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Complexiton Solutions for Complex KdV Equation by Optimal Homotopy Asymptotic Method

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Abstract. In this article an innovative technique named as Optimal Homotopy Asymptotic Method has been explored to treat system of KdV equations computed from complex KdV equation. By developing special form of initial value problems to complex KdV equation, three different types of semi analytic complextion solutions from complex KdV equation have been achieved. First semi analytic position solution received from trigonometric form of initial value problem, second is semi analytic negation solution received by hyperbolic form of initial value problem and third one is special type of semi analytic solution expressed by the combination of trigonometric and hyperbolic functions. It was proved that only first order OHAM solution is accurate to the closed-form solution.

1. Introduction

The popular Navier-Stokes equations are used to study the dynamics of fluids in permeable or impermeable medium. Scientists and researchers are investigating different types of problems using Navier-Stokes equations. There are interesting studies employing Navier-Stokes equations which can be solved through the common techniques of integration. To this end, Benbernou [1] established a Serrin-type regularity criterion in terms of pressure for Leray weak solutions to the Navier-Stokes equations. Involving fluid flow, Gala et al. [2] presented a study that deals with the blow-up criterion for the hydrodynamic system modeling the flow of three-dimensional nematic liquid crystal materials. In another note, Gala et al. [3] considered the regularity problem under the critical condition to the Boussinesq equations with zero heat conductivity. Advance investigative research brings the challenging task in the field of engineering and applied sciences. One of the important tasks is to find out the solution of problem having high nonlinearity arising from the model occur in nature or in industrial. In the presence of advance technology and computer algebraic software like Mathematica, MATLAB, MAPLE etc. still the convergence criteria of such a complicated and high nonlinear problem towards exact form is difficult to evaluate. For this purpose, various powerful techniques have been developed since last decade. Homotopy perturbation method (HPM) [4-6], Adomian decomposition method (ADM) [7, 8], homotopy analysis method (HAM) [9-27], symmetry techniques [28-30] and one of the best among these is optimal homotopy asymptotic method (OHAM) [31]. OHAM has been grown up for many years with excellent applications. The beauty of this method is its simplicity and

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rapid convergence to closed form solution with few iterations. It has been proved tremendously to compute complicated and high order as well as coupled system of nonlinear differential equations. The fundamental idea of OHAM was introduced by Marinca et al. [31] in his perspective nonlinear model related to thin film flow. Work with colleagues, he computed successfully analytical solution of the problem taken from heat transfer model [32], applied this phenomenon to steady flow equation of fourth grade fluid past a permeable plate [33], evaluated the periodic solutions of the problem related to motion of particle on a rotating parabola [34], applied OHAM to nonlinear vibration of an electric machine [35], implemented this approach to oscillators with discontinuities and for fractional power [36], computed explicit solutions by OHAM to large amplitude having nonlinear oscillation of uniform cantilever beam carrying an intermediate lamped mass and rotary inertia [37]. Gossaye et al. [38] practiced OHAM over a nonlinear stretching sheet of the model related to flow and heat transfer of nanofluid and showed the slip effects over this model. Zuhra et al. [39] implemented OHAM on the time dependent problem having high nonlinearity model by Kortewege de Vries. This approach showed high accuracy as compared to closed form solution, yet problem carries 7th order KdV equation. Hosseini et al. [40] used this approach for the influence of velocity and temperature profiles over the non-Newtonian fluid flowing inside the channel with permeable walls. Zuhra et al. [41-42] used this technique in comparison with ADM to solve nonlinear equation with singular two-point boundary value problems and to solve Benjamina Bona Mahoney equation. Roslan et al. [43] did the comparison between HPM and OHAM on the experimental model of MHD flow of Maxwell fluid inside the transpiration channel where the walls of channel are leaky. He proved the best agreement between both the methods. Khan et al. [44] implemented OHAM to solve steady incompressible Berman's model for wall suction/injection and illustrated the impact of Reynolds number through graphs. As overwhelmed by the excellent performance of OHAM, in this article it is recommended to follow OHAM to compute the complex solution of complex KdV equation. Recently many researchers have given their attention to find out the complexiton solutions to complex differential equations like Ma [45], Lou et al. [46] and Chen at al. [47] have been obtained many complexiton solutions by different methods. Hong et al. [48] has been implemented the homotopy perturbation method to achieve the complexiton solutions to KdV equation. In this paper extended OHAM has been imposed on coupled system of KdV equations evaluated from a single complex equation. The complex KdV equation is given through [49] as

$$Z_t - 6ZZ_\xi + Z_{\xi\xi\xi} = 0, \tag{1}$$

where $Z = \psi + \varphi i$. By substituting the value of Z in (1) and collecting real and imaginary parts, it takes the form

$$\begin{aligned} &\psi_t + 6\varphi\varphi_\xi - 6\psi\psi_\xi + \psi_{\xi\xi\xi} = 0,\\ &\varphi_t - 6\psi\phi_\xi - 6\varphi\psi_\xi + \varphi_{\xi\xi\xi} = 0. \end{aligned} \tag{2}$$

(2) can be derived as coupled system of KdV equation.

This paper is arranged as follows: In section 2, the procedure and basic idea of OHAM is presented, section 3 comprises the application of OHAM where three different models are presented according to three different types of initial value problems. Section 4 gives the description on conclusions.

2. Application of Extended OHAM

The formation of OHAM is elaborated here.

Step1: The system of differential equations be

$$\begin{aligned} &L_1(\varpi(\xi,\tau)) + N_1(\varpi(\xi,\tau)) + f(\xi,\tau) = 0,\\ &L_2(\omega(\xi,\tau)) + N_2(\omega(\xi,\tau)) + g(\xi,\tau) = 0, \qquad \xi \in \Omega\\ &B_1\left(\varpi, \frac{\partial\varpi}{\partial\tau}\right) = 0, \quad B_2\left(\omega, \frac{\partial\omega}{\partial\tau}\right) = 0 \qquad \Delta \in \Omega, \end{aligned} \tag{3}$$

where ξ and τ are the spatial and temporal variables, $L_1(\varpi(\xi,\tau))$ and $L_2(\omega(\xi,\tau))$ are the linear components of 1st and 2nd differential equations respectively. $N_1(\varpi(\xi,\tau))$ and $N_2(\omega(\xi,\tau))$ are the nonlinear components

of equation (3). $\varpi(\xi,\tau)$ and $\omega(\xi,\tau)$ are the unknown functions. $f(\xi,\tau)$ and $g(\xi,t)$ are assumed as known functions, $L_1\left(\varpi,\frac{\partial\varpi}{\partial\tau}\right)$ and $L_2\left(\omega,\frac{\partial\omega}{\partial\tau}\right)$ yield the corresponding initial conditions of 2nd differential equations respectively and Δ is the boundary of ξ with domain Ω.
Step 2: According to OHAM, constructing the system of optimal homotopy as

$$\begin{aligned}&\psi(\xi,t;q):B_1\times[0,1]\to R\\&\varphi(\xi,t;q):B_2\times[0,1]\to R\end{aligned}$$

which satisfies

$$\begin{aligned}&H_1(\psi(\xi,\tau;q),q)=(1-q)\left[L_1(\psi(\xi,\tau;q))+f(\xi,\tau)\right]-\\&\qquad H(q)\left[L_1(\psi(\xi,\tau;q))+f(\xi,\tau)+N_1(\psi(\xi,\tau;q))\right]=0,\\&H_2(\varphi(\xi,\tau;q),q)=(1-q)\left[L_2(\varphi(\xi,\tau;q))+g(\xi,\tau)\right]-\\&\qquad H(q)\left[L_2(\varphi(\xi,\tau;q))+g(\xi,\tau)+N_2(\varphi(\xi,\tau;q))\right]=0,\end{aligned}\tag{4}$$

and

$$H_1(q)=\begin{cases}\sum\limits_{j=1}^{\infty}K_jq^j, & q\neq0,\\0, & q=0.\end{cases}\quad and\ H_2(q)=\begin{cases}\sum\limits_{j=1}^{\infty}K_jq^j, & q\neq0,\\0, & q=0.\end{cases}\tag{5}$$

Here the auxiliary functions $H_1(q)$ and $H_2(q)$ are nonzero for $q\neq0$ and $H_1(q)=0$, $H_2(q)=0$ for $q=0$, obviously we have

$$\begin{cases}q=0\Rightarrow H_1(\psi(\xi,\tau;0),0)=\left[L_1(\psi(\xi,\tau;0))+f(\xi,\tau)\right]=0,\\q=0\Rightarrow H_2(\varphi(\xi,\tau;0),0)=\left[L_2(\varphi(\xi,\tau;0))+g(\xi,\tau)\right]=0,\\q=1\Rightarrow H_1(\psi(\xi,\tau;0),0)=H_1(1)\left[L_1(\psi(\xi,\tau;1))+f(\xi,\tau)+N_1(\psi(\xi,\tau;1))\right]=0,\\q=1\Rightarrow H_2(\varphi(\xi,\tau;0),0)=H_1(1)\left[L_2(\varphi(\xi,\tau;1))+g(\xi,\tau)+N_2(\varphi(\xi,\tau;1))\right]=0.\end{cases}\tag{6}$$

