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$\lambda-$Double Statistical Convergence of Functions

Rabia Savaş[a]

[a]*Department of Mathematics, Sakarya University, Sakarya, Turkey*

Abstract. In this paper, the definitions of $\lambda-$double strong summability and $\lambda-$double statistical convergence for real valued measurable functions of two variables defined on $(1,\infty)\times(1,\infty)$ are presented. Using these definitions we present a series of basic results. Additionally, inclusion theorems, extension of existing results in the literature, and their variations have been established.

1. Introduction

The concept of statistical convergence plays an important role in summability theory and functional analysis. The relationship between the summability theory and statistical convergence was introduced by Schoenberg [20]. In a manner similar to Schoenberg's definition, Borwein [1] introduced and studied strongly summable functions. Soon after that, statistically convergent sequences were studied in [6] and [8] extensively. Following Fridy's investigations, Mursaleen in [11] presended a notion of $\lambda-$statistically convergent sequences. Nuray [14] extended Mursaleen's idea to $\lambda-$strongly summable and $\lambda-$statistically convergent functions by taking nonnegative real-valued Lebesgue measurable functions defined on the interval $(1,\infty)$. Recently, Connor and Savas [4] introduced lacunary statistical and sliding window convergence for measurable functions. Moreover, the statistical exhaustiveness was studied by Caserta and Kočinac [3].

Fast [5] and many authors defined statistical convergence as follows: A sequence $x=(x_k)$ is said to be statistically convergent to the number L, if for every $\varepsilon>0$ the following is satisfied:

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n: |x_k-L|\geq\varepsilon\}\right|=0.$$

Whenever this occurs, we write $st-\lim x_k=L$.

In many branches of science and engineering we often come across double sequences, i.e. sequences of matrices and certainly there are situations where either the idea of ordinary convergence does not work or the underlying space does not serve our purpose. In order to deal with such situations we have to introduce a new types of measures which can provide a better tool and a suitable frame work. To that end, we recall that the notion of convergence for double sequences was presented first by Pringsheim in [17] as follows:

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Email address: rabiasavass@hotmail.com (Rabia Savaş)

A double sequence $x = (x_{kl})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kl} - l| < \varepsilon$, whenever $k, l > N_\varepsilon$, in this case we denote such a limit as the following:

$$P - \lim_{k,l\to\infty} x_{kl} = L.$$

More investigations in this direction and more applications of double sequences can be found in [2, 15] where many important references can be found.

The concept of statistical convergence for double sequences was presented in [9] and [10], respectively .

A double sequence $x = (x_{kl})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} |\{k \le m \text{ and } l \le n : |x_{kl} - L| \ge \varepsilon\}| = 0.$$

There are several papers dealing with double statistical convergence (see [12, 16, 18, 19]). Strongly summable sequences arise in the theory of Fourier series, ergodic theory and as well as summability theory. Strongly summable single valued functions were introduced and studied by Borwein in [1]. He defined the following: A nonnegative real-valued Lebesgue measurable function $f(t)$ on the interval $(1, \infty)$ is said to be strongly summable to L if,

$$\lim_{n\to\infty} \frac{1}{n} \int_1^n \left| f(t) - L \right| dt = 0.$$

Definition 1.1. ([7]) Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$. Let $I_n = [n - \lambda_n + 1, n]$.

The following definitions are presented in Nuray [14].

Definition 1.2. ([14]) Let $f(t)$ be a real valued function which is measurable on $(1, \infty)$, if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \int_{n-\lambda_n+1}^{n} \left| f(t) - L \right| dt = 0,$$

then we say that the function $f(t)$ is $\lambda-$strongly summable to L. Whenever this occurs, we write $[W, \lambda] - \lim f(t) = L$ and

$$[W, \lambda] := \{f(t) : \exists L, \ [W\lambda] - \lim f(t) = L\}.$$

If we take $\lambda_n = n$, $[W, \lambda]$ reduces to $[W]$, the space of strongly summable functions.

