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Counting Fuzzy Subgroups of Some Finite Groups by a New Equivalence Relation

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Abstract. The study concerning the classification of the fuzzy subgroups of finite groups is a significant aspect of fuzzy group theory. In early papers, the number of distinct fuzzy subgroups of some non-abelian groups is calculated by the natural equivalence relation. In this paper, we treat to classifying fuzzy subgroups of some groups by a new equivalence relation which has a consistent group theoretical foundation. In fact, we determine exact number of fuzzy subgroups of finite non-abelian groups of order p^3 and special classes of dihedral groups.

1. Introduction

In 1965, Zadeh introduced a fuzzy subset as a function from a non-empty set to a closed unit interval [24]. Then, the theory of fuzzy sets developed in many directions and found applications in a wide variety of fields. In [16], Rosenfeld used this concept to develop the theory of fuzzy groups. Since the notion of fuzzy group is a generalization of the notion of group, many basic properties in group theory extended to fuzzy groups, for more details we refer the reader to [1, 12, 13]. But it is expected that all the results of classical group theory have not their counterparts in the theory of fuzzy groups. Moreover, certain simple results of classical group theory when extended to fuzzy setting do not admit of a trivial proof. In [9], authors discussed some group theoretic facts which can not be extended to the fuzzy setting, in general. For example, they investigated the conditions under which a given fuzzy subgroup of a group G can or can not be realized as a union of two proper fuzzy subgroups. In [15], Ray introduced the notion of isomorphic fuzzy groups and showed that the definition of this notion fulfils the basic requirement that the fuzzy subgroups of two isomorphic groups turns out to be pairwise isomorphic.

The idea of level subsets introduced in [24] and was a useful tool to develop and formulate many results in fuzzy set theory and their applications. Das used this notion to define the notion of a level subgroup of a given fuzzy subgroup [4]. He proved that the family of level subgroups of the given fuzzy subgroups forms a chain. This result permits one to perceive the theory of fuzzy groups within the classical group theory. Also, he showed that the non-empty level subsets of a fuzzy subgroups of a group G are subgroups of G in the ordinary sense [4].

The study of classification of fuzzy subgroups of finite groups is considered as a fundamental general problem and has undergone a rapid development, in the recent years. Many papers have treated the

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classification of the fuzzy subgroups for particular cases of groups with respect to some equivalence relations. The behavior of different equivalence classes and a comparison between some equivalence relations on fuzzy subgroups lattice of a group is given in some papers, for more details, see [2, 10, 11, 14, 18, 22].

Starting point for this discussion is given by the paper [25], where Zhang and Zou considered the number of equivalence classes of fuzzy subgroups of a group G. In [14], for studying the equivalence of fuzzy subgroups, Murali and Makamba have used the equivalence relation $\sim_M$ on the set of all the fuzzy subsets of G that was defined as follows: for two fuzzy subgroups μ and η of G, $\mu \sim_M \eta$ if and only if for all $x, y \in X$, $\mu(x) > \mu(y)$ if and only if $\eta(x) > \eta(y)$ and $\mu(x) = 0$ if and only if $\eta(x) = 0$. They studied the conditions under which the equivalence relation of fuzzy sets can be equivalently described by their level sets. As the equivalence of fuzzy subgroups can be used in the classification of fuzzy groups of some special cyclic groups [14], this method should be examined to determine whether it can be more generally applied. In [8], authors by some examples, showed that the equivalence relation discussed in [14] is only suitable in dealing with the classification of the fuzzy subgroups of a finite group. Afterwards, they proposed a new definition called strong equivalence relation of fuzzy subgroups, briefly S^*-equivalence, showing that any two fuzzy subgroups of an infinite cyclic group with infinite membership values are S^*-equivalence.

Recall here the paper [18], where a natural relation $\sim$ is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: finite cyclic groups and finite elementary abelian p-groups. The equivalence relation $\sim$ develops the equivalence relation in Murali's papers. Also, the above study has been extended to some remarkable classes of non-abelian groups: dihedral groups, symmetric groups, hamiltonian groups, finite p-groups having a cyclic maximal subgroup, dicyclic groups and non-abelian groups of order p^3 and 2^4 in [3, 5–7, 19–21].

Recently, Tărnăuceanu has treated the problem of classifying the fuzzy subgroups of a finite group by a new equivalence relation $\approx$ introduced in [22]. In the present paper we study in order to give a partial answer to the open problem posed in [22]. In fact, we obtain exact number of fuzzy subgroups of the particular cases of dihedral groups and all non-abelian groups of order p^3.

The paper is organized as follows: in Section 2 we present some preliminary results on fuzzy subgroups and recall the main theorems. Section 3 deals with counting the number of distinct fuzzy subgroups of special classes of dihedral groups by using the new equivalence relation $\approx$. In Section 4, we consider the non-abelian groups of order p^3 and calculate the number of their distinct fuzzy subgroups. In the final section, some conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here.

2. Preliminaries

In this section, we recall some necessary definitions and notations, for more details, see [13, 16, 22].

The set $FL(G)$ consisting of all fuzzy subgroups of G forms a lattice with respect to fuzzy set inclusion. For all $t \in [0, 1]$, the level subset corresponding to t is defined as $U(\mu, t) = \{x \in G \mid \mu(x) \geq t\}$. These subsets are useful in the characterization of fuzzy subgroups, in the next manner: let μ be a fuzzy subset of a group G. Then, μ is a fuzzy subgroup of G if and only if its non-empty level subsets are subgroups of G. Without any equivalence on fuzzy subsets of a set, the number of fuzzy subgroups of a finite group is infinite even for the trivial group $\{e\}$. Therefore, the fuzzy subgroups of G must be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of G. One of them, denoted by $\sim$, is introduced in [23] as follows: let μ and η be two fuzzy subsets of G. Then, we define

$$\mu \sim \eta \text{ if and only if } (\mu(x) > \mu(y) \iff \eta(x) > \eta(y), \quad \text{for all } x, y \in G);$$

and two fuzzy subgroups μ and η of G will be called distinct if $\mu \nsim \eta$. In fact, a necessary and sufficient condition for fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ is they determine the same chain of subgroups of G which ends in G. According to this equivalence relation, for counting all distinct fuzzy subgroups of G with respect to $\sim$ it is sufficient to find the number of all chains of subgroups of G that terminate in G and this number is denoted by $F(G)$.