Increasing q through the range $[0,1]$, the solution $\psi(\xi,\tau;q)$ varies from $\varpi_0(\xi,\tau)$ to the final solution $\varpi(\xi,\tau)$ for the 1st differential equation and $\varphi(\xi,\tau;q)$ from initial function $\omega_0(\xi,\tau)$ approaches to final function $\omega(\xi,\tau)$ for 2nd differential equation. $\varpi_0(\xi,\tau)$, $\omega_0(\xi,\tau)$ are estimated from equation (4) for $q=0$:

$$\begin{aligned}&L_1(\varpi_0(\xi,\tau))+f(\xi,\tau)=0,\quad B_1\left(\varpi_0,\tfrac{\partial\varpi}{\partial\tau}\right)=0,\\&L_2(\omega_0(\xi,\tau))+g(\xi,\tau)=0,\quad B_2\left(\omega_0,\tfrac{\partial\omega}{\partial\tau}\right)=0.\end{aligned}\tag{7}$$

$H_1(q)$ and $H_2(q)$ can be expanded in series forms as

$$\begin{aligned}&H_1(q)=qK_{11}+q^2K_{12}+q^3K_{13}+...,\\&H_2(q)=qK_{21}+q^2K_{22}+q^3K_{23}+...\end{aligned}$$

Step 3: To obtain the approximate solutions, expanding by Taylor's series about the parameter q

$$\begin{aligned}&\psi(\xi,\tau;\ q:\ K_{1i})=\varpi_o(\xi,\tau)+\sum_{k\geq1}\varpi_k(\xi,\tau;K_{1i})q^k,i=1,2,...,\\&\varphi(\xi,\tau;\ q:\ K_{2i})=\omega_o(\xi,\tau)+\sum_{k\geq1}\omega_k(\xi,\tau;K_{2i})q^k,i=1,2,...,\end{aligned}\tag{8}$$

where $K_{11},K_{12},K_{13},...,\ K_{21},K_{22},K_{23},...$ become the sources of convergence of equation (8) . If it converges at $q=1$ then by [24-27], it becomes

$$\begin{aligned}&\psi(\xi,\tau;K_{1i})=\varpi_0(\xi,\tau)+\sum_{k=1}^{M}\varpi_k(\xi,\tau;K_{1i})q^k,i=1,2,...,m,\\&\varphi(\xi,\tau;K_{2i})=\omega_0(\xi,\tau)+\sum_{k=1}^{M}\omega_k(\xi,\tau;K_{2i})q^k,i=1,2,...,m.\end{aligned}\tag{9}$$

By inserting equation (9) into (4) and associating the like powers of q, the nonlinear problem can be transferred into a sequence of linear equations; thus the zeroth order (10), first order (11), second order (12) and K^{th} order of the system are obtained as under

$$\begin{aligned} &L_1(\varpi_1(\xi,\tau)) = K_{11}N_{1,0}(\varpi_0(\xi,\tau)), \quad B_1(\varpi_0,\varpi_\tau) = 0,\\ &L_2(\omega_1(\xi,\tau)) = K_{21}N_{2,0}(\omega_0(\xi,\tau)), \quad B_2(\omega_0,\omega_\tau) = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} &L_1(\varpi_2(\xi,\tau)) - L_1(\varpi_1(\xi,\tau)) = K_{12}N_{1,1}(\varpi_0(\xi,\tau)) + K_{11}\begin{pmatrix} L_1(\varpi_1(\xi,\tau)) + \\ N_{1,1}(\varpi_0(\xi,\tau)),\varpi_1(\xi,\tau) \end{pmatrix},\\ &\quad B_1(\varpi_0,\varpi_\tau) = 0,\\ &L_2(\omega_2(\xi,\tau)) - L_2(\omega_1(\xi,\tau)) = K_{22}N_{2,1}(\omega_0(\xi,\tau)) + K_{21}\begin{pmatrix} L_2(\omega_1(\xi,\tau)) + \\ N_{2,1}(\omega_0(\xi,\tau)),\omega_1(\xi,\tau) \end{pmatrix},\\ &\quad B_2(\omega_0,\omega_\tau) = 0, \end{aligned} \tag{11}$$

$$\begin{cases} L_1(\varpi_k(\xi,\tau)) - L_1(\varpi_{k-1}(\xi,\tau)) = K_{1i}N_{1,0}(\varpi_0(\xi,\tau)) + \\ \sum\limits_{i=1}^{k-1} K_{1i}\left(L_1(\varpi_{k-1}(\xi,\tau)) + N_{1,k-1}(\varpi_0(\xi,\tau),\varpi_1(\xi,\tau),...,\varpi_{k-i}(\xi,\tau))\right), \\ B_1(\varpi_0,\varpi_\tau) = 0, \\ L_2(\omega_k(\xi,\tau)) - L_2(\omega_{k-1}(\xi,\tau)) = K_{2i}N_{2,0}(\omega_0(\xi,\tau)) + \\ \sum\limits_{i=1}^{k-1} K_{2i}\left(L_1(\omega_{k-1}(\xi,\tau)) + N_{2,k-1}(\omega_0(\xi,\tau),\omega_1(\xi,\tau),...,\omega_{k-i}(\xi,\tau))\right), \\ B_2(\omega_0,\omega_\tau) = 0, \end{cases} \tag{12}$$

where $k = 2, 3, 4, ...$
Step 3: These all linear problems can be computed for the system and their solutions are used to get K^{th} order solution which include K_{1i} and K_{2i} of the original problem (8). Substituting equation (8) into equation (6), gives the following residuals for system:

$$\begin{aligned} &R_1(\xi,\tau;K_{1i}) = L_1(\tilde{\varpi}^{(m)}(\xi,\tau;K_{1i})) + f(\xi,\tau) + N_1(\tilde{\varpi}^{(m)}(\xi,\tau;K_{1i})),\\ &R_2(\xi,\tau;K_{2i}) = L_2(\tilde{\varphi}^{(m)}(\xi,\tau;K_{2i})) + g(\xi,\tau) + N_2(\tilde{\varphi}^{(m)}(\xi,\tau;K_{1i})). \end{aligned} \tag{13}$$

$R_1(\xi,\tau;K_{1i}) = 0$ and $R_2(\xi,\tau;K_{2i}) = 0$ for some values of K_{1i} and K_{2i} respectively, then
$L_1(\tilde{\varpi}^{(m)}(\xi,\tau;K_{1i}))$ and $L_2(\tilde{\varpi}^{(m)}(\xi,\tau;K_{2i}))$ will be identical with the exact solution. But in usual cases it cannot happened, and in nonlinear problems it is impossible. Therefore, optimal values of the auxiliary constants $K_{11}, K_{12}, ..., K_{1n}$ and $K_{21}, K_{22}, ..., K_{2n}$, are calculated. The purpose of these auxiliary constants is to minimize errors which can be identified by least square process as in [24-25]:

$$\begin{aligned} &J_1(K_{11},K_{12},...,K_{1n}) = \int_0^1 {R_1}^2(\xi,\tau : K_{11},K_{12},...,K_{1m})d\xi,\\ &J_2(K_{21},K_{22},...,K_{2n}) = \int_0^1 {R_2}^2(\xi,\tau : K_{21},K_{22},...,K_{2m})d\xi, \end{aligned} \tag{14}$$

$$\begin{aligned} &\frac{\partial J_1}{\partial K_{11}} = \frac{\partial J_1}{\partial K_{12}} = ... = \frac{\partial J_1}{\partial K_{1m}} = 0,\\ &\frac{\partial J_2}{\partial K_{21}} = \frac{\partial J_2}{\partial K_{22}} = ... = \frac{\partial J_2}{\partial K_{2m}} = 0. \end{aligned} \tag{15}$$

By substitution the known values of the auxiliary constants, the semi analytic solution of OHAM can be determined.

2.1. Application of Extended OHAM

Model 1: Consider the equation (2) with initial values pursued by trigonometric function as follow

$$\begin{cases}\psi(\xi,0) = G(\xi)/F(\xi), \\ \varphi(\xi,0) = H(\xi)/F(\xi),\end{cases} \tag{16}$$

where

$$\begin{aligned}
&F(\xi) = \left(\alpha^2 Cos^2\eta_1 + \beta^2 Cos^2\eta_2\right)^2, \ \eta_1 = w\xi + \lambda_1, \ \eta_2 = w\xi + \lambda_2, \\
&G(\xi) = 2w^2\{(\alpha^2-\beta^2)(\alpha^2 Cos^2\eta_1 - \beta^2 Cos^2\eta_2) + 4\alpha^2\beta^2 Cos(\lambda_1-\lambda_2)Cos\eta_1 Cos\eta_2\}, \\
&H(\xi) = 4\alpha\beta w^2\{(\alpha^2-\beta^2)(\alpha^2 Cos^2\eta_1 - \beta^2 Cos^2\eta_2)Cos(\lambda_1-\lambda_2) - (\alpha^2-\beta^2)Cos^2\eta_1 Cos^2\eta_2\}.
\end{aligned}$$

According to extended OHAM, constructing the following homotopy,

$$\begin{aligned}
&(1-q)\psi_t - H_1(q)\{\psi_t + 6\varphi\varphi_\xi - 6\psi\psi_\xi + \psi_{\xi\xi\xi}\} = 0, \\
&(1-q)\varphi_t - H_2(q)\{\varphi_t - 6\psi\phi_\xi - 6\varphi\psi_\xi + \varphi_{\xi\xi\xi}\} = 0.
\end{aligned} \tag{17}$$

Considering

$$\begin{aligned}
&\psi = \psi_0 + q\psi_1, \quad \varphi = \varphi_0 + q\varphi_1, \\
&H_1(q) = qK_{11}, \quad H_2(q) = qK_{21}.
\end{aligned}$$

