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Almost Sure Exponential Stability of Stochastic Differential Delay Equations

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Abstract. This paper mainly studies whether the almost sure exponential stability of stochastic differential delay equations (SDDEs) is shared with that of the stochastic theta method. We show that under the global Lipschitz condition the SDDE is *pth* moment exponentially stable (for $p \in (0, 1)$) if and only if the stochastic theta method of the SDDE is pth moment exponentially stable and *pth* moment exponential stability of the SDDE or the stochastic theta method implies the almost sure exponential stability of the SDDE or the stochastic theta method, respectively. We then replace the global Lipschitz condition with a finite-time convergence condition and establish the same results. Hence, our new theory enables us to consider the almost sure exponential stability of the SDDEs using the stochastic theta method, instead of the method of Lyapunov functions. That is, we can now perform careful numerical simulations using the stochastic theta method with a sufficiently small step size Δt. If the stochastic theta method is *pth* moment exponentially stable for a sufficiently small $p \in (0, 1)$, we can then deduce that the underlying SDDE is almost sure exponentially stable. Our new theory also enables us to show the *pth* moment exponential stability of the stochastic theta method to reproduce the almost sure exponential stability of the SDDEs.

1. Introduction.

The importance of stochastic differential delay equations (SDDEs) derives from the fact that many of phenomena witnessed around us do not have an immediate effect from the moment of their occurrence. SDDEs have been increasingly used to model the effect of noise and time delay types of complex systems, such as control problems ([1],[2]) and the dynamics of noisy bistable systems with delay ([3]). Stability theory of numerical solutions is one of the central problems in numerical analysis. Stability analysis of numerical methods for stochastic differential equations (SDEs) as well as SDDEs has recently received a lot of attention. This paper avoids using the method of Lyapunov functions.

Explicit solution can hardly be obtained for the SDDEs. There is a quite substantial work that has been done concerning approximate schemes for SDEs and we mention Higham et al.[7] and Mao([4],[6]). But

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this is not the case for SDDEs as it has been pointed out in [5]. There is an extensive literature on stochastic stability, for example Mao([9], [10]), Rodkinaa and Schurz[8].

We are then left two questions:

(P1) If the SDDE is *pth* moment exponentially stable (almost sure exponentially stable), will the numerical method be *pth* moment exponentially stable (almost sure exponentially stable) on the SDDE?

(P2) If the numerical method is *pth* moment exponentially stable (almost sure exponentially stable) of the SDDE, will the SDDE be *pth* moment exponentially stable (almost sure exponentially stable)?

Our aim is to give positive answers to both (P1) and (P2). This paper divides naturally into two parts, the first concerning almost sure exponential stability of the SDDE under global Lipschitz condition , the second part concerning the generalized results of the SDDE.

2. Almost sure exponential stability of stochastic differential delay equations under global Lipschitz condition.

Throughout this paper, we let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), w_2(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote both Euclidean norm in $\mathbf{R}^n$ and the trace norm in $\mathbf{R}^{n\times m}$. Let $C([-\tau,0],\mathbf{R}^n)(\tau > 0)$ denote the family of all continuous functions from $[-\tau,0]$ to $\mathbf{R}^n$. Let $L^2_{\mathcal{F}_t}([-\tau,0],\mathbf{R}^n)$ denote the family of $\mathcal{F}_t$-measurable, $C([-\tau,0],\mathbf{R}^n)$-valued random variables $\phi = \{\phi(r) : r \in [-\tau,0]\}$ such that $\sup_{r\in[-\tau,0]} E|\phi(r)|^2 < \infty$. $C([u-\tau,u],\mathbf{R}^n)$-valued random variable $\eta_u = \{\eta(r) : r \in [u-\tau,u]\} \in L^2_{\mathcal{F}_u}([u-\tau,u],\mathbf{R}^n)$ satisfies $\sup_{r\in[u-\tau,u]} E|\eta(r)|^2 < \infty$. If $x(t)$ is a continuous $\mathbf{R}^n$-valued stochastic process on $t \in [-\tau,\infty)$, $x_t = \{x(t+r), r \in [-\tau,0]\}$ is regarded as a $C([-\tau,0],\mathbf{R}^n)$-valued stochastic process. For $a, b \in \mathbf{R}$, we use $a \vee b$ and $a \wedge b$ for $\max\{a,b\}$ and $\min\{a,b\}$, respectively. Let $\mathbf{Z}^+$ be $\{0,1,2,\cdots\}$.