Now, consider the equivalence relation introduced in [22] which is defined on $FL(G)$ of a finite group G as follows: suppose that ρ is the following action of $Aut(G)$ on $FL(G)$:

$$\rho : FL(G) \times Aut(G) \longrightarrow FL(G), \text{ such that } \rho(\mu, f) = \mu \circ f, \text{ for all } (\mu, f) \in FL(G) \times Aut(G).$$

We denote by $\approx_\rho$ the equivalence relation on $FL(G)$ induced by ρ, namely

$$\mu \approx_\rho \eta \text{ if and only if there exists } f \in Aut(G) \text{ such that } \eta = \mu \circ f.$$

The above action ρ can be described in terms of chains of subgroups of G (for more details, see [22]). Denote by $\bar{C}$ the set consisting of all chains of subgroups of G terminating in G. Then, the action ρ of $Aut(G)$ on $FL(G)$ can be seen as an action of $Aut(G)$ on $\bar{C}$ and $\approx_\rho$ as the equivalence relation induced by this action.

Also, there is a equivalence relation $\approx$ on $FL(G)$ which is described with chains of subgroups of G as follows [22]: let $\mu, \eta \in FL(G)$ and define $\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, $\eta(G) = \{\beta_1, \beta_2, \ldots, \beta_m\}$ such that $\beta_1 > \beta_2 > \ldots > \beta_m$. Then, μ and η determine the following chains of subgroups of G which ends in G:

$$C_\mu : U(\mu, \alpha_1) \subset U(\mu, \alpha_2) \subset \ldots \subset U(\mu, \alpha_n) = G \text{ and } C_\eta : U(\eta, \beta_1) \subset U(\eta, \beta_2) \subset \ldots \subset U(\eta, \beta_m) = G.$$

The equivalence relation $\approx$ on $FL(G)$ is defined as follows:

$$\mu \approx \eta \text{ iff } \exists f \in Aut(G) \text{ such that } f(C_\eta) = C_\mu.$$

Obviously, this is a little more general as $\approx_\rho$. In fact, if $\mu \approx \eta$, then their images are not necessarily equal, but certainly there is a bijection between $Im(\mu)$ and $Im(\eta)$. Moreover, we also remark that $\approx$ generalizes the equivalence relation $\sim$ defined in [18], excepting the case when G is cyclic which we have $\approx = \sim$.

Next, we will focus on computing the number $\mathcal{N}$ of distinct fuzzy subgroups of G with respect to $\approx$, that is the number of distinct equivalence classes of $FL(G)$ modulo $\approx$. An equivalence relation on $\bar{C}$ which is similar with $\approx$ can also be constructed in the following manner: for two chains

$$C_1 : H_1 \subset H_2 \subset \cdots H_m = G \text{ and } C_2 : K_1 \subset K_2 \subset \cdots \subseteq K_n = G$$

of $\bar{C}$, we put

$$C_1 \approx C_2 \text{ iff } m = n \text{ and } \exists f \in Aut(G) \text{ such that } f(H_i) = K_i,\ 1 \le i \le n.$$

In this case the orbit of a chain $C \in \bar{C}$ is $\{f(C) \mid f \in Aut(G)\}$, while the set of all chains in $\bar{C}$ that are fixed by an automorphism f of G is $Fix_{\bar{C}}(f) = \{C \in \bar{C} \mid f(C) = C\}$. Now, the Burnside's lemma leads to the following theorem:

Theorem 2.1. *The number $\mathcal{N}$ of all distinct fuzzy subgroups with respect to $\approx$ of a finite group G is given by the equality*

$$\mathcal{N} = \frac{1}{|Aut(G)|} \sum_{f \in Aut(G)} \left| Fix_{\bar{C}}(f) \right|.$$

Finally, we note that the above formula can successfully be used to calculate $\mathcal{N}$ for any finite group G whose subgroup lattice $L(G)$ and automorphism group $Aut(G)$ are known.

3. The number of fuzzy subgroups in some particular cases of dihedral groups

In this section, we treat to the counting the number of all distinct fuzzy subgroups relative to $\approx$ for two particular cases of dihedral groups.

It is well-known that the finite dihedral group D_{2n} ($n \ge 2$) can be obtained by using its generators: a rotation x of order n and a reflection y of order 2. Under these notations, the presentation of D_{2n} is $D_{2n} =< x, y \mid x^n = y^2 = e, y^{-1}xy = x^{-1} >$. The automorphism group of D_{2n} is [17]:

$$Aut(D_{2n}) = \{f_{\alpha,\beta} \mid 0 \le \alpha \le n-1 \text{ such that } (\alpha, n) = 1, 0 \le \beta \le n-1\},$$

where $f_{\alpha,\beta} : D_{2n} \longrightarrow D_{2n}$ is defined by $f_{\alpha,\beta}(x) = x^{\alpha}$ and $f_{\alpha,\beta}(y) = x^{\beta}y$. Consequently, $|Aut(D_{2n})| = n\varphi(n)$, where φ is Euler's Totient function.