Zero order system

$$\psi_{0,t}(\xi,t) = 0, \quad \varphi_{0,t}(\xi,t) = 0, \tag{18}$$

with initial conditions

$$\psi_0(\xi,0) = G(\xi)/F(\xi), \ \ \varphi_0(\xi,0) = H(\xi)/F(\xi).$$

Its solutions are

$$\begin{aligned}
&\psi_0(\xi,t) = \frac{2w^2\begin{pmatrix} \alpha^4 Sinh^2(w\xi+\lambda_1) - \alpha^2\beta^2 Sinh^2(w\xi+\lambda_1) + 4\alpha^2\beta^2 Cosh(\lambda_1-\lambda_2)Sinh \\ (w\xi+\lambda_1)Sinh(w\xi+\lambda_2) - \alpha^2\beta^2 Sinh^2(w\xi+\lambda_2) + \beta^4 Sinh^2(w\xi+\lambda_2)\end{pmatrix}}{(\alpha^2 Sinh^2(w\xi+\lambda_1)+\beta^2 Sinh^2(w\xi+\lambda_2))^2}, \\
&\varphi_0(\xi,t) = \frac{2w^2\left(\alpha^3\beta Sinh(2(w\xi+\lambda_1))Sinh(\lambda_1-\lambda_2)+\alpha\beta^3 Sinh(\lambda_1-\lambda_2)Sinh(2(w\xi+\lambda_2))\right)}{(\alpha^2 Sinh^2(w\xi+\lambda_1)+\beta^2 Sinh^2(w\xi+\lambda_2)^2}.
\end{aligned} \tag{19}$$

First order system

$$\begin{aligned}
&\psi_{1,t}(\xi,t) = \psi_{0,t} + K_{11}\psi_{0,t} - 6K_{11}\psi_0\,\psi_{0,\xi} + 6K_{11}\varphi_0\,\varphi_{0,\xi} + K_{11}\psi_{0,\xi\xi\xi}, \\
&\varphi_{1,t}(\xi,t) = \varphi_{0,t} + K_{21}\varphi_{0,t} - 6K_{21}\varphi_0\,\psi_{0,\xi} - 6K_{21}\psi_0\,\varphi_{0,\xi} + K_{21}\varphi_{0,\xi\xi\xi},
\end{aligned} \tag{20}$$

with initial conditions

$$\psi_1(\xi,0) = 0, \quad \varphi_1(\xi,0) = 0.$$

Its solutions are

$$
\begin{aligned}
&\psi_1(\xi,t) = -\frac{1}{8(\alpha^2 Cos^2(w\xi+\lambda_1)+\beta^2 Cos^2(w\xi+\lambda_2))^5} t w^5 (2\alpha^2(7\alpha^8+86\alpha^6\beta^2-111\alpha^4\beta^4+188\alpha^2\\
&\beta^6-22\beta^8)Sin(2(w\xi+\lambda_1))+2\alpha^4(\alpha^2+\beta^2)(7\alpha^4+20\alpha^2\beta^2+\beta^4)Sin(4(w\xi+\lambda_1))+6\alpha^{10}Sin(6(w\xi\\
&+\lambda_1))-20\alpha^8\beta^2Sin(6(w\xi+\lambda_1))+54\alpha^6\beta^4Sin(6(w\xi+\lambda_1))+\alpha^{10}Sin(8(w\xi+\lambda_1))+\alpha^8\beta^2Sin(8(w\xi+\\
&\lambda_1))-80\alpha^6\beta^4Sin(2w\xi+6\lambda_1-4\lambda_2)-58\alpha^8\beta^2Sin(2w\xi+4\lambda_1-2\lambda_2)+204\alpha^6\beta^4Sin(2w\xi+4\lambda_1-2\lambda_2\\
&)-138\alpha^4\beta^6Sin(2w\xi+4\lambda_1-2\lambda_2)-6\alpha^8\beta^2Sin(4w\xi+6\lambda_1-2\lambda_2)-6\alpha^6\beta^4Sin(4w\xi+6\lambda_1-2\lambda_2)+16\\
&\alpha^8\beta^2Sin(6w\xi+8\lambda_1-2\lambda_2)-44\alpha^8\beta^2Sin(2(w\xi+\lambda_2))+376\alpha^6\beta^4Sin(2(w\xi+\lambda_2))-222\alpha^4\beta^6Sin(\\
&2(w\xi+\lambda_2))+172\alpha^2\beta^8Sin(2(w\xi+\lambda_2))+14\beta^{10}Sin(2(w\xi+\lambda_2))+2\alpha^6\beta^4Sin(4(w\xi+\lambda_2))+42\alpha^4\\
&\beta^6Sin(4(w\xi+\lambda_2))+54\alpha^2\beta^8Sin(4(w\xi+\lambda_2))+14\beta^{10}Sin(4(w\xi+\lambda_2))+54\alpha^4\beta^6Sin(6(w\xi+\lambda_2))\\
&-20\alpha^2\beta^8Sin(6(w\xi+\lambda_2))+6\beta^{10}Sin(6(w\xi+\lambda_2))+\alpha^2\beta^8Sin(8(w\xi+\lambda_2))+\beta^{10}Sin(8(w\xi+\lambda_2))+\\
&22\alpha^8\beta^2Sin(2(2w\xi+\lambda_1+\lambda_2))+102\alpha^6\beta^4Sin(2(2w\xi+\lambda_1+\lambda_2))+102\alpha^4\beta^6Sin(2(2w\xi+\lambda_1+\lambda_2))\\
&+22\alpha^2\beta^8Sin(2(2w\xi+\lambda_1+\lambda_2))+6\alpha^6\beta^4Sin(4(2w\xi+\lambda_1+\lambda_2))+6\alpha^4\beta^6Sin(4(2w\xi+\lambda_1+\lambda_2))+\\
&34\alpha^8\beta^2Sin(2(3w\xi+2\lambda_1+\lambda_2))-60\alpha^6\beta^4Sin(2(3w\xi+2\lambda_1+\lambda_2))+66\alpha^4\beta^6Sin(2(3w\xi+2\lambda_1+\lambda_2))\\
&+4\alpha^8\beta^2Sin(2(4w\xi+3\lambda_1+\lambda_2))+4\alpha^6\beta^4Sin(2(4w\xi+3\lambda_1+\lambda_2))+66\alpha^6\beta^4Sin(2(3w\xi+\lambda_1+2\lambda_2))-\\
&60\alpha^4\beta^6Sin(2(3w\xi+\lambda_1+2\lambda_2))+34\alpha^2\beta^8Sin(2(3w\xi+\lambda_1+2\lambda_2))+4\alpha^4\beta^6Sin(2(4w\xi+\lambda_1+3\lambda_2))+\\
&4\alpha^2\beta^8Sin(2(4w\xi+\lambda_1+3\lambda_2))-138\alpha^6\beta^4Sin(2w\xi-2\lambda_1+4\lambda_2)+204\alpha^4\beta^6Sin(2w\xi-2\lambda_1+4\lambda_2)-\\
&58\alpha^2\beta^8Sin(2w\xi-2\lambda_1+4\lambda_2)-80\alpha^4\beta^6Sin(2w\xi-4\lambda_1+6\lambda_2)-6\alpha^4\beta^6Sin(4w\xi-2\lambda_1+6\lambda_2)-6\alpha^2\\
&\beta^8Sin(4w\xi-2\lambda_1+6\lambda_2)+16\alpha^2\beta^8Sin(6w\xi-2\lambda_1+8\lambda_2))K_1,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\varphi_1(\xi,t) = -\frac{1}{16(\alpha^2 Cos^2(w\xi+\lambda_1)+\beta^2 Cos^2(w\xi+\lambda_2))^5} t w^5 \alpha(\alpha-\beta)\beta(\alpha+\beta)(-2(7\alpha^6+117\alpha^4\beta^2-\\
&46\alpha^2\beta^4+20\beta^6)Sin(2(w\xi+\lambda_1))-2(7\alpha^6+72\alpha^4\beta^2+37\alpha^2\beta^4)Sin(4(w\xi+\lambda_1))-6\alpha^6Sin(6(w\xi+\lambda_1\\
&))-18\alpha^4\beta^2Sin(6(w\xi+\lambda_1))-\alpha^6Sin(8(w\xi+\lambda_1))+20\alpha^4\beta^2Sin(2w\xi+6\lambda_1-4\lambda_2)+2\alpha^6\beta^2Sin(2w\xi+\\
&5\lambda_1-3\lambda_2)+162\alpha^4\beta^4Sin(2w\xi+5\lambda_1-3\lambda_2)-28\alpha^6\beta^2Sin(4w\xi+7\lambda_1-3\lambda_2)-30\alpha^6Sin(2w\xi+4\lambda_1-2\\
&\lambda_2)-38\alpha^4\beta^2Sin(2w\xi+4\lambda_1-2\lambda_2)-92\alpha^2\beta^4Sin(2w\xi+4\lambda_1-2\lambda_2)-19\alpha^6Sin(4w\xi+6\lambda_1-2\lambda_2)+9\\
&\alpha^4\beta^2Sin(4w\xi+6\lambda_1-2\lambda_2)-4\alpha^6Sin(6w\xi+8\lambda_1-2\lambda_2)+142\alpha^6\beta^2Sin(\lambda_1-\lambda_2)+16\alpha^4\beta^4Sin(\lambda_1-\lambda_2\\
&)+142\alpha^2\beta^6Sin(\lambda_1-\lambda_2)-5\alpha^6Sin(2(\lambda_1-\lambda_2))-139\alpha^4\beta^2Sin(2(\lambda_1-\lambda_2))+139\alpha^2\beta^4Sin(2(\lambda_1-\lambda_2))\\
&+5\beta^6Sin(2(\lambda_1-\lambda_2))+\quad 142\alpha^6\beta^2Sin(3(\lambda_1-\lambda_2))+150\alpha^4\beta^4Sin(3(\lambda_1-\lambda_2))+142\alpha^2\beta^6Sin(3(\lambda_1-\\
&\lambda_2))+67\alpha^4\beta^2Sin(4(\lambda_1-\lambda_2))-67\alpha^2\beta^4Sin(4(\lambda_1-\lambda_2))+134\alpha^4\beta^4Sin(5(\lambda_1-\lambda_2))+14\alpha^8Sin(2w\xi+\\
&3\lambda_1-\lambda_2)+160\alpha^6\beta^2Sin(2w\xi+3\lambda_1-\lambda_2)-42\alpha^4\beta^4Sin(2w\xi+3\lambda_1-\lambda_2)+132\alpha^2\beta^6Sin(2w\xi+3\lambda_1-\\
&\lambda_2)+14\alpha^8Sin(4w\xi+5\lambda_1-\lambda_2)+12\alpha^6\beta^2Sin(4w\xi+5\lambda_1-\lambda_2)-30\alpha^4\beta^4Sin(4w\xi+5\lambda_1-\lambda_2)+6\alpha^8Sin\\
&(6w\xi+7\lambda_1-\lambda_2)-26\alpha^6\beta^2Sin(6w\xi+7\lambda_1-\lambda_2)+\alpha^8Sin(8w\xi+9\lambda_1-\lambda_2)-40\alpha^6Sin(2(w\xi+\lambda_2))+92\\
&\alpha^4\beta^2Sin(2(w\xi+\lambda_2))-234\alpha^2\beta^4Sin(2(w\xi+\lambda_2))-14\beta^6Sin(2(w\xi+\lambda_2))-74\alpha^4\beta^2Sin(4(w\xi+\lambda_2))-\\
&144\alpha^2\beta^4Sin(4(w\xi+\lambda_2))-14\beta^6Sin(4(w\xi+\lambda_2))-18\alpha^2\beta^4Sin(6(w\xi+\lambda_2))-6\beta^6Sin(6(w\xi+\lambda_2))-\\
&\beta^6Sin(8(w\xi+\lambda_2))+14\alpha^8Sin(2w\xi+\lambda_1+\lambda_2)+26\alpha^6\beta^2Sin(2w\xi+\lambda_1+\lambda_2)-26\alpha^2\beta^6Sin(2w\xi+\lambda_1+\\
&\lambda_2)-14\beta^8Sin(2w\xi+\lambda_1+\lambda_2)-51\alpha^6Sin(2(2w\xi+\lambda_1+\lambda_2))-43\alpha^4\beta^2Sin(2(2w\xi+\lambda_1+\lambda_2))-43\alpha^2\\
&\beta^4Sin(2(2w\xi+\lambda_1+\lambda_2))-51\beta^6Sin(2(2w\xi+\lambda_1+\lambda_2))+38\alpha^6\beta^2Sin(3(2w\xi+\lambda_1+\lambda_2))-38\alpha^2\beta^6Si\\
&n(3(2w\xi+\lambda_1+\lambda_2))-11\alpha^4\beta^2Sin(4(2w\xi+\lambda_1+\lambda_2))-11\alpha^2\beta^4Sin(4(2w\xi+\lambda_1+\lambda_2))-26\alpha^6Sin(2(3\\
&w\xi+2\lambda_1+\lambda_2))-50\alpha^4\beta^2Sin(2(3w\xi+2\lambda_1+\lambda_2))-40\alpha^2\beta^4Sin(2(3w\xi+2\lambda_1+\lambda_2))+14\alpha^8Sin(4w\xi\\
&+3\lambda_1+\lambda_2)+56\alpha^6\beta^2Sin(4w\xi+3\lambda_1+\lambda_2)-30\alpha^4\beta^4Sin(4w\xi+3\lambda_1+\lambda_2)-16\alpha^2\beta^6Sin(4w\xi+3\lambda_1+\\
&\lambda_2)-5\alpha^6Sin(2(4w\xi+3\lambda_1+\lambda_2))-7\alpha^4\beta^2Sin(2(4w\xi+3\lambda_1+\lambda_2))+6\alpha^8Sin(6w\xi+5\lambda_1+\lambda_2)+12\alpha^6\\
&\beta^2Sin(6w\xi+5\lambda_1+\lambda_2)-58\alpha^4\beta^4Sin(6w\xi+5\lambda_1+\lambda_2)+\alpha^8Sin(8w\xi+7\lambda_1+\lambda_2)+2\alpha^6\beta^2Sin(8w\xi+\\
&7\lambda_1+\lambda_2)-40\alpha^4\beta^2Sin(2(3w\xi+\lambda_1+2\lambda_2))-50\alpha^2\beta^4Sin(2(3w\xi+\lambda_1+2\lambda_2))-26\beta^6Sin(2(3w\xi+\lambda_1\\
&+2\lambda_2))-132\alpha^6\beta^2Sin(2w\xi-\lambda_1+3\lambda_2)+42\alpha^4\beta^4Sin(2w\xi-\lambda_1+3\lambda_2)-160\alpha^2\beta^6Sin(2w\xi-\lambda_1+3\lambda_2\\
&)-14\beta^8Sin(2w\xi-\lambda_1+3\lambda_2)+16\alpha^6\beta^2Sin(4w\xi+\lambda_1+3\lambda_2)+30\alpha^4\beta^4Sin(4w\xi+\lambda_1+3\lambda_2)-56\alpha^2\beta^6\\
&Sin(4w\xi+\lambda_1+3\lambda_2)-14\beta^8Sin(4w\xi+\lambda_1+3\lambda_2)-7\alpha^2\beta^4Sin(2(4w\xi+\lambda_1+3\lambda_2))-5\beta^6Sin(2(4w\xi+\\
&\lambda_1+3\lambda_2))+2\alpha^6\beta^2Sin(8w\xi+5\lambda_1+3\lambda_2)-92\alpha^4\beta^2Sin(2w\xi-2\lambda_1+4\lambda_2)-38\alpha^2\beta^4Sin(2w\xi-2\lambda_1+\\
&4\lambda_2)-30\beta^6Sin(2w\xi-2\lambda_1+4\lambda_2)-162\alpha^4\beta^4Sin(2w\xi-3\lambda_1+5\lambda_2)-2\alpha^2\beta^6Sin(2w\xi-3\lambda_1+5\lambda_2)+
\end{aligned}
$$

