A Note on the FIP Property for Extensions of Commutative Rings

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Abstract. A ring extension \( R \subset S \) is said to be FIP if it has only finitely many intermediate rings between \( R \) and \( S \). The main purpose of this paper is to characterize the FIP property for a ring extension, where \( R \) is not (necessarily) an integral domain and \( S \) may not be an integral domain. Precisely, we establish a generalization of the classical Primitive Element Theorem for an arbitrary ring extension. Also, various sufficient and necessary conditions are given for a ring extension to have or not to have FIP, where \( S = R[\alpha] \) with \( \alpha \) a nilpotent element of \( S \).

1. Introduction

All rings considered below are commutative and unital; all inclusions of rings are unital. For a ring \( R \), we frequently use \( \text{Spec}(R) \) (respectively, \( \text{Max}(R) \)) to denote the set of all prime (respectively, maximal) ideals of \( R \). If \( R \subset S \) is an extension of rings, we will denote by \([R, S]\) the set of all \( R \)-subalgebras of \( S \) (that is, the set of rings \( T \) such that \( R \subset T \subset S \)), by \( (R : S) = \{x \in R : xS \subseteq R\} \) the conductor of \( R \) in \( S \). In particular, if \( [R, S] = \{R, S\} \), we say that \( R \subset S \) is a minimal extension \([6,9]\). Recall from [1] that a ring extension \( R \subset S \) is said to have (or to satisfy) FIP (for the “finitely many intermediate algebras property”) if \([R, S]\) is finite. The initial work on the FIP property in [1] was motivated in part by a desire to generalize the Primitive Element Theorem, a classical result in field theory: If \( K \subset L \) is a finite-dimensional field extension, \( L = K[\alpha] \) for some element \( \alpha \in L \) if and only if \([K, L]\) is finite. One example of a FIP extension would be any minimal ring extension , and whenever that condition holds, then \( S = R[\alpha] \) for each \( \alpha \in S \setminus R \). The key connection between the above ideas is that if a ring extension \( R \subset S \) has FIP, then any maximal chain \( R = R_0 \subset R_1 \subset \ldots \subset R_n = S \) is finite and results from juxtaposing \( n \) minimal extensions \( R_i \subset R_{i+1} \), \( 0 \leq i \leq n - 1 \). The FIP property was introduced and studied in [1] and, along with various related properties, has been treated in many other papers [2–5, 8–11]. In particular, Section 3 of [1] was devoted to the study of ring extension \( R \subset S \) satisfying FIP when \( R \) is a field. That work culminated in [1, Theorem 3.8] which gave a generalization of the Primitive Element Theorem. Later, Dobbs et al. in [2] completed this study in the case where \( R \) is replaced by an arbitrary Artinian reduced ring (cf. [2, Theorem III.2] and [2, Theorem III.5]). The present paper heavily relies on [1] and [2]; we will freely use the characterizations of the FIP extensions that were given there. The plan of this article is as follows: Section 2 was central to the work in [1, Section 3] and that led to the above-mentioned generalizations of the classical Primitive Element. The main result is the following: Let \( R \) be an infinite ring all of whose residue class fields are infinite and let \( R \subset S \) be an extension such that \( S/C \)
is a reduced ring, where \( C = (R : S) \). Then \( R \subseteq S \) has FIP if and only if \( R/C \) is an Artinian ring and \( S = R[\alpha] \) for some \( \alpha \in S \) where \( \alpha \) is algebraic over \( R \). (Recall that a ring is said to be reduced if it has no nonzero nilpotent elements.) As a consequence, we recover the result obtained by Anderson et al. in [1, Lemma 3.5].