Suppose that $f_{\alpha,\beta} \in Aut(D_{2n})$, then by $Fix(f_{\alpha,\beta})$, we denote the set consisting of all subgroups of D_{2n} that are invariant relative to $f_{\alpha,\beta}$, that is $Fix(f_{\alpha,\beta}) = \{H \leq D_{2p} \mid f_{\alpha,\beta}(H) = H\}$. It is clear that for every divisor r of n, the subgroup $H_0^r =< x^{\frac{n}{r}} >\cong \mathbb{Z}_r$ of D_{2n} belongs to $Fix(f_{\alpha,\beta})$. Also for $1 \leq i \leq \frac{n}{r}$, the subgroup $H_i^r =< x^{\frac{n}{r}}, x^{i-1}y >\cong D_{2r}$ belongs to $Fix(f_{\alpha,\beta})$ if and only if $\frac{n}{r}$ divides $(\alpha - 1)(i - 1) + \beta$ [17].

3.1. Classifying fuzzy subgroups of Dihedral group of order p

Consider $D_{2p} =< x, y \mid x^p = y^2 = e, y^{-1}xy = x^{-1} >$, where p is an odd prime number. As we already have seen, $|Aut(D_{2p})| = p(p-1)$, more precisely $Aut(D_{2p}) = \{f_{\alpha,\beta} \mid 1 \leq \alpha \leq p-1, 0 \leq \beta \leq p-1\}$.

All subgroups of D_{2p} are $H_0^1 = \{e\}$, $H_1^p = D_{2p}$, $H_0^p =< x >\cong \mathbb{Z}_p$, $H_i^1 =< x^{i-1}y >\cong \mathbb{Z}_2$, where $1 \leq i \leq p$.

By definition $Fix(f_{\alpha,\beta})$, the corresponding sets of subgroups of D_{2p} that are invariant relative to the above automorphisms can be described by a direct computation as follows:

$$
\begin{aligned}
&Fix(f_{1,0}) = L(D_{2p}); \quad Fix(f_{1,j}) = \{H_0^1, H_0^p, H_1^p\}, \text{ where } 1 \leq j \leq p-1;\\
&Fix(f_{i,0}) = \{H_0^1, H_0^p, H_1^1, H_1^p\}, \text{ where } 2 \leq i \leq p-1;\\
&Fix(f_{2,p-1}) = Fix(f_{3,p-2}) = Fix(f_{4,p-3}) = \cdots = Fix(f_{p-1,2}) = \{H_0^1, H_0^p, H_2^1, H_1^p\};\\
&Fix(f_{2,p-2}) = Fix(f_{3,p-4}) = Fix(f_{4,p-6}) = \cdots = Fix(f_{p-1,4}) = \{H_0^1, H_0^p, H_3^1, H_1^p\};\\
&Fix(f_{2,p-3}) = Fix(f_{3,p-6}) = Fix(f_{4,p-9}) = \cdots = Fix(f_{p-1,6}) = \{H_0^1, H_0^p, H_4^1, H_1^p\};\\
&\vdots\\
&Fix(f_{2,2}) = Fix(f_{3,4}) = Fix(f_{4,6}) = \cdots = Fix(f_{p-1,p-4}) = \{H_0^1, H_0^p, H_{p-1}^1, H_1^p\};\\
&Fix(f_{2,1}) = Fix(f_{3,2}) = Fix(f_{4,3}) = \cdots = Fix(f_{p-1,p-2}) = \{H_0^1, H_0^p, H_p^1, H_1^p\}.
\end{aligned}
$$

All chains of subgroups of G that terminate in G are [19]:

$$
\begin{aligned}
&C_0 : H_1^1 =< y >\subseteq H_1^p = D_{2p};\\
&C_1 : H_2^1 =< xy >\subseteq H_1^p = D_{2p};\\
&\vdots\\
&C_{p-1} : H_p^1 =< x^{p-1}y >\subseteq H_1^p = D_{2p};\\
&C_p : H_0^p =< x >\subseteq H_1^p = D_{2p};\\
&C_{p+1} : H_1^p = D_{2p}.
\end{aligned}
$$

For $0 \leq i \leq p+1$, the chain C_{0i} is obtained by adding $H_0^1 = \{e\}$ to the end of chain C_i. By using some elementary computation, we find that

$$
\begin{aligned}
&Fix_{\bar{C}}(f_{1,0}) = \{C_i, C_{0i} \mid 0 \leq i \leq p+1\};\\
&Fix_{\bar{C}}(f_{1,j}) = \{C_p, C_{p+1}, C_{0p}, C_{0(p+1)}\}, \text{ where } 1 \leq j \leq p-1;\\
&Fix_{\bar{C}}(f_{i,0}) = \{C_0, C_p, C_{p+1}, C_{00}, C_{0p}, C_{0(p+1)}\}, \text{ where } 2 \leq i \leq p-1;\\
&Fix_{\bar{C}}(f_{2,p-1}) = Fix_{\bar{C}}(f_{3,p-2}) = \cdots = Fix_{\bar{C}}(f_{p-1,2}) = \{C_1, C_p, C_{p+1}, C_{01}, C_{0p}, C_{0(p+1)}\};\\
&Fix_{\bar{C}}(f_{2,p-2}) = Fix_{\bar{C}}(f_{3,p-4}) = \cdots = Fix_{\bar{C}}(f_{p-1,4}) = \{C_2, C_p, C_{p+1}, C_{02}, C_{0p}, C_{0(p+1)}\};\\
&Fix_{\bar{C}}(f_{2,p-3}) = Fix_{\bar{C}}(f_{3,p-6}) = \cdots = Fix_{\bar{C}}(f_{p-1,6}) = \{C_3, C_p, C_{p+1}, C_{03}, C_{0p}, C_{0(p+1)}\};\\
&\vdots\\
&Fix_{\bar{C}}(f_{2,2}) = Fix_{\bar{C}}(f_{3,4}) = \cdots = Fix_{\bar{C}}(f_{p-1,p-4}) = \{C_{p-2}, C_p, C_{p+1}, C_{0(p-2)}, C_{0p}, C_{0(p+1)}\};\\
&Fix_{\bar{C}}(f_{2,1}) = Fix_{\bar{C}}(f_{3,2}) = \cdots = Fix_{\bar{C}}(f_{p-1,p-2}) = \{C_{p-1}, C_p, C_{p+1}, C_{0(p-1)}, C_{0p}, C_{0(p+1)}\}.
\end{aligned}
$$

According to Theorem 2.1, we conclude that $\mathcal{N} = \frac{1}{p(p-1)}(2p + 4 + 4(p-1) + 6p(p-2)) = 6$.