$30\alpha^4\beta^4 Sin(4w\xi-\lambda_1+5\lambda_2)-12\alpha^2\beta^6 Sin(4w\xi-\lambda_1+5\lambda_2)-14\beta^8 Sin(4w\xi-\lambda_1+5\lambda_2)+58\alpha^4\beta^4 Sin(6w\xi+\lambda_1+5\lambda_2)-12\alpha^2\beta^6 Sin(6w\xi+\lambda_1+5\lambda_2)-6\beta^8 Sin(6w\xi+\lambda_1+5\lambda_2)-2\alpha^2\beta^6 Sin(8w\xi+3\lambda_1+5\lambda_2)+20\alpha^2\beta^4 Sin(2w\xi-4\lambda_1+6\lambda_2)+9\alpha^2\beta^4 Sin(4w\xi-2\lambda_1+6\lambda_2)-19\beta^6 Sin(4w\xi-2\lambda_1+6\lambda_2)+28\alpha^2\beta^6 Sin(4w\xi-3\lambda_1+7\lambda_2)+26\alpha^2\beta^6 Sin(6w\xi-\lambda_1+7\lambda_2)-6\beta^8 Sin(6w\xi-\lambda_1+7\lambda_2)-2\alpha^2\beta^6 Sin(8w\xi+\lambda_1+7\lambda_2)-\beta^8 Sin(8w\xi+\lambda_1+7\lambda_2)-4\beta^6 Sin(6w\xi-2\lambda_1+8\lambda_2)-\beta^8 Sin(8w\xi-\lambda_1+9\lambda_2))K_1.$

Adding (20) and (21) in the form of

$$\begin{aligned}\tilde{\psi}(\xi,t)&=\psi_0(\xi,t)+\psi_1(\xi,t;K_{11}),\\ \tilde{\varphi}(\xi,t)&=\varphi_0(\xi,t)+\varphi_1(\xi,t;K_{21}).\end{aligned} \tag{22}$$