Consider an n-dimensional stochastic differential delay equation (SDDE).

$$dy(t) = f(y(t), y(t-\tau))dt + g(y(t), y(t-\tau))dw(t) \tag{1}$$

on $t \in [0,\infty)$ with initial data $y_0 = \xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$,where $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ and $g : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$.

Throughout this paper, unless otherwise specified, we impose the following hypothesis as standing hypothesis:

(H1) Both f and g satisfy the global Lipschitz condition.That is, there is a positive constant K such that

$$|f(x,y) - f(\hat{x},\hat{y})|^2 \vee |g(x,y) - g(\hat{x},\hat{y})|^2 \leq K(|x-\hat{x}|^2 + |y-\hat{y}|^2) \tag{2}$$

for $x, y, \hat{x}, \hat{x} \in \mathbf{R}^n$.

Moreover, $f(0,0) = 0$ and $g(0,0) = 0$.

We should point out that the reason we assume that $f(0,0) = 0$ and $g(0,0) = 0$ is because this paper is concerned with the stochastic stability of the trivial solution $y(t) \equiv 0$. It is also easy to see that this hypothesis implies this condition

$$|f(x,y)|^2 \vee |g(x,y)|^2 \leq K(|x|^2 + |y|^2) \ \ \forall x, y \in \mathbf{R}^n. \tag{3}$$

2.1. The pth moment stability of the SDDE

As we know, under (H1) for any initial data $y_0 = \xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$ the SDDE (1) has a unique global solution $y(t)$ on $t \geq -\tau$, see [4]. We shall denote the solution by $y(t;0,\xi)$. So we immediately get this following lemma.

Lemma 2.1. *Let* (H1) *hold. Let* $p \in (0,1)$ *and* $T > 0$, *then*

$$\sup_{t\in[-\tau,T]} E|y(t;0,\xi)|^p \leq H(T,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p. \tag{4}$$

In particularly, if $y_{t_0} = \xi \in L^2_{\mathcal{F}_{t_0}}([-\tau, 0], \mathbf{R}^n)$*, we have*

$$\sup_{t\in[t_0-\tau,t_0+T]} E|y(t;t_0,\eta_{t_0})|^p \leq H(T,p,K) \sup_{r\in[t_0-\tau,t_0]} E|\eta(r)|^p, t \geq t_0, \tag{5}$$

where

$$H(T,p,K) = 3^{\frac{p}{2}}[1 + K(T+1)\tau]^{\frac{p}{2}} e^{3pK(T+1)T}.$$

Proof. We write $y(t) = y(t; 0, \xi)$ It is straightforward to show that

$$y(t) = y(0) + \int_0^t f(y(s), y(s-\tau))ds + \int_0^t g(y(s), y(s-\tau))dw(s).$$

We get

$$|y(t)|^2 \leq 3|\xi(0)|^2 + 3|\int_0^t f(y(s), y(s-\tau))ds|^2 + 3|\int_0^t g(y(s), y(s-\tau))dw(s)|^2.$$

Hence

$$\begin{aligned} E|y(t)|^2 \leq& 3E|\xi(0)|^2 + 3E|\int_0^t f(y(s), y(s-\tau))ds|^2 + 3E|\int_0^t g(y(s), y(s-\tau))dw(s)|^2 \\ \leq& 3E|\xi(0)|^2 + 3K(t+1)\int_0^t E|y(s)|^2 ds + 3K(t+1)\int_0^t E|y(s-\tau)|^2 ds \\ =& 3E|\xi(0)|^2 + 3K(t+1)\int_0^t E|y(s)|^2 ds + 3K(t+1)\int_{-\tau}^{t-\tau} E|y(s)|^2 ds \\ \leq& [3 + 3K(t+1)\tau] \sup_{r\in[-\tau,0]} E|\xi(r)|^2 + 6K(t+1)\int_0^t E|y(s)|^2 ds. \end{aligned}$$