3.2. Classifying fuzzy subgroups of Dihedral group of order p_1p_2

Consider $D_{2p_1p_2} =< x, y \mid x^{p_1p_2} = y^2 = e, y^{-1}xy = x^{-1} >$, where p_1 and p_2 are distinct prime numbers. As we already have seen, $|Aut(D_{2p_1p_2})| = p_1p_2(p_1p_2 - p_1 - p_2 + 1)$, more precisely

$$Aut(D_{2p_1p_2}) = \{f_{\alpha,\beta} \mid 1 \leq \alpha \leq p_1p_2 - 1, (\alpha, \beta) = 1, 0 \leq \beta \leq p_1p_2 - 1\}.$$

The structure of the subgroup lattice of $D_{2p_1p_2}$ is as follows: $H_0^1 = \{e\}$, $H_0^{p_1} =< x^{p_2} >\cong \mathbb{Z}_{p_1}$, $H_0^{p_2} =< x^{p_1} >\cong \mathbb{Z}_{p_2}$, $H_0^{p_1p_2} =< x >\cong \mathbb{Z}_{p_1p_2}$, $H_1^{p_1p_2} = D_{2p_1p_2}$, $H_i^1 =< x^{i-1}y >\cong \mathbb{Z}_2$, where $1 \le i \le p_1p_2$, $H_i^{p_1} =< x^{p_2}, x^{i-1}y >\cong D_{2p_1}$, where $1 \le i \le p_2$ and $H_i^{p_2} =< x^{p_1}, x^{i-1}y >\cong D_{2p_2}$, where $1 \le i \le p_1$.

All chains of subgroups of $D_{2p_1p_2}$ that terminate in $D_{2p_1p_2}$ are [19]:

$C_1 : D_{2p_1p_2}$;
$C_2 : H_0^{p_1} \subset D_{2p_1p_2}$;
$C_3 : H_0^{p_2} \subset D_{2p_1p_2}$;
$C_i : H_{i-3}^{p_1} \subset D_{2p_1p_2}, 4 \le i \le p_2 + 3$;
$C_i : H_{i-p_2-3}^{p_2} \subset D_{2p_1p_2}, p_2 + 4 \le i \le p_1 + p_2 + 3$;
$C'_i : H_0^{p_2} \subset H_{i-p_2-3}^{p_2} \subset D_{2p_1p_2}, p_2 + 4 \le i \le p_1 + p_2 + 3$;
$C_i : H_{i-p_1-p_2-3}^1 \subset D_{2p_1p_2}, p_1 + p_2 + 4 \le i \le p_1p_2 + p_1 + p_2 + 3$;
$C_{p_1ij} : H_{i+p_2j}^1 \subset H_i^{p_1} \subset D_{2p_1p_2}, 1 \le i \le p_2, 0 \le j \le p_1 - 1$;
$C_{p_2ij} : H_{i+p_1j}^1 \subset H_i^{p_2} \subset D_{2p_1p_2}, 1 \le i \le p_1, 0 \le j \le p_2 - 1$.

$C'_1 : H_0^{p_1p_2} \subset D_{2p_1p_2}$;
$C'_2 : H_0^{p_1} \subset H_0^{p_1p_2} \subset D_{2p_1p_2}$;
$C'_3 : H_0^{p_2} \subset H_0^{p_1p_2} \subset D_{2p_1p_2}$;
$C'_i : H_0^{p_1} \subset H_{i-3}^{p_1} \subset D_{2p_1p_2}, 4 \le i \le p_2 + 3$;

Also, for $1 \le i \le p_1p_2 + p_1 + p_2 + 3$ and $1 \le j \le p_1 + p_2 + 3$, the chains C_{0i} and C'_{0j} are obtained by adding $H_0^1 = \{e\}$ to the end of chains C_i and C'_j, respectively. Similarly, the chains C_{0p_1ij} and C_{0p_2ij} are constructing of chains C_{p_1ij} and C_{p_2ij}, respectively.

Clearly, $H_0^1, H_0^{p_1}, H_0^{p_2}, H_0^{p_1p_2}, H_1^{p_1p_2} \in Fix(f_{\alpha,\beta})$, for all $f_{\alpha,\beta} \in Aut(D_{2p_1p_2})$. Hence for $i = 1, 2, 3$, the chains C_i and C'_i are invariant relative to every element of $Aut(D_{2p_1p_2})$. By a direct computation, the following results are obtained.

For $4 \le i \le p_2 + 3$, the subgroup $H_{i-3}^{p_1}$ is invariant relative to $\varphi(p_1p_2)p_1$ elements of $Aut(D_{2p_1p_2})$. These isomorphisms preserve the chains C_i and C'_i, where $4 \le i \le p_2 + 3$.

Also, for $p_2 + 4 \le i \le p_1 + p_2 + 3$, the subgroup $H_{i-p_2-3}^{p_2}$ is invariant relative to $\varphi(p_1p_2)p_2$ elements of $Aut(D_{2p_1p_2})$ which leads to chains C_i and C'_i are invariant relative to these isomorphisms, where $p_2 + 4 \le i \le p_1 + p_2 + 3$.