Putting (22) in (13), then applying the least square method (24-25), the values can be computed as

$$K_{11}=\quad 0.000121948170060871,\ K_{21}=-0.000811676555101159. \tag{23}$$

By substituting (23) in (22), the semi analytic solution to OHAM is achieved. The related close solution of model (16) is given below

$$\begin{aligned}\psi(\xi,t)&=\frac{2w^2\{(\alpha^2-\beta^2)(\alpha^2Cos^2\eta'_1-\beta^2Cos^2\eta'_2)+4\alpha^2\beta^2Cos(\lambda_1-\lambda_2)Cos\eta'_1Cos\eta'_2\}}{\left(\alpha^2Cos^2\eta'_1+\beta^2Cos^2\eta'_2\right)^2},\\ \varphi(\xi,t)&=\frac{4\alpha\beta w^2\{(\alpha^2-\beta^2)(\alpha^2Cos^2\eta'_1-\beta^2Cos^2\eta'_2)Cos(\lambda_1-\lambda_2)-(\alpha^2-\beta^2)Cos^2\eta'_1Cos^2\eta'_2\}}{\left(\alpha^2Cos^2\eta'_1+\beta^2Cos^2\eta'_2\right)^2},\end{aligned} \tag{24}$$

where $\eta'_1=w\xi+4w^3t+\lambda_1,\ \eta'_2=w\xi+4w^3+\lambda_2$. In order to prove the high accuracy of semi analytic solution by proposed technique, the numerical simulation is illustrated in Tables (1, 2) and Figs. (1-8).
Model 2: Consider (2) with initial condition in the hyperbolic function

$$\psi(\xi,0)=B(\xi)/A(\xi),\ \ \varphi(\xi,0)=C(\xi)/A(\xi), \tag{25}$$

where

$$\begin{aligned}&A(\xi)=(\alpha^2Sinh^2(\eta_1)^2+\beta^2Sinh^2(\eta_2))^2,\qquad \eta_1=w\xi+\lambda_1;\quad \eta_2=w\xi+\lambda_2,\\ &B(\xi)=2w^2((\alpha^2-\beta^2)(\alpha^2(Sinh^2(\eta_1))-\beta^2(Sinh^2(\eta_2))+4\alpha^2\beta^2Cosh(\eta_1-\eta_2)Sinh(\eta_1)Sinh(\eta_2)),\\ &C(\xi)=2\alpha\beta w^2(\alpha^2Sinh(2\eta_1)+\beta^2Sinh(2\eta_2))Sinh(\eta_1-\eta_2).\end{aligned}$$

Closed form solution is

$$\begin{aligned}\psi(\xi,t)&=\frac{\left(\begin{array}{c}2w^2((\alpha^2-\beta^2)(\alpha^2(Sinh^2(\eta'_1))-\beta^2(Sinh^2(\eta'_2))\\ +4\alpha^2\beta^2Cosh(\eta'_1-\eta'_2)Sinh(\eta'_1)Sinh(\eta'_2))\end{array}\right)}{(\alpha^2Sinh^2(\eta'_1)^2+\beta^2Sinh^2(\eta'_2))^2},\\ \varphi(\xi,t)&=\frac{2\alpha\beta w^2(\alpha^2Sinh(2\eta'_1)+\beta^2Sinh(2\eta'_2))Sinh(\eta'_1-\eta'_2)}{(\alpha^2Sinh^2(\eta'_1)^2+\beta^2Sinh^2(\eta'_2))^2},\\ &\eta'_1=w\xi-4w^3t+\lambda_1;\quad \eta'_2=w\xi-4w^3t+\lambda_2.\end{aligned} \tag{26}$$

Zero order system

$$\psi_{0,t}(\xi,t)=0,\quad \varphi_{0,t}(\xi,t)=0, \tag{27}$$

with initial conditions

$$\begin{cases}\psi_0(\xi,0)=B(\xi)/A(\xi),\\ \varphi_0(\xi,0)=C(\xi)/A(\xi).\end{cases}$$

Its solution is

$$\psi_0(\xi,t)=\frac{2w^2\begin{pmatrix}(\alpha^4Sinh^2(w\xi+\lambda_1)-\alpha^2\beta^2Sinh^2(w\xi+\lambda_1)+4\alpha^2\beta^2Cosh(\lambda_1-\lambda_2)Sinh\\(w\xi+\lambda_1)Sinh(w\xi+\lambda_2)-\alpha^2\beta^2Sinh^2(w\xi+\lambda_2)+\beta^4Sinh^2(w\xi+\lambda_2))\end{pmatrix}}{(\alpha^2Sinh^2(w\xi+\lambda_1)+\beta^2Sinh^2(w\xi+\lambda_2))^2},$$

$$\varphi_0(\xi,t)=\frac{2w^2(\alpha^3\beta Sinh(2(w\xi+\lambda_1))Sinh(\lambda_1-\lambda_2)+\alpha\beta^3Sinh(\lambda_1-\lambda_2)Sinh(2(w\xi+\lambda_2)))}{(\alpha^2Sinh^2(w\xi+\lambda_1)+\beta^2Sinh^2(w\xi+\lambda_2))^2}. \tag{28}$$

The solution of first order problem (20) is

$$\begin{aligned}\psi_1(\xi,t)=&-\frac{1}{8(\alpha^2Sinh^2(w\xi+\lambda_1)+\beta^2Sinh^2(w\xi+\lambda_2))^5}tw^5(-2\alpha^2(7\alpha^8+86\alpha^6\beta^2-111\alpha^4\beta^4+188\\
&\alpha^2\beta^6-22\beta^8)Sinh(2(w\xi+\lambda_1))+2\alpha^4(\alpha^2+\beta^2)(7\alpha^4+20\alpha^2\beta^2+\beta^4)Sinh(4(w\xi+\lambda_1))-6\alpha^{10}Sin\\
&h[6(w\xi+\lambda_1)]+20\alpha^8\beta^2Sinh(6(w\xi+\lambda_1))-54\alpha^6\beta^4Sinh(6(w\xi+\lambda_1))+\alpha^{10}Sinh(8(w\xi+\lambda_1))+\alpha^8\\
&\beta^2Sinh(8(w\xi+\lambda_1))+80\alpha^6\beta^4Sinh(2w\xi+6\lambda_1-4\lambda_2)+58\alpha^8\beta^2Sinh(2w\xi+4\lambda_1-2\lambda_2)-204\alpha^6\beta^4\\
&Sinh(2w\xi+4\lambda_1-2\lambda_2)+138\alpha^4\beta^6Sinh(2w\xi+4\lambda_1-2\lambda_2)-6\alpha^8\beta^2Sinh(4w\xi+6\lambda_1-2\lambda_2)-6\alpha^6\beta^4\\
&Sinh(4w\xi+6\lambda_1-2\lambda_2)-16\alpha^8\beta^2Sinh(6w\xi+8\lambda_1-2\lambda_2)+44\alpha^8\beta^2Sinh(2(w\xi+\lambda_2))-376\alpha^6\beta^4Sin\\
&h(2(w\xi+\lambda_2))+222\alpha^4\beta^6Sinh(2(w\xi+\lambda_2))-172\alpha^2\beta^8Sinh(2(w\xi+\lambda_2))-14\beta^{10}Sinh(2(w\xi+\lambda_2))\\
&+2\alpha^6\beta^4Sinh(4(w\xi+\lambda_2))+42\alpha^4\beta^6Sinh(4(w\xi+\lambda_2))+54\alpha^2\beta^8Sinh(4(w\xi+\lambda_2))+14\beta^{10}Sinh(4(w\\
&\xi+\lambda_2))-54\alpha^4\beta^6Sinh(6(w\xi+\lambda_2))+20\alpha^2\beta^8Sinh(6(w\xi+\lambda_2))-6\beta^{10}Sinh(6(w\xi+\lambda_2))+\alpha^2\beta^8Sin\\
&h(8(w\xi+\lambda_2))+\beta^{10}Sinh(8(w\xi+\lambda_2))+22\alpha^8\beta^2Sinh(2(2w\xi+\lambda_1+\lambda_2))+102\alpha^6\beta^4Sinh(2(2w\xi+\lambda_1\\
&+\lambda_2))+102\alpha^4\beta^6Sinh[2(2w\xi+\lambda_1+\lambda_2))+22\alpha^2\beta^8Sinh(2(2w\xi+\lambda_1+\lambda_2))+6\alpha^6\beta^4Sinh(4(2w\xi+\lambda_1\\
&+\lambda_2))+6\alpha^4\beta^6Sinh(4(2w\xi+\lambda_1+\lambda_2))-34\alpha^8\beta^2Sinh(2(3w\xi+2\lambda_1+\lambda_2))+60\alpha^6\beta^4Sinh(2(3w\xi+2\\
&\lambda_1+\lambda_2))-66\alpha^4\beta^6Sinh(2(3w\xi+2\lambda_1+\lambda_2))+4\alpha^8\beta^2Sinh(2(4w\xi+3\lambda_1+\lambda_2))+4\alpha^6\beta^4Sinh(2(4w\xi\\
&+3\lambda_1+\lambda_2))-66\alpha^6\beta^4Sinh(2(3w\xi+\lambda_1+2\lambda_2))+60\alpha^4\beta^6Sinh(2(3w\xi+\lambda_1+2\lambda_2))-34\alpha^2\beta^8Sinh(2(\\
&3w\xi+\lambda_1+2\lambda_2))+4\alpha^4\beta^6Sinh(2(4w\xi+\lambda_1+3\lambda_2))+4\alpha^2\beta^8Sinh(2(4w\xi+\lambda_1+3\lambda_2))+138\alpha^6\beta^4Sin\\
&h(2w\xi-2\lambda_1+4\lambda_2)-204\alpha^4\beta^6Sinh(2w\xi-2\lambda_1+4\lambda_2)+58\alpha^2\beta^8Sinh(2w\xi-2\lambda_1+4\lambda_2)+80\alpha^4\beta^6Si\\
&nh(2w\xi-4\lambda_1+6\lambda_2)6\alpha^4\beta^6Sinh(4w\xi-2\lambda_1+6\lambda_2)-6\alpha^2\beta^8Sinh(4w\xi-2\lambda_1+6\lambda_2)-16\alpha^2\beta^8Sinh(6\\
&w\xi-2\lambda_1+8\lambda_2))K_1\end{aligned}$$