This, by the Gronwall inequality, yields

$$E|y(t)|^2 \leq [3 + 3K(t+1)\tau] \sup_{r\in[-\tau,0]} E|\xi(r)|^2 e^{6K(t+1)t}. \tag{6}$$

Hence, when $p \in (0, 1)$,

$$E|y(t)|^p \leq (E|y(t)|^2)^{\frac{p}{2}} \leq 3^{\frac{p}{2}}[1 + K(t+1)\tau]^{\frac{p}{2}} e^{3pK(t+1)t} \sup_{r\in[-\tau,0]} E|\xi(r)|^p.$$

In other words, we have

$$H(t,p,K) = 3^{\frac{p}{2}}[1 + K(t+1)\tau]^{\frac{p}{2}} e^{3pK(t+1)t}, \quad p \in (0,1). \tag{7}$$

Thus

$$E|y(t)|^p \leq H(t,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p \quad \forall t > 0.$$

In this paper we often need to introduce the solution to the SDDE (1) for initial data $y_{t_0} = \eta_{t_0} \in L^2_{\mathcal{F}_{t_0}}([t_0 - \tau, t_0], \mathbf{R}^n)$. We shall denote this solution by $y(t; t_0, \eta_{t_0})$. Moreover, for any $t_0 \geq 0$, we can denote $y(t) = y(t; t_0, \eta_{t_0})$ as the solution of the SDDE (1) on $t \geq t_0 - \tau$ with the initial data y_{t_0}, we hence get

$$y(t) = y(t_0) + \int_{t_0}^t f(y(s), y(s-\tau))ds + \int_{t_0}^t g(y(s), y(s-\tau))dw(s).$$

We get

$$|y(t)|^2 \le 3|y(t_0)|^2 + 3|\int_{t_0}^{t} f(y(s), y(s-\tau))ds|^2 + 3|\int_{t_0}^{t} g(y(s), y(s-\tau))dw(s)|^2.$$

We obtain

$$\begin{aligned}E|y(t)|^2 \le& 3E|y(t_0)|^2 + 3E|\int_{t_0}^{t} f(y(s), y(s-\tau))ds|^2 + 3E|\int_{t_0}^{t} g(y(s), y(s-\tau))dw(s)|^2\\ \le& 3E|y(t_0)|^2 + 3K(t-t_0+1)\int_{t_0}^{t} E|y(s)|^2 ds + 3K(t-t_0+1)\int_{t_0}^{t} E|y(s-\tau)|^2 ds\\ =& 3E|y(t_0)|^2 + 3K(t-t_0+1)\int_{t_0}^{t} E|y(s)|^2 ds + 3K(t-t_0+1)\int_{t_0-\tau}^{t-\tau} E|y(s)|^2 ds\\ \le& [3+3K(t-t_0+1)\tau] \sup_{r\in[t_0-\tau,t_0]} E|\eta(r)|^2 + 6K(t-t_0+1)\int_{t_0}^{t} E|y(s)|^2 ds.\end{aligned}$$

This, by the Gronwall inequality, yields

$$E|y(t)|^2 \le [3+3K(t-t_0+1)\tau] \sup_{r\in[t_0-\tau,t_0]} E|\eta(r)|^2 e^{6K(t-t_0+1)(t-t_0)}.$$

Hence, when $p \in (0,1)$,

$$E|y(t)|^p \le (E|y(t)|^2)^{\frac{p}{2}} \le 3^{\frac{p}{2}}[1+K(t-t_0+1)\tau]^{\frac{p}{2}} e^{3pK(t-t_0+1)(t-t_0)} \sup_{r\in[t_0-\tau,t_0]} E|\eta(r)|^p.$$

In other words, we have

$$H(t-t_0,p,K) = 3^{\frac{p}{2}}[1+K(t-t_0+1)\tau]^{\frac{p}{2}} e^{3pK(t-t_0+1)(t-t_0)}, \ \ p \in (0,1).$$

Thus

$$E|y(t)|^p \le H(t-t_0,p,K) \sup_{r\in[t_0-\tau,t_0]} E|\eta(r)|^p, \ \ t \ge t_0.$$

The proof is hence complete. □

In this section we consider the pth moment exponential stability of the origin, which we define as follows.