Suppose that $p_1 + p_2 + 4 \le i \le p_1p_2 + p_1 + p_2 + 3$ and $0 \le \alpha \le p_1p_2 - 1$ with $(\alpha, n) = 1$. Then, there is an unique $0 \le \beta \le p_1p_2 - 1$ such that $H_{i-p_1-p_2-3}^1 \in Fix(f_{\alpha,\beta})$. This implies that $\varphi(p_1p_2)$ elements of $Aut(D_{2p_1p_2})$ preserve $H_{i-p_1-p_2-3}^1$. Thus, for $p_1 + p_2 + 4 \le i \le p_1p_2 + p_1 + p_2 + 3$, the chain C_i is invariant relative to these isomorphisms.

For $1 \le i \le p_2$ and $0 \le j \le p_1 - 1$, the chain C_{p_1ij} consists the subgroups $H_i^{p_1}$ and $H_{i+p_2j}^1$. Furthermore, if $H_{i+p_2j}^1 \in Fix(f_{\alpha,\beta})$, then $H_i^{p_1} \in Fix(f_{\alpha,\beta})$, where $f_{\alpha,\beta} \in Aut(G)$. Therefore, if $H_{i+p_2j}^1 \in Fix(f_{\alpha,\beta})$, then the chain C_{p_1ij} is invariant respect to $f_{\alpha,\beta}$. This yields that $\varphi(p_1p_2)$ elements of $Aut(D_{2p_1p_2})$ preserve the chain C_{p_1ij}. In the same way, for all $1 \le i \le p_1$ and $0 \le j \le p_2 - 1$, the chain C_{p_2ij} is invariant relative to the $\varphi(p_1p_2)$ elements of $Aut(D_{2p_1p_2})$.

Note that $\sum_{f\in Aut(G)} \left|Fix_{\bar{C}}(f)\right| = \sum_{C\in\bar{C}} \left|\left\{f_{\alpha,\beta} \in Aut(G) \mid f(C) = C\right\}\right|$. So, by the above results and Theorem 2.1 we obtain

$$\begin{aligned}\mathcal{N} &= \frac{2}{\varphi(p_1p_2)p_1p_2}(6\varphi(p_1p_2)p_1p_2 + 2\varphi(p_1p_2)p_1p_2 + 2\varphi(p_1p_2)p_1p_2 + \varphi(p_1p_2)p_1p_2 \\ &\quad +\varphi(p_1p_2)p_1p_2 + \varphi(p_1p_2)p_1p_2) \\ &= \frac{2}{\varphi(p_1p_2)p_1p_2}(13\varphi(p_1p_2)p_1p_2) = 26.\end{aligned}$$

The above results lead to the following theorem.

Theorem 3.1. *For distinct prime numbers $p_1 \neq 2$, p_2, the number $\mathcal{N}$ of all distinct fuzzy subgroups of the dihedral groups D_{2p_1} and $D_{2p_1p_2}$ with respect to $\approx$ is* 6 *and* 26*, respectively.*

4. Counting fuzzy subgroups of finite groups of order p^3

In this section, we compute explicitly the number $\mathcal{N}$ of all distinct fuzzy subgroups with respect to $\approx$ for non-abelian groups of order p^3.

There are two non-abelian groups of order p^3. In the process of determining $\mathcal{N}$ for these groups, we distinguish two cases.

Case 1: $p = 2$.
In this case, these groups are the dihedral group D_8 and the quaternion group Q_8. In [22, Theorem 4.3.1], the number of all distinct fuzzy subgroups of D_8 with respect to $\approx$ is determined and we have $\mathcal{N}(D_8) = 16$.

Now, consider the quaternion group Q_8 having the presentation

$$Q_8 =< x, y \mid x^4 = e, x^2 = y^2, y^{-1}xy = x^{-1} > .$$

The automorphism group of Q_8 is well-known, namely

$$Aut(Q_8) = \{f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta} \mid \alpha = 1, 3, 0 \leq \beta \leq 3\},$$

where $f_{\alpha,\beta}, g_{\alpha,\beta}, h_{\alpha,\beta} : Q_8 \longrightarrow Q_8$ are defined by $f_{\alpha,\beta} = \begin{cases} x \longrightarrow x^{\alpha} \\ y \longrightarrow x^{\beta}y \end{cases}$,

$$g_{\alpha,\beta} = \begin{cases} x \longrightarrow x^{\alpha}y \\ y \longrightarrow x^{\beta}y, \text{ if } \beta \text{ is even} \\ y \longrightarrow x^{\beta}, \text{ if } \beta \text{ is odd} \end{cases} \quad \text{and} \quad h_{\alpha,\beta}(x) = \begin{cases} x \longrightarrow x^{\alpha-1}y \\ y \longrightarrow xy^{\beta} \end{cases}.$$

Therefore, $|Aut(Q_8)| = 24$. The subgroup lattice of Q_8 consists of the trivial subgroup $\{e\}$, three subgroups isomorphic to $\mathbb{Z}_4$, namely $H_1 =< x >, H_2 =< y >, H_3 =< xy >$, a subgroup of order 2, namely $H_4 =< x^2 >\cong \mathbb{Z}_2$ and Q_8. All chains of subgroups of G that terminate in G are [6]:

$$C_1 : Q_8; \quad C_i : H_4 \subseteq H_{i-1} \subseteq Q_8, \text{ where } 2 \leq i \leq 4; \quad C_i : H_{i-4} \subseteq Q_8, \text{ where } 5 \leq i \leq 8.$$