Definition 1.3. ([14]) Let $f(t)$ be a real valued function which is measurable on $(1, \infty)$. If for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ t \in I_n : \left| f(t) - L \right| \ge \varepsilon \right\} \right| = 0,$$

where the vertical bars indicate the Lebesgue measure of the enclosed set, then we say that the function $f(t)$ is $\lambda-$statistically convergent to L. In this case we write $S_\lambda - \lim f(t) = L$ and

$$(S, \lambda) := \{f(t) : \exists L, \ S_\lambda - \lim f(t) = L\}.$$

If we take $\lambda_n = n$, (S, λ) reduces to S, the set of statistically convergent functions.

Recently, Mursaleen et al. [13] studied the generalized statistical convergence and statistical core of double sequences.

Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ are two nondecreasing sequences of positive real numbers such that each tending to ∞ and

$$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1$$

and

$$\mu_{n+1} \leq \mu_n + 1, \mu_1 = 1.$$

The collection of such sequences (λ, μ) will be denoted by Δ.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers. Then the (λ, μ) density of K is defined as

$$\delta_{\lambda,\mu} = P - \lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} \left|\{m - \lambda_m + 1 \leq j \leq m, n - \mu_n + 1 \leq k \leq n : (j,k) \in K\}\right|$$

provided that the limit exists, where $I_m = [m - \lambda_m + 1, m]$ and $J_n = [n - \mu_n + 1, n]$ and $\lambda_{mn} = \lambda_m \mu_n$.

Throughout this paper we shall denote $(k \in I_m, l \in J_n)$ by $(k, l) \in I_{mn}$.

Definition 1.4. A double sequence $x = (x_{kl})$ is said to be $(\lambda, \mu)-$statistically convergent to the number L if for every $\varepsilon > 0$

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} |\{(k,l) \in I_{mn} : |x_{kl} - L| \geq \varepsilon\}| = 0.$$

We denote this by $S_{\lambda,\mu} - \lim x = L$.

2. Main Results

In this section we shall present new definitions which are useful in the sequel of this paper. By a two variable function $f(t,s)$, we shall mean a real valued measurable function of two variables in the interval $(1,\infty) \times (1,\infty)$.

Definition 2.1. A function $f(t,s)$ of two variables is said to be strongly double summable to L if

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \int_1^m \int_1^n \left|f(t,s) - L\right| dtds = 0.$$

$[V]_2$ will denote the space of all strongly double summable functions.

Definition 2.2. Let $\lambda, \mu \in \Delta$. A function $f(t,s)$ of two variables is said to be $\lambda-$strongly double summable to L if

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \int_{m-\lambda_m+1}^{m} \int_{n-\mu_n+1}^{n} \left|f(t,s) - L\right| dtds = 0.$$

Whenever this occurs, we write $f(t,s) \to L([V, \lambda, \mu])$. The set of all $\lambda-$strongly double summable functions will be denoted by simply $[V, \lambda, \mu]$. If we take $\lambda_{mn} = mn$ then $[V, \lambda, \mu]$ reduces to $[V]_2$, the space of all strongly double summable functions.

Definition 2.3. A function $f(t,s)$ of two variables is said to be double statistically convergent to L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{(t,s) : t \le m,\ s \le n : \left|f(t,s) - L\right| > \varepsilon\right\}\right| = 0,$$

where the vertical bars indicate the Lebesgue measure of the enclosed set. In this case we write $S_f - \lim f(t,s) = L$ and

$$S_f := \left\{f(t,s) : \exists L,\ S_f - \lim f(t,s) = L\right\}.$$

Definition 2.4. Let $\lambda, \mu \in \Delta$. The two variable function $f(t,s)$ is said to be $\lambda\mu-$ double statistically convergent to L, if for every $\varepsilon > 0$,

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(t,s) \in I_{mn} : \left|f(t,s) - L\right| \ge \varepsilon\right\}\right| = 0,$$

where the vertical bars indicate the Lebesgue measure of the enclosed set. In this case we write $S_f^{\lambda,\mu} - \lim f(t,s) = L$ and

$$(S,\lambda,\mu) := \left\{f(t,s) : \exists L,\ S_f^{\lambda,\mu} - \lim f(t,s) = L\right\}.$$

If we take $\lambda_{mn} = mn$, then (S,λ,μ) is the same as S_f, the set of double statistically convergent functions.