Section 3 studies when FIP holds for ring extensions \( R \subseteq S \) such that \( S = R[\alpha] \), where \( \alpha \) is a nilpotent element. We establish some necessary and sufficient conditions for which a ring extension of this form has FIP. The first of these appears in Theorem 3.4 which states: Let \( R \) be a reduced ring and assume that \( S = R[\alpha] \) where \( \alpha \) is a nilpotent element of \( S \). Suppose that \( R/(R : S) \) is an infinite ring. Then \( R \subseteq S \) is a minimal extension if and only if \( (R : S) \in \text{Max}(R) \) and \( \alpha^2 \in (R : S) \). Also, we obtain a characterization of \([R, S]\) which satisfies FIP, in term of finite maximal chains. We present the following result in Theorem 3.5: If \( S = R[\alpha] \) where \( \alpha \in S \) satisfies \( \alpha^2 = 0 \), then \( R \subseteq S \) has FIP if and only if there exists a finite maximal chain from \( R \) to \( S \). As consequence of this result, we establish that if \( S = R[\alpha] \) where \( \alpha^2 = 0 \) and \( (R : S) \) is a maximal ideal of \( R \) or \( \beta \) has only finitely many ideals, then \( R \subseteq S \) has FIP. Another context for which we find a complete answer is given in Theorem 3.9: If \( R \) is a finite domain and \( S = R[\alpha, \beta] \), where \( \alpha^2 = \beta^2 = 0 \). Then \( R \subseteq S \) has FIP if and only if there exists a finite maximal chain from \( R \) to \( S \) and either \( S = R[\alpha] \) or \( S = R[\beta] \). Finally, any unexplained terminology is standard as in [12] and [13].

2. A generalized Primitive Element Theorem

Consider a ring extension \( R \subseteq S \) that has FIP. Recall from [1, Proposition 2.2 (a), (b)] that \( S \) must be a finite-type \( R \)-algebra and algebraic over \( R \). Moreover, in case \( R \) contains an infinite field, we have that \( S = R[\alpha] \) for some \( \alpha \in S \) that is algebraic over \( R \) (cf. [1, Corollary 3.2] and [1, Lemma 3.5]). Our primary interest in this section is to complete this study, we generalize the last cited results.

Proposition 2.1. Let \( R \subseteq S \) be an extension of rings such that:

(i) \( R/C \) is a finite ring, where \( C = (R : S) \);

(ii) \( S = R[\alpha] \) for some \( \alpha \in S \).

Then \( R \subseteq S \) has FIP if and only if \( \alpha \) is integral over \( R \).

Proof. For the “only if” part, since \( R/C \) is a finite ring, we have \( \dim(R/C) = 0 \) (the Krull dimension of \( R/C \)). Moreover, as \( R \subseteq C \) has FIP, then so is \( R/C \subseteq S/C \). It follows from [1,Proposition 3.4 (b)] that \( S/C \) is integral over \( R/C \). Whence, \( S/C \) is an integral over \( R/C \), in particular \( \alpha \) is integral over \( R \). Conversely, we assume that \( \alpha \) is integral over \( R \), then \( S/C = (R/C)[\pi] \) where \( \pi = \alpha + C \subseteq S/C \) is integral over \( R/C \). Thus, \( S/C \) is a finitely generated \( R/C \)-module and since \( R/C \) is a finite ring, hence \( S/C \) is also finite. Then, \( R/C \subseteq S/C \) has FIP. This prove that \( R \subseteq S \) has FIP.

\( \Box \)

Corollary 2.2. If \( S = \mathbb{Z}[\alpha] \) where \( \alpha \in S \) is integral over \( \mathbb{Z} \), then \( \mathbb{Z} \subseteq S \) has FIP if and only if \( (\mathbb{Z} : S) \neq 0 \).

Proof. Suppose that \( \mathbb{Z} \subseteq S \) has FIP and assume, by way of contradiction, that \( (\mathbb{Z} : S) = 0 \). Since \( S \) is a finitely generated \( \mathbb{Z} \)-module and each non unit of \( \mathbb{Z} \) is a non-zero-divisor of \( \mathbb{Z} \), then [3, Theorem 2.1] ensures that there exists a infinite chain of intermediate rings between \( \mathbb{Z} \) and \( S \). This contradicts the fact that \( \mathbb{Z} \subseteq S \) has FIP. Conversely, it suffice to notice that since \( (\mathbb{Z} : S) \neq 0 \), then \( \mathbb{Z}/(\mathbb{Z} : S) \) is finite. Hence, the result follows from Proposition 2.1.

\( \Box \)

To prove our main result, Theorem 2.4, we need the following lemma.