$$\begin{aligned}\varphi_1(\xi,t)=&-\frac{1}{4(\alpha^2Sinh^2(w\xi+\lambda_1)+\beta^2Sinh^2(w\xi+\lambda_2))^5}tw^5\alpha\beta(71\alpha^8+8\alpha^6\beta^2+276\alpha^4\beta^4+8\alpha^2\beta^6\\
&+71\beta^8-2\alpha^2(\alpha^2+\beta^2)(37\alpha^4-8\alpha^2\beta^2+78\beta^4)Cosh(2(w\xi+\lambda_1))-8(\alpha^8-5\alpha^6\beta^2+8\alpha^4\beta^4)Cosh\\
&(4(w\xi+\lambda_1))+10\alpha^8Cosh(6(w\xi+\lambda_1))+10\alpha^6\beta^2Cosh(6(w\xi+\lambda_1))+\alpha^8Cosh(8(w\xi+\lambda_1))-82\alpha^6\\
&\beta^2Cosh(2w\xi+4\lambda_1-2\lambda_2)-82\alpha^4\beta^4Cosh(2w\xi+4\lambda_1-2\lambda_2)-28\alpha^6\beta^2Cosh(4w\xi+6\lambda_1-2\lambda_2)+2\\
&76\alpha^6\beta^2Cosh(2(\lambda_1-\lambda_2))+16\alpha^4\beta^4Cosh(2(\lambda_1-\lambda_2))+276\alpha^2\beta^6Cosh(2(\lambda_1-\lambda_2))+134\alpha^4\beta^4Cos\\
&h(4(\lambda_1-\lambda_2))-156\alpha^6\beta^2Cosh(2(w\xi+\lambda_2))-140\alpha^4\beta^4Cosh(2(w\xi+\lambda_2))-58\alpha^2\beta^6Cosh(2(w\xi+\lambda_2\\
&))-74\beta^8Cosh(2(w\xi+\lambda_2))-64\alpha^4\beta^4Cosh(4(w\xi+\lambda_2))+40\alpha^2\beta^6Cosh(4(w\xi+\lambda_2))-8\beta^8Cosh(4\\
&(w\xi+\lambda_2))+10\alpha^2\beta^6Cosh(6(w\xi+\lambda_2))+10\beta^8Cosh(6(w\xi+\lambda_2))+\beta^8Cosh(8(w\xi+\lambda_2))-44\alpha^6\beta^2C\\
&osh(2(2w\xi+\lambda_1+\lambda_2))+80\alpha^4\beta^4Cosh(2(2w\xi+\lambda_1+\lambda_2))-44\alpha^2\beta^6Cosh(2(2w\xi+\lambda_1+\lambda_2))+6\alpha^4\beta^4\\
&Cosh(4(2w\xi+\lambda_1+\lambda_2))+30\alpha^6\beta^2Cosh(2(3w\xi+2\lambda_1+\lambda_2))+30\alpha^4\beta^4Cosh(2(3w\xi+2\lambda_1+\lambda_2))+4\\
&\alpha^6\beta^2Cosh(2(4w\xi+3\lambda_1+\lambda_2))+30\alpha^4\beta^4Cosh(2(3w\xi+\lambda_1+2\lambda_2))+30\alpha^2\beta^6Cosh(2(3w\xi+\lambda_1+2\lambda_2\\
&))+4\alpha^2\beta^6Cosh(2(4w\xi+\lambda_1+3\lambda_2))-82\alpha^4\beta^4Cosh(2w\xi-2\lambda_1+4\lambda_2)-82\alpha^2\beta^6Cosh(2w\xi-2\lambda_1+\\
&4\lambda_2)-28\alpha^2\beta^6Cosh(4w\xi-2\lambda_1+6\lambda_2)Sinh(\lambda_1-\lambda_2)K_1,\end{aligned} \tag{29}$$

Adding (28), (29) and applying the same procedure as above (22-23), it becomes

$$K_{11} = 0.0003375499289662265,$$
$$K_{21} = -0.03758072639218757.$$

By putting in (22), semi analytic complexiton solution by OHAM can be produced. The effectiveness of OHAM can be observed from Tables (3, 4) and Figs. (9-16).

Model 3: Taking equation (2) with initial value problem having the collection of trigonometric function and hyperbolic function.

$$\begin{aligned}\psi(\xi,0) &= A(\xi)/B(\xi),\\ \varphi(\xi,0) &= C(\xi)/B(\xi),\end{aligned} \tag{30}$$

where

$$\begin{aligned}A(\xi) &= -(272Cos(8\xi) + 240Cos(8\xi)Cosh(2\xi) + 272Cosh(2\xi) - 128Sin(8\xi)Sinh(2\xi) + 240),\\ B(\xi) &= \tfrac{835}{8} - \tfrac{17}{2}Cos(8\xi) + \tfrac{1}{8}Cos(16\xi) + 136Cosh(2\xi) - 8Cos(8\xi)Cosh(2\xi) + 32Cosh(4\xi),\\ C(\xi) &= -(1862Sin(4\xi)Cosh(\xi) + 30Cosh(\xi)Sin(12\xi) + 240Cos(4\xi)Sinh(\xi)\\ &\quad +16Sinh(\xi)Cos(12\xi) + 480Sin(4\xi)Cosh(3\xi) + 256Cos(4\xi)Sinh(3\xi)).\end{aligned}$$

Closed form solution is

$$\begin{aligned}\psi(\xi,t) &= {}^{A(\xi,t)}\!/_{B(\xi,t)},\\ \varphi(\xi,t) &= {}^{C(\xi,t)}\!/_{B(\xi,t)},\end{aligned}$$

where

$$\begin{aligned}&A(\xi,t) = -(272Cos(8\xi+108t) + 240Cos(8\xi+108t)Cosh(2\xi+94t) + 272Cosh\\ &(2\xi+94t) - 128Sin(8\xi+104t)Sinh(2\xi+94t) + 240),\\ &B(\xi,t) = \tfrac{835}{8} - \tfrac{17}{2}Cos(8\xi+104t) + \tfrac{1}{8}Cos(16\xi+208t) + 136Cosh(2\xi+94t) - 8C\\ &os(8\xi+104t)Cosh(2\xi+94t) + 32Cosh(4\xi+188t),\\ &C(\xi,t) = -(1862Sin(4\xi+52t)Cosh(\xi) + 30Cosh(\xi+47t)Sin(12\xi+156t) + 240C\\ &os(4\xi+52t)Sinh(\xi+47t) + 16Sinh(\xi+47t)Cos(12\xi+156t) + 480Sin(4\xi+52t)\\ &Cosh(3\xi+141t) + 256Cos(4\xi+52t)Sinh(3\xi+141t)).\end{aligned} \tag{31}$$

The solution of zero order (18) with initial conditions

$$\psi_0(\xi,0) = A(\xi)/B(\xi), \quad \varphi_0(\xi,0) = C(\xi)/B(\xi)$$

is

$$\begin{aligned}\psi_0(\xi,t) &= \frac{128(15+17Cos(8\xi)+17Cosh(2\xi)+15Cos(8\xi)Cosh(2\xi)-8Sin(8\xi)Sinh(2\xi))}{-835+68Cos(8\xi)-Cos(16\xi)-1088Cosh(2\xi)+64Cos(8\xi)Cosh(2\xi)-256Cosh(4\xi)},\\ \varphi_0(\xi,t) &= \frac{\left(\begin{array}{l}16(931Cosh(\xi)Sin(4\xi)+240Cosh(3\xi)Sin(4\xi)+15Cosh(\xi)Sin(12\xi)\\ +120Cos(4\xi)Sinh(\xi)+8Cos(12\xi)Sinh(\xi)+128Cos(4\xi)Sinh(3\xi))\end{array}\right)}{-835+68Cos(8\xi)-Cos(16\xi)-1088Cosh(2\xi)+64Cos(8\xi)Cosh(2\xi)-256Cosh(4\xi)}.\end{aligned} \tag{32}$$