Definition 2.2. *Let $p > 0$. The SDDE (1) is said to be exponentially stable in the pth moment if there is a pair of positive constants λ, M such that, for initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$,*

$$E|y(t;0,\xi)|^p \le M \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\lambda t}, \quad t \ge 0. \tag{8}$$

We refer to λ as a rate constant and M as a growth constant.

In this paper we often need to introduce the solution to the SDDE (1) for initial data $y_r = \xi \in L^2_{\mathcal{F}_r}([-\tau,0],\mathbf{R}^n)$. Hypotheses (H1) guarantee the existence and uniqueness of this solution which is denoted by $y(t;r,y_r)$. It is easy to observe that the solutions to the SDDE (1) have the following flow property:

$$y(t;0,\xi) = y(t;r,y_r), \quad t \ge r \ge 0.$$

Moreover, due to the autonomous property of the SDDE (1), the exponential stability in pth moment (8) implies

$$E|y(t;r,\xi)|^p \le M \sup_{s\in[-\tau,0]} E|\xi(s)|^p e^{-\lambda(t-r)}, \quad t \ge r \ge 0. \tag{9}$$

2.2. The pth moment stability of the theta method.

Given a free parameter $\theta \in [0,1]$, the numerical solutions by the stochastic theta method are defined by

$$\begin{aligned} x((k+1)\Delta t) =& x(k\Delta t) + (1-\theta) f(x(k\Delta t), x((k-N)\Delta t))\Delta t \\ &+ \theta f(x((k+1)\Delta t), x((k+1-N)\Delta t))\Delta t + g(x(k\Delta t), x((k-N)\Delta t))\Delta w_k, \end{aligned} \tag{10}$$

with initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau,0], \mathbf{R}^n)$, where $x(k\Delta t) = x(k\Delta t; 0, \xi)$, $k \in \mathbf{Z}^+$, $\Delta w_k = w((k+1)\Delta t) - w(k\Delta t)$, $\Delta t = \frac{\tau}{N}$.

Let us introduce two continuous-time step process, define $z_1(t) = z_1(t; 0, \xi)$, $z_2(t) = z_2(t; 0, \xi)$,

$$z_1(t) = \sum_{k=-N}^{\infty} x(k\Delta t)\mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t), z_2(t) = \sum_{k=-N}^{\infty} x((k+1)\Delta t)\mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t),$$

with $\mathbf{1}_G$ denoting the indicator function for the set G. In our analysis we find it convenient to work with continuous-time approximate and hence we find

$$x(t) = \begin{cases} \xi, & t \in [-\tau, 0], \\ x(0) + (1-\theta)\displaystyle\int_0^t f(z_1(s), z_1(s-\tau))ds + \theta \displaystyle\int_0^t f(z_2(s), z_2(s-\tau))ds & \\ + \displaystyle\int_r^t g(z_1(s), z_1(s-\tau))dw(s), & t \geq 0, \end{cases}$$

where $x(t) = x(t; 0, \xi)$.

As for the exact solution $y(t; 0, \xi)$, the numerical solutions by the stochastic theta method have the following flow property too:

$$x(t; 0, \xi) = x(t; r, x_r) \quad 0 \leq r < t < \infty$$

provided r is the multiple of Δt.

Lemma 2.3. *Let (H1) hold. Let $p \in (0,1)$ and let Δt be sufficiently small for $K(\theta\Delta t)^2 < \frac{1}{10}$. Then the discrete process $\{x(k\Delta t; 0, \xi)\}_{k \in Z^+}$ defined by the stochastic theta method (10) satisfies*

$$\sup_{-\tau \leq k\Delta t \leq \tau + T} E|x(k\Delta t; 0, \xi)|^p \leq \tilde{H}(T, p, K) \sup_{r \in [-\tau, 0]} E|\xi(r)|^p, \tag{11}$$

and the discrete process $\{x(k\Delta t; u, \eta_u)\}_{k \in Z^+}$ defined by the stochastic theta method (10) satisfies

$$\sup_{u-\tau \leq k\Delta t \leq u+\tau+T} E|x(k\Delta t; u, \eta_u)|^p \leq \tilde{H}(T, p, K) \sup_{r \in [u-\tau, u]} E|\eta(r)|^p, \tag{12}$$