Lemma 2.3. Let \( R \subseteq S \) be an extension of rings. Denote \( C = (R : S) \). If \( R \subseteq S \) has FIP, then \( R/C \) is a reduced ring if and only if \( C \) is the intersection of finitely many maximal ideals of \( R \).
Proof. It is clear that if $C$ is the intersection of finitely many maximal ideals of $R$, then $R/C$ is a finite direct sum of fields. Thus $R/C$ is a reduced ring. Conversely, because $R \subset S$ has FIP, hence $R \subset S$ has FCP (in the sense of [4]). It follows from [4, Theorem 4.2] that $R/C$ is an Artinian ring. Since $R/C$ is a reduced Artinian ring, Wedderburn-Artin Theory (cf. [13, Theorem 3, page 209]) expresses $R/C$ uniquely as the internal direct product of finitely many fields $K_i$, that is, $R/C = K_1 \times \ldots \times K_n$. Let Max$(R/C) = \{N_1, \ldots, N_n\} = \{M_1/C, \ldots, M_n/C\}$, where $M_i \in \text{Max}(R)$ and $C \subseteq M_i$ for each $i = 1, \ldots, n$. As $N_1 \cap \ldots \cap N_n = 0$, then $(M_1/C) \cap \ldots \cap (M_n/C) = (M_1 \cap \ldots \cap M_n)/C = 0$. Thus $C = M_1 \cap \ldots \cap M_n$. □

Theorem 2.4 below provides a generalization of the Primitive Element Theorem.

Theorem 2.4. Let $R$ be an infinite ring all of whose residue class fields are infinite. Let $R \subset S$ be an extension such that $S/C$ is a reduced ring, where $C = (R : S)$. Then $R \subset S$ has FIP if and only if $R/C$ is an Artinian ring and $S = R[\alpha]$ for some $\alpha \in S$ where $\alpha$ is algebraic over $R$.

Proof. Notice by [2, Proposition II.4] that $R \subset S$ has FIP if and only if $R/C \subset S/C$ has FIP. For the “only if” part, since $S/C$ is a reduced ring, then $R/C$ is also a reduced ring. It follows from Lemma 2.3 that $C = \bigcap_{i=1}^{n} M_i$, where $M_i \in \text{Max}(R)$ for each $i$. By the Chinese Remainder Theorem, $R/C = K_1 \times \ldots \times K_n$ such that $K_i$ is an infinite field for each $i$, and hence $R/C$ is an Artinian ring. It remains to prove that $S = R[\alpha]$ for some $\alpha \in S$. By virtue of [4, Proposition 3.7 (d)], we can identify $S/C$ with $S_1 \times \ldots \times S_n$ such that $K_i \subseteq S_i$ and $R/C \subset S/C$ satisfies FIP if and only if $S_i = K_i[\beta]$ satisfies FIP if and only if $K_i \subset S_i$ satisfies FIP for each $i$. Notice that since $S/C$ is a reduced ring, then so is $S_i$. Then, we conclude form [1, Lemma 3.5] that $R/C \subset S/C$ satisfies FIP if and only if $S_i = K_i[\beta]$ for each $i$. Denote $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, then it is easy to verify that $K_i[\beta_1] \times \ldots \times K_i[\beta_n] \cong (K_i \times \ldots \times K_i)((\beta_1, \ldots, \beta_n)) = R/C[\beta]$. Therefore, $R/C \subset S/C$ satisfies FIP if and only if $S/C = R/C[\beta]$, where $\beta$ is algebraic over $R/C$. This implies that $R \subset S$ satisfies FIP if and only if $S = R[\alpha]$ for some $\alpha \in S$ which is algebraic over $R$ and satisfies $\bar{\alpha} = \alpha + C = \beta$.

For the “if” part, assume that $S = R[\alpha]$ for some $\alpha \in S$ where $\alpha$ is algebraic over $R$ and $R/C$ is an Artinian ring. Since, in addition, $R/C$ is reduced, hence Wedderburn-Artin Theory (cf. [13, Theorem 3, page 209]) expresses $R/C$ uniquely as the internal direct product of finitely many fields $K_i$, that is, $R/C = K_1 \times \ldots \times K_n$. Again [4, Proposition 3.7 (d)], the ring $S/C$ can be uniquely expressed as a product of rings $S_1 \times \ldots \times S_n$ where $K_i \subseteq S_i$ for each $i \in \{1, \ldots, n\}$. Moreover, since $S/C = R/C[\alpha]$ where $\bar{\alpha} = \alpha + C$, hence reasoning as in the proof of the “only if” part, we deduce that $S_i = K_i[\beta_i]$ where $\bar{\alpha} = (\beta_1, \ldots, \beta_n)$ and $\beta_i$ is algebraic over $K_i$. Hence, if $K_i$ is a finite field, then $S_i$ is a finite $K_i$-vector space. Then, $S_i$ is finite and so $K_i \subseteq S_i$ has FIP. Now, if $K_i$ is infinite field, then [1, Lemma 3.5] ensures that $K_i \subseteq S_i$ has FIP. By globalization, we deduce that $K_i \subseteq S_i$ has FIP for each $i \in \{1, \ldots, n\}$. Then, $R/C \subset S/C$ has FIP [4, Proposition 3.7 (d)]. Finally, according to [2, Proposition II.4], we conclude that $R \subset S$ has FIP, which completes the proof. □

In view of Theorem 2.4, the “if” implication is valid, for if $R/C$ is an Artinian ring. The following example will show that the hypothesis “$R/C$ is an Artinian ring” cannot be omitted in the above theorem.

