin, 2003.
[4] D. L. Davis, D. W. Robinson, Generalized inverses of morphisms, Linear Algebra Appl. 5 (1972) 319-328.
[5] M. P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506-514.
[6] J. M. Miao, D. W. Robinson, Group and Moore-Penrose inverses of regular morphisms with kernel and cokernel, Linear Algebra Appl. 110 (1988) 263-270.
[7] W. K. Nicholson, Strongly clean rings and Fitting's lemma, Comm. Algebra 27 (1999) 3583-3592.
[8] R. Puystjen, R. E. Hartwig, The group inverse of a companion matrix, Linear Multilinear Algebra 43 (1997) 137-150.
[9] R. Puystjens, D.W. Robinson, The Moore-Penrose inverse of a morphism in additive category, Comm. Algebra 12(3) (1984) 287-299.
[10] D. W. Robinson, R. Puystjens, Generalized inverses of morphisms with kernels, Linear Algebra Appl. 96 (1987) 65-85.
[11] H. You, J. L. Chen, Generalized inverses of a sum of morphisms, Linear Algebra Appl. 338 (2001) 261-273.
[12] G. R. Wang, Y. M. Wei, S. Qiao, Generalized inverses: Theory and computations, Science Press, Beijing/New York, 2004.
[13] Y. M. Yu, G. R. Wang, The generalized inverse $A^{(2)}_{T,S}$ over commutative rings, Linear Multilinear Algebra 53 (2005) 293-302.
[14] Y. M. Yu, G. R. Wang, The generalized inverse $A^{(2)}_{T,S}$ of a matrix over an associative ring, J. Aust. Math. Soc. 83 (2007) 423-437.
[15] Y. M. Wei, A characterization and representation of the generalized inverse $A^{(2)}_{T,S}$ amd its applications, Linear Algebra Appl. 280 (1998) 97-96.