Also, for $1 \leq i \leq 8$ the chain C'_i is obtained by adding $\{e\}$ to the end of chain C_i. By using some elementary computation, we get $Fix_{\bar{C}}(g_{j,k}) = Fix_{\bar{C}}(h_{j,k}) = \{C_i, C'_i \mid i = 1, 8\}$, where $j, k = 1, 3$, and

$$\begin{aligned}
&Fix_{\bar{C}}(f_{1,0}) = Fix_{\bar{C}}(f_{1,2}) = Fix_{\bar{C}}(f_{3,0}) = Fix_{\bar{C}}(f_{3,2}) = \{C_i, C'_i \mid 1 \leq i \leq 8\};\\
&Fix_{\bar{C}}(f_{1,1}) = Fix_{\bar{C}}(f_{1,3}) = Fix_{\bar{C}}(f_{3,1}) = Fix_{\bar{C}}(f_{3,3}) = \{C_i, C'_i \mid i = 1, 2, 5, 8\};\\
&Fix_{\bar{C}}(g_{1,0}) = Fix_{\bar{C}}(g_{1,2}) = Fix_{\bar{C}}(g_{3,0}) = Fix_{\bar{C}}(g_{3,2}) = \{C_i, C'_i \mid i = 1, 3, 6, 8\};\\
&Fix_{\bar{C}}(h_{1,0}) = Fix_{\bar{C}}(h_{1,2}) = Fix_{\bar{C}}(h_{3,0}) = Fix_{\bar{C}}(h_{3,2}) = \{C_i, C'_i \mid i = 1, 4, 7, 8\}.
\end{aligned}$$

By applying Theorem 2.1, we conclude that $\mathcal{N} = \frac{2}{24}(8 \cdot 2 + 4 \cdot 8 + 12 \cdot 4) = 8$.

Theorem 4.1. *The number $\mathcal{N}$ of all distinct fuzzy subgroups with respect to $\approx$ of the quaternion group Q_8 is equal to* 8.

Case 2: p is an odd prime number.
In this case, the finite non-abelian groups of order p^3 are $G_{1p^3} = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_p$ and $G_{2p^3} = (\mathbb{Z}_p \times \mathbb{Z}_p) \ltimes \mathbb{Z}_p$. Our next goal is to determine the exact number $\mathcal{N}$ for these groups.

4.1. *Classifying fuzzy subgroups of G_{1p^3}*

The representation of $G_{1p^3} = \mathbb{Z}_{p^2} \ltimes \mathbb{Z}_p$ is as follows:

$$G_{1p^3} =< x, y \mid x^{p^2} = y^p = e, \ y^{-1}xy = x^{p+1} > .$$

For $0 \le i, j \le p-1$, the order of every element $x^{pi}y^j$ excepting identity is p. Consequently, G_{1p^3} includes p^2-1 elements of order p and p^3-p^2 elements of order p^2.

In [5, Section 2], the number of all distinct fuzzy subgroups with respect to $\sim$ of G_{1p^3} is given by the equality $F(G_{1p^3}) = 8p+8$. This number is based on maximal subgroups of G_{1p^3} which are qualified as follows: $M_0 =< x >= \{x^{pi+j} \mid 0 \le i, j \le p-1\} \cong \mathbb{Z}_{p^2}$, $M_k =< x^k y >= \{x^{pi+kj}y^j \mid 0 \le i, j \le p-1\} \cong \mathbb{Z}_{p^2}$, where $1 \le k \le p-1$ and $M_p =< x^p, y >= \{x^{pi}y^j \mid 0 \le i, j \le p-1\} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Other nontrivial subgroups of G_{1p^3} are $N_k =< x^{pk}y >= \{x^{pki}y^i \mid 0 \le i \le p-1\} \cong \mathbb{Z}_p$, where $0 \le k \le p-1$ and $N_p =< x^p >\cong \mathbb{Z}_p$. Then, all chains of subgroups of G that terminate in G are [5]:

$$\begin{array}{ll} C_i : M_i \subseteq G_{1p^3}; & C_{0i} : N_p \subset M_i \subseteq G_{1p^3}, \text{ where } 0 \le i \le p; \\ C_{p+i} : N_{i-1} \subset G_{1p^3}; & C_{0(p+i)} : N_{i-1} \subset M_p \subset G_{1p^3}, \text{ where } 1 \le i \le p; \\ C_{2p+1} : N_p \subset G_{1p^3}; & C_{0(2p+1)} : G. \end{array}$$

Also, for $0 \le i \le 2p+1$, the chains C'_i and C'_{0i} are obtained by adding $\{e\}$ to the end of chains C_i and C_{0i}, respectively.

The automorphism group of G_{1p^3} is:

$$Aut(G_{1p^3}) = \{f_{i,j,k,t} \mid 0 \le i, k, t \le p-1,\ 1 \le j \le p-1\},$$

where $f_{i,j,k,t} = \begin{cases} x \longrightarrow x^{pi+j}y^k \\ y \longrightarrow x^{pt}y \end{cases}$. It follows that $|Aut(G_{1p^3})| = p^3(p-1)$.

Obviously, $C_r \in Fix_{\bar{C}}(f_{i,j,k,t})$ leads to C_{0r}, C'_r and C'_{0r} are invariant relative to $f_{i,j,k,t}$, where $0 \le r \le 2p+1$. Thus, it is enough to consider conditions which cause the chains C_r are invariant relative to an automorphism $f_{i,j,k,t}$. A simple verification shows that $f_{i,j,k,t}(N_p) = N_p$ and $f_{i,j,k,t}(M_p) = M_p$. Therefore, for all $0 \le i, k, t \le p-1$ and $1 \le j \le p-1$ we find that

$$\mathcal{A} = \{C_r, C_{0r}, C'_r, C'_{0r} \mid r = p, 2p+1\} \subseteq Fix_{\bar{C}}(f_{i,j,k,t}).$$