Example 2.5. Let $R$ be an infinite-dimensional valuation domain with a height 1 prime ideal $P$. Pick $\alpha \in P$ where $\alpha \neq 0$ and set $S = qf(R)$ the quotient field of $R$. It is clear that $C = (R : S) = 0$, and hence $R/C \cong R$ is not Artinian. Also $S/C \cong S$ is a reduced ring. On the other hand, [12, Theorem 19] ensures that $S = R[\alpha^{-1}]$. But $R \subset S$ does not have FIP since $[R_{\alpha}, p \in \text{Spec}(R)]$ is an infinite set of intermediate rings between $R$ and $qf(R)$.

Corollary 2.6. ([11, Lemma 3.5]) Let $R$ be an infinite field, and let $R \subset S$ be an extension such that $S$ is a reduced ring. Then $R \subset S$ has FIP if and only if $S = R[\alpha]$ for some $\alpha \in S$ such that $\alpha$ is algebraic over $R$.

Proof. Since $R$ is quasi-local with maximal ideal 0, then $R/0 \cong R$ is infinite. Moreover, as $(R : S) = 0$, hence $S/(R : S) \cong S$ is a reduced ring. Therefore, the conclusion follows readily from Theorem 2.4. □

3. When the generator is a nilpotent element

Consider a ring extension $R \subset S$. In view of the central role that nilpotent elements have played in the study of the FIP property for a ring extension (cf. [1, Theorem 3.8] and Section IV of [2]), we devote
this section to completing this study and to investigating when \( R \subset S \) has FIP where \( S = \mathbb{R}[\alpha] \) with \( \alpha \) is a nilpotent element of \( S \). We begin with two results giving useful sufficient conditions for FIP to fail.

**Proposition 3.1.** Let \( R \subset S \) be a ring extension such that \( S = \mathbb{R}[\alpha] \) where \( \alpha \) is a nilpotent element of \( S \). If \( (R : S) \in \text{Spec}(R) \setminus \text{Max}(R) \), then \( R \subset S \) does not have FIP.

**Proof.** Since \((R : S) \in \text{Spec}(R) \setminus \text{Max}(R)\), then \( R/(R : S) \) is a integral domain (not a field), and we have \( S/(R : S) = (R/(R : S))[\overline{\alpha}] \) where \( \overline{\alpha} = \alpha + (R : S) \). We prove that \((0 : \overline{\alpha}) = [\overline{r} \in R/(R : S)|\overline{r}\overline{\alpha} = 0] = 0 \). Let \( \overline{r} \in R/(R : S) \) such that \( \overline{r}\overline{\alpha} = 0 \), hence \( \overline{r}\overline{\alpha} = 0 \). It follows that \( r \in (R : S) \). As \((R : S) \) is a prime ideal of \( R \) and \( \alpha \notin (R : S) \), we conclude that \( r \in (R : S) \). This implies that \( \overline{r} = 0 \), and so \((0 : \overline{\alpha}) = 0 \). According to \[2, \text{Proposition IV.1]\], we have that \( R/(R : S) \subset (R : S) \) does not have FIP, and so is \( R \subset S \). \( \Box \)

The following result is a generalization of \[2, \text{Proposition IV.1}\].

**Corollary 3.2.** Let \( R \) be an integral domain that is not a field, and \( R \subset S \) such that \( S = \mathbb{R}[\alpha] \) where \( \alpha \) is a nilpotent element of \( S \). If \( (R : S) \neq 0 \), then \( R \subset S \) does not have FIP.

**Proposition 3.3.** Let \( R \subset S \) be an extension such that \( S = \mathbb{R}[\alpha] \) where \( \alpha \) is a nilpotent element of \( S \). Denote \( C = (R : S) \). If \( C \in \text{Max}(R) \), then \( R \subset S \) has FIP if and only if \( R/C \) is finite or \( R/C \) is an infinite field and \( \alpha \in C \).