$\forall T \geq 0$, where

$$\tilde{H}(T, p, K) = \{[13 + 10K(T+1)\tau][13 + 10K(\tau+1)\tau]\}^{\frac{p}{2}} \sup_{r \in [-\tau, 0]} E|\xi(r)|^p e^{10pK(\tau+1)\tau} e^{15pK(T+1)T}.$$

Proof. For convenience, we write $x(k\Delta t) = x(k\Delta t; 0, \xi)$. It is easy to see that $z_1(k\Delta t) = z_2((k-1)\Delta t) = x(k\Delta t)$. We divided the whole proof into two steps.

Step 1. For any $0 \leq k\Delta t < (k+1)\Delta t \leq \tau$, it is easily shown that

$$\begin{aligned} x((k+1)\Delta t) =& \xi(0) + \int_0^{(k+1)\Delta t} (1-\theta) f(z_1(s), z_1(s-\tau))ds \\ &+ \int_0^{(k+1)\Delta t} \theta f(z_2(s), z_2(s-\tau))ds + \int_0^{(k+1)\Delta t} g(z_1(s), z_1(s-\tau))dw(s). \end{aligned}$$

Noting that

$$\int_0^{(k+1)\Delta t} f(z_2(s), z_2(s-\tau))ds = \int_0^{k\Delta t} f(z_2(s), z_2(s-\tau))ds + \int_{k\Delta t}^{(k+1)\Delta t} f(z_2(s), z_2(s-\tau))ds$$
$$= \int_{\Delta t}^{(k+1)\Delta t} f(z_1(s), z_1(s-\tau))ds + f(x((k+1)\Delta t), x((k+1-N)\Delta t))\Delta t,$$

we get

$$x((k+1)\Delta t) = \xi(0) - \theta f(\xi(0), \xi(-\tau))\Delta t + \theta f(x((k+1)\Delta t), x((k+1-N)\Delta t))\Delta t$$
$$+ \int_0^{(k+1)\Delta t} f(z_1(s), z_1(s-\tau))ds + \int_0^{(k+1)\Delta t} g(z_1(s), z_1(s-\tau))dw(s).$$

Hence

$$|x((k+1)\Delta t)|^2 \le 5\{ |\xi(0)|^2 + (\theta\Delta t)^2 |f(\xi(0), \xi(-\tau))|^2$$
$$+ (\theta\Delta t)^2 |f(x((k+1)\Delta t), x((k+1-N)\Delta t))|^2 + |\int_0^{(k+1)\Delta t} f(z_1(s), z_1(s-\tau))ds|^2$$
$$+ |\int_0^{(k+1)\Delta t} g(z_1(s), z_1(s-\tau))dw(s)|^2\}.$$

By the condition (3) as well as the Hölder inequality and the property of the Itô isometry, we can show

$$E|x((k+1)\Delta t)|^2$$
$$\le 5\{ E|\xi(0)|^2 + K(\theta\Delta t)^2 E|\xi(0)|^2 + K(\theta\Delta t)^2 E|\xi(-\tau)|^2$$
$$+ K(\theta\Delta t)^2 E|x((k+1)\Delta t)|^2 + K(\theta\Delta t)^2 E|x((k+1-N)\Delta t)|^2$$
$$+ K((k+1)\Delta t + 1)\int_0^{(k+1)\Delta t} E|z_1(s)|^2 ds + K((k+1)\Delta t + 1)\int_0^{(k+1)\Delta t} E|z_1(s-\tau)|^2 ds\}$$
$$\le 5\{ E|\xi(0)|^2 + K(\theta\Delta t)^2 E|\xi(0)|^2 + K(\theta\Delta t)^2 E|\xi(-\tau)|^2$$
$$+ K(\theta\Delta t)^2 E|x((k+1)\Delta t)|^2 + K(\theta\Delta t)^2 E|x((k+1-N)\Delta t)|^2$$
$$+ K((k+1)\Delta t + 1)\int_0^{(k+1)\Delta t} E|z_1(s)|^2 ds + K((k+1)\Delta t + 1)\int_{-\tau}^{(k+1)\Delta t-\tau} E|z_1(s)|^2 ds\}.$$