Similarly the solution of first order (20) is given as

$$\begin{aligned}&\psi_1(\xi,t) = \frac{1}{(-17+Cos(8\xi)-16Cosh(2\xi))^5}64t(8(38895949Cosh(2\xi) + 8(3207799 + 2032861\\ &Cosh(4\xi) + 404192Cosh(6\xi) + 25856Cosh(8\xi)))Sin(8\xi) + 8(2220880 + 3023619Cosh(2\xi) + 8\\ &65504Cosh(4\xi) + 64640Cosh(6\xi))Sin(16\xi) + 8(37848 + 41633Cosh(2\xi) + 4040Cosh((\xi))Sin(\\ &24\xi) + 4(272 + 101Cosh(2\xi))Sin(32\xi) - 14600381Sinh(2\xi) + 1121Cos(32\xi)Sinh(2\xi) + 4Cos(1\\ &6\xi)(2911079 + 3642896Cosh(2\xi) + 717440Cosh(4\xi))Sinh(2\xi) - 10545440Sinh(4\xi) + 8Cos(2\\ &4\xi)(28271Sinh(2\xi) + 11210Sinh(4\xi)) - 2183424Sinh[6\xi] - 69632Sinh(8\xi) + 8Cos(8\xi)(12366\\ &089Sinh(2\xi) + 11455446Sinh(4\xi) + 4271488Sinh(6\xi) + 573952Sinh(8\xi)))K_1,\end{aligned}$$

$$\begin{aligned}\varphi_1(\xi,t) = {} & \frac{1}{(-17+Cos(8\xi)-16Cosh(2\xi))^5}t(4Cos(\xi)(-6294467654Cos(4\xi)-101Cos(36\xi)\\ & -Cos(28\xi)(300221+90496Cosh(2\xi))-4Cos(20\xi)(17465753+18477600Cosh(2\xi)+17323\\ & 52Cosh(4\xi))-4Cos(12\xi)(225467165+304199520Cosh(2\xi)+82047744Cosh(4\xi)+579174\\ & 4Cosh(6\xi)))-256Cos(4\xi)(59194593Cosh(3\xi)+16(1243551Cosh(5\xi)+188176Cosh(7\xi)+64\\ & 64Cosh(9\xi)))+(2(5564002719+8567306944Cosh(2\xi)+3782360064Cosh(4\xi)+850051072\\ & Cosh(6\xi)+73465856Cosh(8\xi))Sin(4\xi)+4(964976875+1397434976Cosh(2\xi)+496193792\\ & Cosh(4\xi)+64282624Cosh(6\xi))Sin(12\xi)+4(71374793+90548704Cosh(2\xi)+19227392Cos\\ & h(4\xi))Sin(20\xi)+19(56941+52864Cosh(2\xi))Sin(28\xi)+1121Sin(36\xi))Sinh(\xi))K_1,\end{aligned} \tag{33}$$

where

K_{11}= 0.006555167326912838,
K_{21}= 0.04021604072725644.

By substituting K_{11}, K_{21} in (22), the complexiton solution of OHAM is obtained whose accuracy can be examined from Table (5, 6) and Figs. (17-22).

3. Results and discussion:

The procedure of OHAM elaborated in section 2 has been implemented to three different models of section 3 which give the significant results to each of these models. For the precision of solution by OHAM, the closed form solution in tables and figures related to each model is used. By using the parameters $\alpha = 2,\ \beta = 4,\ \lambda_1 = 0,\ \lambda_2 = 1$ *and* $w = 2$, the Tables (1-2) and Figs. (1-8) are constructed. Tables (1-2) display the comparison of semi analytic position solution by OHAM with exact position solution. This comparison is made precise by absolute error column. Figures (1-8) show the individual plot of exact solution, the solution by OHAM, convergence of OHAM solution to exact and the plot of the range of absolute error for the functions $\psi(\xi,t)$ and $\varphi(\xi,t)$ respectively. With the parameters, Tables (3-4) are developed for both functions $\psi(\xi,t)$ and $\varphi(\xi,t)$. Absolute error column gives the accuracy of OHAM to exact solution at various values. Figures (9-16) declare the negation solutions of closed form and obtained by OHAM. Tables (5-6) and Figs. (17-22) come into existence due to the results from the formation of special type of initial value problems which is the combination of trigonometric and hyperbolic functions. Tables (5-6) show the absolute error column that OHAM solution is very close to the exact solution. All tables and plots proved that OHAM is too identical to the exact solution in each case at every point within the domain.

4. Conclusions

Coupled system of KdV is computed from the complex KdV equation. Three types of semi analytical OHAM solutions have been achieved based on trigonometric form of initial value problem, semi analytic negation solution based on hyperbolic form of initial value problem and another type of semi analytic solution based on two forms. In each case the solution obtained by OHAM give identical to exact form. This method is smooth, reliable and easy to use throughout the domain so it is predicted that OHAM is perfect for complex nonlinear problems.

Table 1: Comparison of OHAM solution with exact solution for $\psi(\xi, t)$ with parameters $\alpha = 2,\ \beta = 4,\ \lambda_1 = 0,\ \lambda_2 = 1$ *and* $w = 2$.

ξ	**Exact Solution**	**OHAM Solution**	**Absolute Error OHAM**
-2	118.306	118.306	2.84217×10^{-14}
-1	24.8155	24.8155	3.55271×10^{-15}
0	76.3785	76.3785	0
1	115.719	115.719	0
2	-81.9772	-81.9772	1.42109×10^{-14}

Table 1 shows the exact and OHAM position solutions.

Table 2: Comparison of OHAM solution with exact solution for $\varphi(\xi, t)$ with parameters $\alpha = 2,\ \beta = 4,\ \lambda_1 = 0,\ \lambda_2 = 1$ *and* $w = 2$.

ξ	**Exact Solution**	**OHAM Solution**	**Absolute Error OHAM**
-2	703.776	703.776	1.13687×10^{-13}
-1	630.014	630.014	0
0	115.607	115.607	1.42109×10^{-14}
1	775.607	775.607	1.13687×10^{-14}
2	-99.1379	-99.1379	5.68434×10^{-14}

Table 2 shows the exact and OHAM position solutions.

Table 3: Comparison of OHAM solution with exact solution for $\varphi(\xi, t)$ with parameters $\alpha = 1,\ \beta = 5,\ \lambda_1 = 0,\ \lambda_2 = 1$ *and* $w = 2$.

ξ	**Exact Solution**	**OHAM Solution**	**Absolute Error OHAM**
-3	0.0035057	0.0035057	8.67362×10^{-19}
-2	0.189177	0.189177	5.55112×10^{-19}
-1	-14.5962	-14.5962	1.77636×10^{-15}
0	0.583848	0.583848	1.11022×10^{-16}
1	0.0111486	0.0111486	0

Table 3 shows the exact and OHAM negation solutions.

Table 4: Comparison of OHAM solution with exact solution for $\varphi(\xi, t)$ with parameters $\alpha = 1,\ \beta = 5,\ \lambda_1 = 0,\ \lambda_2 = 1$ *and* $w = 2$.

ξ	**Exact Solution**	**OHAM Solution**	**Absolute Error OHAM**
-3	0.00489342	0.00489342	8.673622×10^{-19}
-2	0.2734750	0.2734750	5.55112×10^{-17}
-1	22.8807	22.8807	1.06581×10^{-14}
0	-0.915229	-0.915229	3.33067×10^{-16}
1	-0.0155838	-0.0155838	1.73472×10^{-18}

Table 4 shows the exact and OHAM negation solutions.

Table 5: shows the exact and OHAM analytical solutions for for $\varphi(\xi, t)$.

ξ	**Exact Solution**	**OHAM Solution**	**Absolute Error OHAM**
-3	-0.0376820	-0.0376820	0
-2	-0.0413012	-0.0413012	0
-1	-0.4233450	-0.42334507	0
0	-4	-4	0
1	-0.4233450	-0.4233450	0

Table 5 shows the exact and OHAM analytical solutions.

Table 6: shows the exact and OHAM analytical solutions for $\varphi(\xi, t)$.

ξ	**Exact Solution**	**OHAM Solution**	**Absolute Error OHAM**
-3	-0.06712	-0.06712	0
-2	1.83654	1.83654	0
-1	0	0	0
0	-4	-4	0
1	5.15004	5.15004	0

Table 6 shows the exact and OHAM analytical solutions.

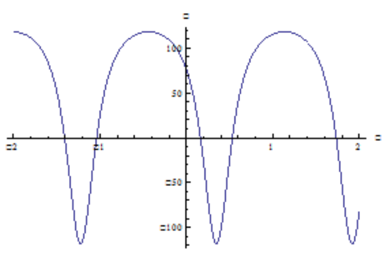

Figure 1

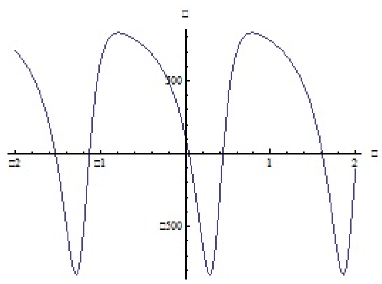

Figure 2

Figure 1-2: Exact position solution (24) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 2, \quad \beta = 4, \quad \lambda_1 = 0, \quad \lambda_2 = 1 \quad and \quad w = 2$ at $t = 0$.

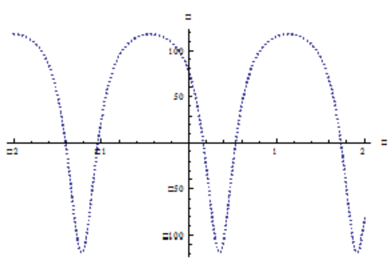

Figure 3

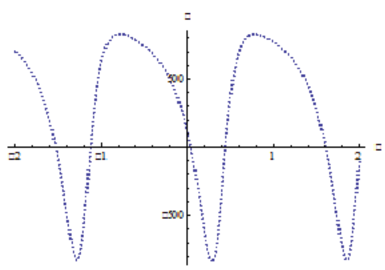

Figure 4

Figure 3-4: Semi analytic position solution obtained by OHAM (model 1) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 2, \quad \beta = 4, \quad \lambda_1 = 0, \quad \lambda_2 = 1 \quad and \quad w = 2$ at $t = 0$.