Theorem 2.5. *Let $\lambda = (\lambda_{mn}) \in \Delta$ and $f(t,s)$ be a real valued measurable function of two variables. Then*

1. $[V,\lambda,\mu] \subset (S,\lambda,\mu)$ *and the inclusion is proper;*
2. *If $f(t,s)$ is bounded and $S_f^{\lambda\mu} - f(t,s) = L$ then $[V,\lambda,\mu] - \lim f(t,s) = L$ and hence $[V]_2 - \lim f(t,s) = L$ provided $f(t,s)$ is not eventually constant.*

Proof. 1. Let $\varepsilon > 0$ and $[V,\lambda,\mu] = \lim f(t,s) = L$, and note

$$\begin{aligned} \int_{(t,s)\in I_{mn}} \left|f(t,s) - L\right| dtds &\ge \int_{(t,s)\in I_{mn}, \left|f(t,s)-L\right|\ge\varepsilon} \left|f(t,s) - L\right| dtds \\ &\ge \varepsilon \left|\left\{(t,s) \in I_{mn} : \left|f(t,s) - L\right| \ge \varepsilon\right\}\right|. \end{aligned}$$

Therefore $[V,\lambda,\mu] - \lim f(t,s) = L$ implies $S_f^{\lambda\mu} - \lim f(t,s) = L$.

Define a function $f(t,s)$ by

$$f(t,s) = \begin{cases} ts, m - \sqrt{\lambda_m} + 1 \le t \le m,\ n - \sqrt{\mu_n} + 1 \le s \le n; \\ 0, otherwise. \end{cases}$$

Then $f(t,s)$ is not a bounded function and for every ε $(0 < \varepsilon \le 1)$

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(t,s) \in I_{mn} : \left|f(t,s) - 0\right| \ge \varepsilon\right\}\right| = P - \lim_{m,n\to\infty} \frac{\sqrt{\lambda_{mn}}}{\lambda_{mn}} = 0.$$

Thus, $S_f^{\lambda\mu} - \lim f(t,s) = 0$. However,

$$P - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \int_{m-\lambda_m+1}^{m} \int_{n-\mu_n+1}^{n} \left|f(t,s) - 0\right| dtds = \infty.$$

Therefore $f(t,s) \notin [V,\lambda,\mu]$. Thus, the inclusion is proper.

2. Suppose that $S_f^{\lambda\mu} - \lim f(t,s) = L$ and $f(t,s)$ be a bounded function, say $\left|f(t,s)-L\right| \leq M$ for all t,s. Given $\varepsilon > 0$, we are granted

$$\begin{aligned}
\frac{1}{\lambda_{mn}} \int\limits_{(t,s)\in I_{mn}} \left|f(t,s)-L\right| dtds &= \frac{1}{\lambda_{mn}} \int\limits_{(t,s)\in I_{mn}, \left|f(t,s)-L\right| \geq \varepsilon} \left|f(t,s)-L\right| dtds \\
&\quad + \frac{1}{\lambda_{mn}} \int\limits_{(t,s)\in I_{mn}, \left|f(t,s)-L\right| < \varepsilon} \left|f(t,s)-L\right| dtds \\
&\leq \frac{M}{\lambda_{mn}} \left|\left\{(t,s)\in I_{mn} : \left|f(t,s)-L\right| \geq \varepsilon\right\}\right| + \varepsilon
\end{aligned}$$

which implies that $[V,\lambda,\mu] - \lim f(t,s) = L$. Also, we have the following since $\frac{\lambda_{mn}}{mn} \leq 1$ for all m,n,

$$\begin{aligned}
\frac{1}{mn} \int_1^m \int_1^n \left|f(t,s)-L\right| dtds &= \frac{1}{mn} \int_1^{m-\lambda_m} \int_1^{n-\mu_n} \left|f(t,s)-L\right| dtds \\
&\quad + \frac{1}{mn} \int\limits_{(t,s)\in I_{mn}} \left|f(t,s)-L\right| dtds \\
&\leq \frac{1}{mn} \int_1^{m-\lambda_m} \int_1^{n-\mu_n} \left|f(t,s)-L\right| dtds \\
&\quad + \frac{1}{mn} \int\limits_{(t,s)\in I_{mn}} \left|f(t,s)-L\right| dtds \\
&\leq \frac{2}{\lambda_m \mu_n} \int\limits_{(t,s)\in I_{mn}} \left|f(t,s)-L\right| dtds.
\end{aligned}$$