**Proof.** Notice by \[2, \text{Proposition II.4}\] that \( R \subset S \) has FIP if and only if \( R/C \subset S/C \) has FIP. We have \( S/C = R/C[\overline{\alpha}] \) where \( \overline{\alpha} = \alpha + C \). If \( R/C \) is finite, then \( S/C \) is also finite since \( S/C \) is a \( R/C \)-vector space. Thus \( R/C \subset S/C \) has FIP, and so is \( R \subset S \). Now, if \( R/C \) is an infinite field, then \[1, \text{Lemma 3.6 (b)}\] ensures that \( R/C \subset S/C \) has FIP if and only if \( \overline{\alpha} \neq 0 \), that is, \( R \subset S \) has FIP if and only if \( \alpha \in C \). \( \Box \)

The following result is a characterization of minimal extensions where \( S \) is the form \( \mathbb{R}[\alpha] \) for some nilpotent element \( \alpha \in S \).

**Theorem 3.4.** Let \( R \subset S \) be a reduced ring and let \( S = \mathbb{R}[\alpha] \) where \( \alpha \) is a nilpotent element of \( S \). Suppose that \( R/(R : S) \) is a infinite ring. Then \( R \subset S \) is a minimal extension if and only if \( (R : S) \in \text{Max}(R) \) and \( \alpha \notin (R : S) \).

**Proof.** If \( R \subset S \) is a minimal (integral) extension, then \( C = (R : S) \in \text{Max}(R) \) and from Proposition 3.3 we have \( \alpha \in C \). It follows that \( R/C \) is a infinite field and \( S/C = R/C[\overline{\alpha}] \) where \( \overline{\alpha} = \alpha + C \), and so \( \overline{\alpha} \neq 0 \). Hence, the proof of \[1, \text{Lemma 3.6 (b)}\] shows that \( [R/C, S/C] = [R/C, R/C[\overline{\alpha}]], S/C = R/C[\overline{\alpha}] \). Moreover, \( R/C \subset S/C \) is a minimal extension since \( R \subset S \) is a minimal extension, we conclude that either \( R/C = R/C[\overline{\alpha}] \) or \( R/C[\overline{\alpha}] = R/C[\overline{\alpha}] \). Then, either \( R = \mathbb{R}[\alpha] \) or \( \mathbb{R}[\alpha] = R[\alpha] \). Suppose that \( R[\alpha] = \mathbb{R}[\alpha] \) and let \( n \geq 2 \) be the index of nilpotency for \( \alpha \). Hence, \( \alpha = r_0 + r_1 \alpha + r_2 \alpha^2 + \ldots + r_{n-1} \alpha^{2(n-1)} \), for some \( r_0, r_1, \ldots, r_{n-1} \in R \). Thus, \( r_0 = \alpha - (r_1 \alpha^2 + r_2 \alpha^3 + \ldots + r_{n-1} \alpha^{2(n-1)}) \) is a nilpotent element, and so \( r_0 = 0 \) since \( R \) is reduced. This implies that \( \alpha = a(r_1 \alpha + r_2 \alpha^2 + \ldots + r_{n-1} \alpha^{2(n-1)}) \), hence \( (r_1 \alpha + r_2 \alpha^2 + \ldots + r_{n-1} \alpha^{2(n-1)}) = 1 \), a contradiction since \( (r_1 \alpha + r_2 \alpha^2 + \ldots + r_{n-1} \alpha^{2(n-1)}) \) is a nilpotent element. Therefore, \( R = \mathbb{R}[\alpha] \), and hence \( \alpha \notin R \). Now, we prove that \( \alpha \in C \). Let \( x \notin S \), then \( x = a_0 + a_1 \alpha + a_2 \alpha^2 + \ldots + a_{n-1} \alpha^{n-1} \) for some \( a_0, a_1, \ldots, a_{n-1} \in R \). Hence, \( a \alpha = a_0 \alpha + a_1 \alpha^2 + a_2 \alpha^3 + \ldots + a_{n-1} \alpha^{n} \). Notice that any power of \( \alpha \) is a product of a power of \( \alpha^2 \) and a power of \( \alpha^3 \). As \( \alpha \alpha = \alpha^2 \in \alpha \), it follows that \( a \alpha^2 \in \alpha \), and hence \( \alpha^2 \in C \). Conversely, since \( \alpha \alpha \in C \), then \( S/C = R/C[\overline{\alpha}] \) where \( \overline{\alpha^2} = 0 \). As, in addition, \( R/C \) is a infinite field since \( C \) is a maximal ideal of \( R \), then the end of the proof of \[1, \text{Lemma 3.6 (b)}\] ensures that \( R/C \subset S/C \) is a minimal extension, this implies that \( R \subset S \) is also a minimal extension \[9, \text{Corollary 1.4}\]. \( \Box \)

We are now in position to give a characterization of \([R, S]\) which satisfies FIP, in term of finite maximal chains.