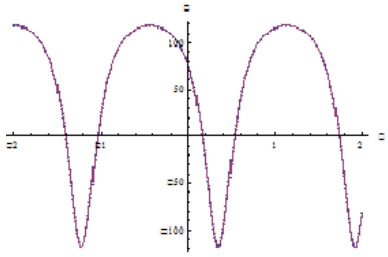

Figure 5

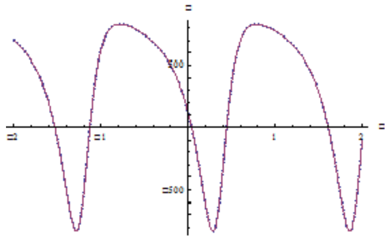

Figure 6

Figure 5-6: Comparison of Exact position solution (24) and OHAM solution (model 1) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 2, \quad \beta = 4, \quad \lambda_1 = 0, \ \lambda_2 = 1 \ \ and \ \ w = 2$ at $t = 0$.

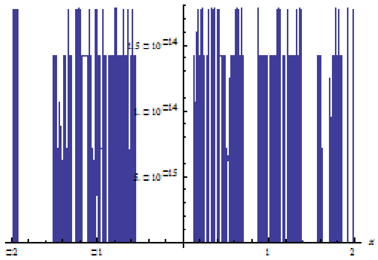

Figure 7

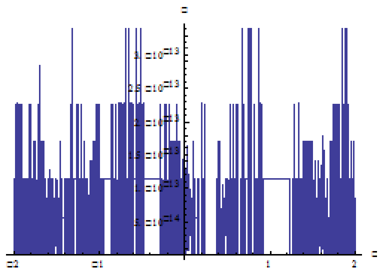

Figure 8

Figure 7-8: Absolute error between exact position solution and OHAM solution (model 1) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 2, \quad \beta = 4, \quad \lambda_1 = 0, \ \lambda_2 = 1 \ \ and \ \ w = 2$ at $t = 0$.

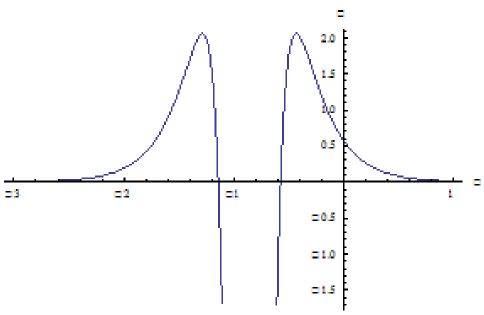

Figure 9

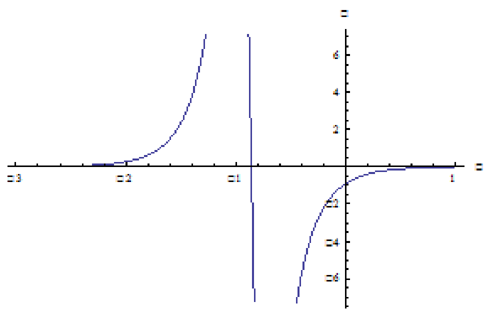

Figure 10

Figure 9-10: Exact negation solution (26) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 2, \quad \beta = 4, \quad \lambda_1 = 0, \ \lambda_2 = 1 \ \ and \ \ w = 2$ at $t = 0$.

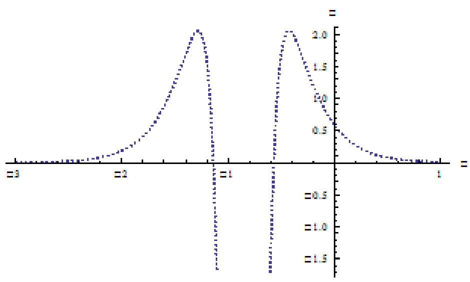

Figure 11

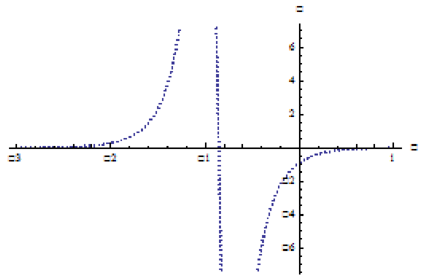

Figure 12

Figure 11-12: Semi analytic negation solution by OHAM (model 2) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 2, \quad \beta = 4, \quad \lambda_1 = 0, \ \lambda_2 = 1 \ \ and \ \ w = 2$ at $t = 0$.

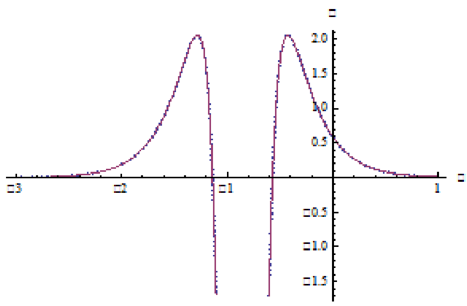

Figure 13

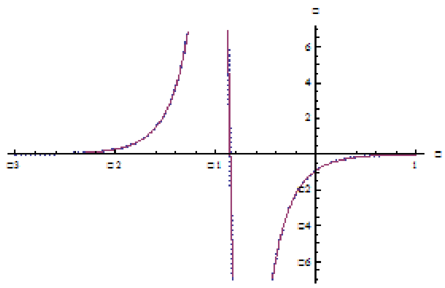

Figure 14

Figure 13-14: Comparison of exact negation solution and semi analytic negation solution by OHAM (model 2) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 1, \quad \beta = 4, \quad \lambda_1 = 0, \ \lambda_2 = 1 \ \ and \ \ w = 2$ at $t = 0$.

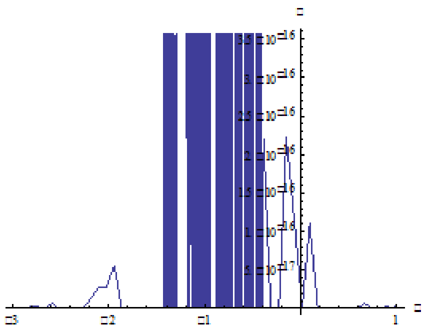

Figure 15

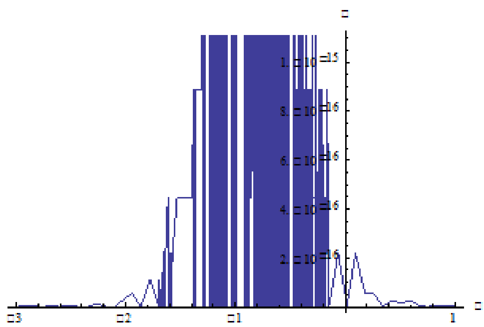

Figure 16

Figure 15-16: Absolute error between exact negation solution and OHAM solution (model 3) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by using the parameters $\alpha = 1, \quad \beta = 4, \quad \lambda_1 = 0, \ \lambda_2 = 1 \ \ and \ \ w = 2$ at $t = 0$.

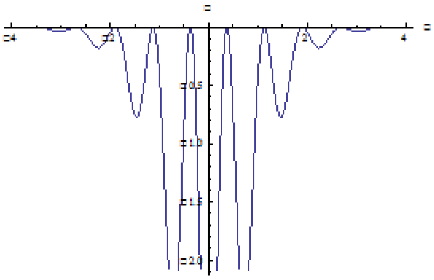

Figure 17

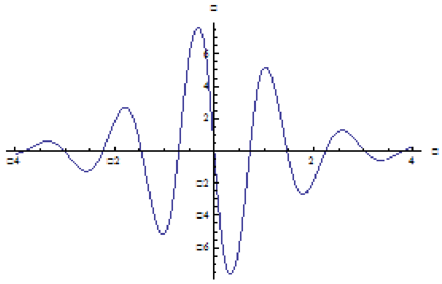

Figure 18

Figure 17-18: Exact analytical solution (31) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by at $t = 0$.

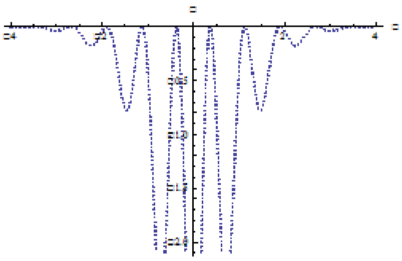

Figure 19

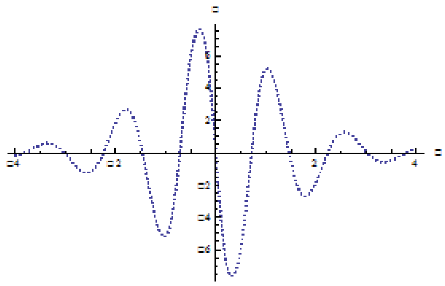

Figure 20

Figure 19-20: OHAM analytical solution (model 3) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by at $t = 0$.

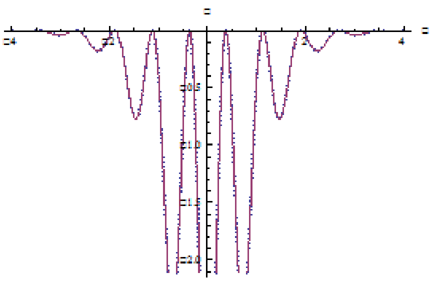

Figure 21

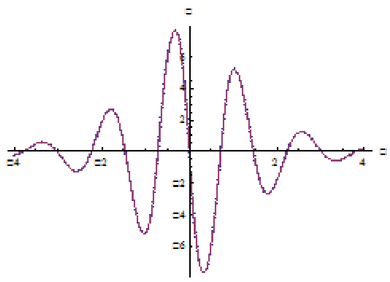

Figure 22

Figure 21-22: Comparison of exact analytical solution and semi analytical solution by OHAM (model 3) for complex KdV system (a) $\psi(\xi, t)$ (b) $\varphi(\xi, t)$ by at $t = 0$.

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