Hence, $[V]_2 - \lim f(t,s) = L$ since $[V,\lambda,\mu] - \lim f(t,s) = L$. □

Theorem 2.6. *$S_f \subset (S,\lambda,\mu)$ if and only if*

$$P - \lim_{m,n\to\infty} \inf \frac{\lambda_{mn}}{mn} > 0. \tag{1}$$

Proof. For $\varepsilon > 0$,

$$\left\{t \leq m,\ s \leq n : \left|f(t,s)-L\right| \geq \varepsilon\right\} \supset \left\{(t,s)\in I_{mn} : \left|f(t,s)-L\right| \geq \varepsilon\right\}.$$

Therefore,

$$\begin{aligned}
\frac{1}{mn} \left|\left\{t \leq m,\ s \leq n : \left|f(t,s)-L\right| \geq \varepsilon\right\}\right| &\geq \frac{1}{mn} \left|\left\{(t,s)\in I_{mn} : \left|f(t,s)-L\right| \geq \varepsilon\right\}\right| \\
&\geq \frac{\lambda_{mn}}{mn} \frac{1}{\lambda_{mn}} \left|\left\{(t,s)\in I_{mn} : \left|f(t,s)-L\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Hence by using (1) and taking the limit as $m,n\to\infty$ in the Pringsheim sense, we are granted that $f(t,s) \to L(S_f)$ implies $f(t,s) \to L(S,\lambda,\mu)$.

Conversely, suppose that $P-\lim_{m,n\to\infty}\inf\frac{\lambda_{mn}}{mn}=0$. We can choose indices $\left(m_j\right)$ and (n_k) such that $\frac{\lambda_{m_j,n_k}}{m_j,n_k}<\frac{1}{jk}$. Define a function $f(t,s)$ by 1 if $(t,s)\in I_{m_j,n_k}j=k=1,2,...$ and 0 otherwise. Then $f(t,s)\in[V]_2$ and hence $f(t,s)\in S_f$. But $f(t,s)\notin[V,\lambda,\mu]$ and Theorem 2.5 (2) implies that $f(t,s)\notin(S,\lambda,\mu)$. Thus (1) is necessary. $\square$

Finally, we conclude this paper with the following definition and two theorems without proofs. The proofs are omitted them because the techniques are similar to those used in Theorem 2.5.

Definition 2.7. Let $\lambda,\mu\in\Delta$, p be a real number. Then the two variable function $f(t,s)$ is said to be λ_p-strongly double summable L, if

$$P-\lim_{m,n\to\infty}\frac{1}{\lambda_m\mu_n}\int\limits_{m-\lambda_m+1}^{m}\int\limits_{n-\mu_n+1}^{n}\left|f(t,s)-L\right|^p dtds=0.$$

In this case we write $\left[V_p,\lambda,\mu\right]-\lim f(t,s)=L$ and

$$\left[V_p,\lambda,\mu\right]:=\left\{f(t,s):\exists L,\left[V_p,\lambda,\mu\right]-\lim f(t,s)=L\right\}.$$

If we take, $\lambda_m=m,\mu_n=n$, then $\left[V_p,\lambda,\mu\right]$ is the same as $\left[V_p\right]$, the set of strongly $p-$Cesàro summable functions.

Theorem 2.8. *Let $1\leq p<\infty$. If a two variable function $f(t,s)$ is λ_p-strongly double summable to L, then it is $\lambda\mu-$double statistically convergent to L.*

Theorem 2.9. *Let $1\leq p<\infty$. If a bounded two variable function $f(t,s)$ is $\lambda\mu-$double statistically convergent to L, then it is λ_p-strongly double summable to L.*

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