**Theorem 3.5.** If \( R \subset S \) is an extension of rings such that \( S = \mathbb{R}[\alpha] \) with \( \alpha^2 = 0 \), then the following conditions are equivalent:

(i) \( R \subset S \) has FIP;

(ii) There exists a finite maximal chain from \( R \) to \( S \).
Proof. (i) ⇒ (ii) The result is clear since the condition “$R \subset S$ has FIP”, implies that any maximal chain from $R$ to $S$ is finite.

(ii) ⇒ (i) Since $S = R + Ra$, therefore [7, Proposition 4.12] gives a bijection between $[R, S]$ and the set of ideals of $R$ containing $C = (R : S)$. On the other hand, by assumption, there is a finite maximal chain $R = R_0 \subset R_1 \subset \ldots \subset R_n = S$ in $[R, S]$. For each $i = 0, \ldots, n-1$, denote $C_i = (R : R_{i+1})$ and $m_i = C_i \cap R$. Since $R_i \subset R_{i+1}$ is both minimal and integral, hence $C_i \in \text{Max}(R)$ and so $m_i \in \text{Max}(R)$ [6, Thorme 2.2]. Moreover, it is clear that $C \subseteq C_i$ for each $i$, thus $C \subseteq \bigcap_{i=0}^{n-1} m_i$. By iteration, we get

\[
\prod_{i=0}^{n-1} m_i R \subseteq \prod_{i=0}^{n-2} m_i R_{n-1} \subseteq \ldots \subseteq m_0 R_1 \subseteq R.
\]

Then, $\prod_{i=0}^{n-1} m_i \subseteq C \subseteq \bigcap_{i=0}^{n-1} m_i$. Hence, the $m_i$ are precisely the uniquely ideals of $R$ containing $C$. Therefore, $|[R, S]| = \left|\prod_{i=0}^{n-1} m_i\right|$, this prove that $R \subset S$ has FIP. \hfill \Box

The proof of Theorem 3.5 established the following result.

**Proposition 3.6.** Let $R \subset S$ be a ring extension such that $S = R[a]$ where $a^2 = 0$. If $(R : S)$ is a maximal ideal of $R$ or $R$ has only finitely many ideals, then $R \subset S$ has FIP. Moreover, $R \subset S$ is a minimal extension if and only if $(R : S) \in \text{Max}(R)$.

**Remark 3.7.** If $S = R[a]$ where $a$ is a nilpotent element of $S$ of index $n \neq 2$, then Theorem 3.5 does not follow in general. For instance, let $R$ be any infinite field $K$ of characteristic 2 and take $S = K[X]/(X^4) = K[x]$ where $x = X + (X^4)$ and $x^4 = 0$. Then, $\{1, x, x^2, x^3\}$ is a $K$-vector space basis of $S$. As $\dim_K(S) < \infty$, then any maximal chain of intermediate rings between $K$ and $S$ is finite, while the failure to satisfy FIP can be seen by applying [1, Lemma 3.6(a)].

We next give the following lemma which be used often later. Lemma 3.8 provides a generalization of [1, Lemma 2.6 (c)].

**Lemma 3.8.** Let $R \subset S$ be an extension. If $R$ is infinite domain and $R \subset S$ has FIP, then $S$ does not contain two nilpotent elements of index 2 which are algebraically independent over $R$.