By using some elementary computation, we conclude that for all $0 \le i, k, t \le p-1$ and $2 \le j \le p-1$, there exist $0 \le r_. \le p-1$ and $1 \le s_. \le p$ such that

$$\mathcal{B} = \left\{C_{r_.}, C_{0r_.}, C'_{r_.}, C'_{0r_.}, C_{p+s_.}, C_{0(p+s_.)}, C'_{p+s_.}, C'_{0(p+s_.)}\right\} \subseteq Fix_{\bar{C}}(f_{i,j,k,t}),$$

which leads to $Fix_{\bar{C}}(f_{i,j,k,t}) = \mathcal{A} \cup \mathcal{B}$. Also, for all $0 \le i \le p-1$,

$$\begin{array}{l} Fix_{\bar{C}}(f_{i,1,0,0}) = \{C_r, C_{0r}, C'_r, C'_{0r} \mid 0 \le r \le 2p+1\}; \\ Fix_{\bar{C}}(f_{i,1,0,t}) = \mathcal{A} \cup \{C_r, C_{0r}, C'_r, C'_{0r} \mid 0 \le r \le p-1\}, \text{where } 1 \le t \le p-1; \\ Fix_{\bar{C}}(f_{i,1,k,0}) = \left\{C_{p+r}, C_{0(p+r)}, C'_{p+r}, C'_{0(p+r)} \mid 0 \le r \le p+1\right\}, \text{where } 1 \le k \le p-1; \\ Fix_{\bar{C}}(f_{i,1,k,t}) = \mathcal{A}, \text{where } 1 \le k, t \le p-1. \end{array}$$

Therefore, for all $0 \le i \le p-1$ we have

$$\begin{array}{l} \left|Fix_{\bar{C}}(f_{i,j,k,t})\right| = 16, \text{where } 0 \le k, t \le p-1 \text{ and } 2 \le j \le p-1; \\ \left|Fix_{\bar{C}}(f_{i,1,0,0})\right| = 8p+8; \\ \left|Fix_{\bar{C}}(f_{i,1,0,t})\right| = 4p+8, \text{where } 1 \le t \le p-1; \\ \left|Fix_{\bar{C}}(f_{i,1,k,0})\right| = 4p+8, \text{where } 1 \le k \le p-1; \\ \left|Fix_{\bar{C}}(f_{i,1,k,t})\right| = 8, \text{where } 1 \le k, t \le p-1. \end{array}$$

According to Theorem 2.1, the number $\mathcal{N}$ of all distinct fuzzy subgroups of the group G_{1p^3} with respect to $\approx$ is equal to

$$\mathcal{N} = \frac{1}{p^3(p-1)}\left(16p^3(p-2) + p(8p+8) + 2p(p-1)(4p+8) + 8p(p-1)^2\right) = 16.$$

4.2. Classifying fuzzy subgroups of G_{2p^3}

The representation of $G_{2p^3} = (\mathbb{Z}_p \times \mathbb{Z}_p) \ltimes \mathbb{Z}_p$ is:

$$G_{2p^3} =< x, y, z \mid x^p = y^p = z^p = e,\ xz = zx, yz = zy, y^{-1}xy = xz > .$$

According to the representation of G_{2p^3}, for every $0 \leq i, j, k \leq p-1$, we have

$$x^i y^j = y^j x^i z^{ij} \text{ and } (x^i y^j)^k = x^{ki} y^{kj} z^{-k(k-1)ij/2}, \text{ for all } k \in \mathbb{N}.$$

The above equalities show that the order of all of elements of G_{2p^3} except identity is p.

The group G_{2p^3} has $p+1$ maximal subgroups such that they are of prime index p and isomorph to $\mathbb{Z}_p \times \mathbb{Z}_p$. The structure of them is as follows:
$M_i =< x^i y, z >= \left\{(x^i y)^j z^k \mid 0 \leq j, k \leq p-1\right\}$, where $0 \leq i \leq p-1$, and $M_p =< x, z >= \left\{x^j z^k \mid 0 \leq j, k \leq p-1\right\}$.
According to these maximal subgroups, the number of all distinct fuzzy subgroups of G_{2p^3} with respect to $\sim$ is equal to $F(G_{2p^3}) = 4p^2 + 8p + 8$, [7, Section 3].

Other nontrivial subgroups of G_{2p^3} are $N_{ij} =< x^i y z^j >\cong \mathbb{Z}_p$, where $0 \leq i, j \leq p-1$, $N_{pj} =< xz^j >\cong \mathbb{Z}_p$, where $0 \leq j \leq p-1$ and $N =< z >$.

By the above results, all chains of subgroups of G_{2p^3} that terminate in G_{2p^3} are [7]:

$C_1 : G_{2p^3}$; $\quad C_2 : N \subset G_{2p^3}$;
$C_{\mathcal{U}_{1i}} : M_i \subset G_{2p^3}$; $\quad C_{\mathcal{U}_{2i}} : N \subset M_i \subset G_{2p^3}$, where $0 \leq i \leq p$;
$C_{\mathcal{V}_{1ij}} : N_{ij} \subset G_{2p^3}$; $\quad C_{\mathcal{V}_{2ij}} : N_{ij} \subset M_i \subset G_{2p^3}$, where $0 \leq i \leq p$ and $0 \leq j \leq p-1$.

Also, for $0 \leq i \leq p$, $0 \leq j \leq p-1$ and $k = 1, 2$, the chains C'_k, $C'_{\mathcal{U}_{ki}}$ and $C'_{\mathcal{V}_{kij}}$ are obtained by adding $\{e\}$ to the end of chains C_k, $C_{\mathcal{U}_{ki}}$ and $C_{\mathcal{V}_{kij}}$, respectively.

The automorphism group of G_{2p^3} is:

$$Aut(G_{2p^3}) = \{f_{a,b}^{j,r,s}, g_{a,b,m}^{j,r,s}, h_{a,b}^{j,r,s} \mid 1 \leq a, b, m \leq p-1, 0 \leq j, r, s \leq p-1\},$$

where $f_{a,b}^{j,r,s} = \begin{cases} x \longrightarrow x^a z^j \\ y \longrightarrow x^r y^b z^s \\ z \longrightarrow z^{ab} \end{cases}$, $g_{a,b,m}^{j,r,s} = \begin{cases} x \longrightarrow (xy^m)^a z^j \\ y \longrightarrow (xy^m)^r y^b z^s \\ z \longrightarrow z^{ab} \end{cases}$ and $h_{a,b}^{j,r,s} = \begin{cases} x \longrightarrow y^a z^j \\ y \longrightarrow x^b y^r z^s \\ z \longrightarrow z^{-ab} \end{cases}$. Consequently, $|Aut(G_{2p^3})| = p^3(p-1)^2(p+1)$.