**Proof.** If the assertion fails, $S$ contains two nilpotent elements $a$ and $\beta$ of index 2 which are algebraically independent over $R$. We consider two cases:

Case 1. $a\beta = 0$, then $\{1, a, \beta\}$ is a basis of $R[a, \beta]$ as a finitely generated $R$-module. For each $r \in R$, consider $T_r = \{ a + ba + r\beta : a, b \in R \}$. It is clear that $R \subseteq T_r \subseteq S$ for each $r$. Moreover, since $a$ and $\beta$ are nilpotent elements of index 2, on easy verifies that each $T_r$ is a ring. Also, $T_r \neq T_{r'}$ for each $r \neq r'$. Indeed, if $T_r = T_{r'}$, then $a + r\beta = a_0 + b_0a + r'b_0\beta$ for some $a_0, b_0 \in R$. Since $\{1, a, \beta\}$ is a basis of $R[a, \beta]$, it follows that $a_0 = 0$, $b_0 = 1$ and $r = b_0r'$. This yields that $r = r'$. Since $R$ is infinite, $\{T_r, r \in R\}$ is an infinite collection of intermediate rings between $R$ and $S$, contradicting that $R \subset S$ has FIP.

Case 2. $a\beta \neq 0$. First, suppose that $a\beta$ is algebraically independent with $a$ and $\beta$ over $R$, then $\{1, a, \beta, a\beta\}$ is a basis of $R[a, \beta]$ as a finitely generated $R$-module. For each $r \in R$, consider $T_r = \{ a + ba + r\beta : a, b \in R \}$. Reasoning as in the first case, we show that $\{T_r, r \in R\}$ describes an infinite family of rings, contradicting that $R \subset S$ has FIP. In the remaining case, $a\beta = r_0a + r_1\beta$ where $r_0, r_1 \in R$. Let $r \in R$, consider $T_r = \{ a + r\beta : a,b,c \in R \}$. Then, $T_r$ is intermediate ring between $R$ and $S$. Moreover, $T_r \neq T_{r'}$ for each $r \neq r'$. Indeed, if $r + r' = a_0 + r'\beta + a_0 + r'\beta$ for some $a_0, b_0, c_0 \in R$ such that $b_0 \neq c_0$. Then, $T_r$ is intermediate ring between $R$ and $S$. Moreover, $T_r \neq T_{r'}$ for each $r \neq r'$. Indeed, if $r + r' = a_0 + r'\beta + a_0 + r'\beta$ for some $a_0, b_0, c_0 \in R$ where $b_0 \neq c_0$. Since $\{1, a, \beta\}$ is a basis of $R[a, \beta]$ as a finitely generated $R$-module, then $a_0 = 0$ and $r = r'\beta = r'\beta$. Because $R$ is integral domain, it follows that $b_0 = c_0$, the desired contradiction. Therefore, $\{T_r, r \in R\}$ is an infinite collection of intermediate rings between $R$ and $S$, contradicting that $R \subset S$ has FIP. \hfill \Box

Again, by combining Lemma 3.8 and Theorem 3.5, we obtain directly another characterization of $[R, S]$ which satisfies FIP where $S = R[a, \beta]$ and $a^2 = \beta^2 = 0$. 
**Theorem 3.9.** Let $R \subset S$ be an extension such that $R$ is infinite domain and $S = R[\alpha, \beta]$, where $\alpha^2 = \beta^2 = 0$. Then $R \subset S$ has FIP if and only if there exists a finite maximal chain from $R$ to $S$ and either $S = R[\alpha]$ or $S = R[\beta]$.

We close this section by the following proposition.

**Proposition 3.10.** Let $R = R_1 \times \cdots \times R_n$ be a finite product of rings and let $R \subset S$ be a ring extension. Using [2, Lemma III.3], identify $S$ with $S_1 \times \cdots \times S_n$. For each $i \in \{1, \ldots, n\}$, consider the following three conditions (which depend on $i$):

1. $R_i$ is finite and $S_i$ is a finitely generated $R_i$-module;
2. $R_i$ is infinite ring all of whose residue class fields are infinite, $S_i/C_i$ is a reduced ring where $C_i = (R_i : S_i)$, $R_i/C_i$ is Artinian and $S_i = R_i[\alpha_i]$ for some $\alpha_i \in S_i$, which is algebraic over $R_i$.
3. $R_i$ is infinite, $(R_i : S_i) \in \text{Max}(R_i)$ and $S_i = R_i[\alpha_i]$ for some $\alpha_i \in S_i$ which satisfies $\alpha_i^3 \in (R_i : S_i)$.

If for each $i \in \{1, \ldots, n\}$, at least one of the conditions (1), (2), (3) holds, then $R \subset S$ has FIP.

**Proof.** Combine [2, Proposition III.4 (a)] with [4, Proposition 5.1], Theorem 2.4 and Proposition 3.3.

**References**