It is clear that every automorphism of G_{2p^3} is preserving the subgroup N. Consequently, for $k = 1, 2$, the chains C_k and C'_k are invariant relative to the every automorphism of G_{2p^3} i.e., $\{C_1, C_2, C'_1, C'_2\} \subseteq Fix_{\bar{C}}(\xi)$, where $\xi \in Aut(G_{2p^3})$.

Note that for $\xi \in Aut(G_{2p^3})$, it is difficult to determine the size of $Fix_{\bar{C}}(\xi)$. Therefore, by the equality $\sum_{\xi \in Aut(G_{2p^3})} \left|Fix_{\bar{C}}(\xi)\right| = \sum_{C \in \bar{C}} \left|\{\xi \in Aut(G) \mid f(C) = C\}\right|$ and Theorem 2.1, we apply the following equality for determining $\mathcal{N}$:

$$\mathcal{N} = \frac{1}{|Aut(G_{2p^3})|} \sum_{C \in \bar{C}} \left|\left\{\xi \in Aut(G_{2p^3}) \mid \xi(C) = C\right\}\right|.$$

In order to do this, suppose that $\mathcal{A}_{a,b} = \{f_{a,b}^{j,r,s} \mid 0 \leq j, r, s \leq p-1\}$, $\mathcal{B}_{a,b,m} = \{g_{a,b,m}^{j,r,s} \mid 0 \leq j, r, s \leq p-1\}$ and $\mathcal{D}_{a,b} = \{h_{a,b,m}^{j,r,s} \mid 0 \leq j, r, s \leq p-1\}$, where $1 \leq a, b, m \leq p-1$.

Take $k = 1, 2$ and consider the following table which is derived by direct computation. In this table the first column presents the number of automorphisms of G_{2p^3} which preserve the chains of the second column.

Table 1: The description of the number of automorphisms of G_{2p^3}

Number	Chains
All elements of $Aut(G_{2p^3})$,	C_k, C'_k,
All elements of $\mathcal{A}_{a,b}$,	$C_{\mathcal{U}_{kp}}, C'_{\mathcal{U}_{kp}}$,
p^2 elements of $\mathcal{A}_{a,b}$,	$C_{\mathcal{V}_{kpj}}, C'_{\mathcal{V}_{kpj}}$, where $0 \leq j \leq p-1$,
p^2 elements of $\mathcal{A}_{a,b}$,	$C_{\mathcal{U}_{kj}}, C'_{\mathcal{U}_{kj}}$, where $0 \leq j \leq p-1$,
p elements of $\mathcal{A}_{a,b}$,	$C_{\mathcal{V}_{kij}}, C'_{\mathcal{V}_{kij}}$, where $0 \leq i, j \leq p-1$,
p^2 elements of $\mathcal{B}_{a,b,m}$,	$C_{\mathcal{U}_{ki}}, C'_{\mathcal{U}_{ki}}$, where $0 \leq i \leq p-1$, $i \neq m$,
p elements of $\mathcal{B}_{a,b,m}$,	$C_{\mathcal{V}_{kij}}, C'_{\mathcal{V}_{kij}}$, where $0 \leq i, j \leq p-1$, $i \neq m$,
p^2 elements of $\mathcal{D}_{a,b}$,	$C_{\mathcal{U}_{ki}}, C'_{\mathcal{U}_{ki}}$, where $1 \leq i \leq p-1$,
p elements of $\mathcal{D}_{a,b}$,	$C_{\mathcal{V}_{kij}}, C'_{\mathcal{V}_{kij}}$, where $1 \leq i \leq p-1$ and $0 \leq j \leq p-1$.

By the above results, we conclude that

$$\sum_{j=0}^{p-1}\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}\left|Fix_{\bar{C}}(g_{a,b,m}^{j,r,s})\right| = \sum_{j=0}^{p-1}\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}\left|Fix_{\bar{C}}(h_{a,b}^{j,r,s})\right| = 4(3p-2)p^2$$

and

$$\sum_{j=0}^{p-1}\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}\left|Fix_{\bar{C}}(f_{a,b}^{j,r,s})\right| = 20p^3.$$

Therefore,

$$\mathcal{N} = \frac{4}{p^3(p-1)^2(p+1)}\Big((p-1)^2p^3(3p-2) + (p-1)^2(5p^3)\Big) = 12.$$

Then, we get the following theorem:

Theorem 4.2. *For an odd prime number p, the number $\mathcal{N}$ of all distinct fuzzy subgroups of the groups $\mathbb{Z}_{p^2} \ltimes \mathbb{Z}_p$ and $(\mathbb{Z}_p \times \mathbb{Z}_p) \ltimes \mathbb{Z}_p$ with respect to $\approx$ is* 16 *and* 12*, respectively.*

5. Conclusion

The study of fuzzy subgroups of finite groups can be made with respect to some natural equivalence relations on the fuzzy subgroup lattices, as the equivalence relations introduced and used in [2, 8, 10, 11, 14, 18, 22]. Obviously, other such relations can be introduced and investigated. In this paper, we treated the classifying the fuzzy subgroups of some finite non-abelian groups, by the new equivalence relation $\approx$, introduced in [22]. In order to do this, we studied subgroup structure and automorphism group of groups of order p^3 and special classes of dihedral groups and determined the exact number of distinct fuzzy subgroups of them. Clearly, the study started in the present paper can successfully be extended to other remarkable classes of finite non-abelian groups. This will surely constitute the subject of some further research.

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