This, together with the condition $K(\theta\Delta t)^2 < \frac{1}{10}$, yields

$$E|x((k+1)\Delta t)|^2 \le 11E|\xi(0)|^2 + E|\xi(-\tau)|^2 + E|\xi((k+1-N)\Delta t)|^2$$
$$+ 20K((k+1)\Delta t + 1)\int_0^{(k+1)\Delta t} E|z_1(s)|^2 ds + 10K((k+1)\Delta t + 1)\int_{-\tau}^{0} E|z_1(s)|^2 ds$$
$$\le [13 + 10K(\tau+1)\tau] \sup_{r\in[-\tau,0]} E|\xi(r)|^2 + 20K(\tau+1)\Delta t \sum_{j=0}^{k} E|x(j\Delta t)|^2.$$

By the discrete Gronwall inequality, we hence obtain

$$\sup_{0\le(k+1)\Delta t\le\tau} E|x((k+1)\Delta t)|^2 \le [13 + 10K(\tau+1)\tau] \sup_{r\in[-\tau,0]} E|\xi(r)|^2 e^{20K(\tau+1)\tau}.$$

Step 2. For any $\tau \le k\Delta t < (k+1)\Delta t \le \tau + T$, it is easily shown that

$$x((k+1)\Delta t) = x(\tau) + \int_\tau^{(k+1)\Delta t} (1-\theta) f(z_1(s), z_1(s-\tau))ds$$
$$+ \int_\tau^{(k+1)\Delta t} \theta f(z_2(s), z_2(s-\tau))ds + \int_\tau^{(k+1)\Delta t} g(z_1(s), z_1(s-\tau))dw(s).$$

Noting that

$$\int_{\tau}^{(k+1)\Delta t} f(z_2(s), z_2(s-\tau))ds = \int_{\tau}^{k\Delta t} f(z_2(s), z_2(s-\tau))ds + \int_{k\Delta t}^{(k+1)\Delta t} f(z_2(s), z_2(s-\tau))ds$$
$$= \int_{\Delta t+\tau}^{(k+1)\Delta t} f(z_1(s), z_1(s-\tau))ds + f(x((k+1)\Delta t), x((k+1-N)\Delta t))\Delta t,$$

we get

$$x((k+1)\Delta t) = x(\tau) - \theta f(x(\tau), \xi(0))\Delta t + \theta f(x((k+1)\Delta t), x((k+1-N)\Delta t))\Delta t$$
$$+ \int_{\tau}^{(k+1)\Delta t} f(z_1(s), z_1(s-\tau))ds + \int_{\tau}^{(k+1)\Delta t} g(z_1(s), z_1(s-\tau))dw(s).$$

Hence

$$|x((k+1)\Delta t)|^2 \leq 5\{ |x(\tau)|^2 + (\theta\Delta t)^2|f(x(\tau), \xi(0))|^2 + |\int_{\tau}^{(k+1)\Delta t} g(z_1(s), z_1(s-\tau))dw(s)|^2$$
$$+ (\theta\Delta t)^2|f(x((k+1)\Delta t), x((k+1-N)\Delta t))|^2 + |\int_{\tau}^{(k+1)\Delta t} f(z_1(s), z_1(s-\tau)ds|^2\}.$$

By the condition (3) as well as the Hölder inequality and the property of the Itô isometry, we can show

$$E|x((k+1)\Delta t)|^2 \leq 5\{ E|x(\tau)|^2 + K(\theta\Delta t)^2 E|x(\tau)|^2 + K(\theta\Delta t)^2 E|\xi(0)|^2$$
$$+ K(\theta\Delta t)^2 E|x((k+1)\Delta t)|^2 + K(\theta\Delta t)^2 E|x((k+1-N)\Delta t)|^2$$
$$+ K((k+1)\Delta t - \tau + 1)\int_{\tau}^{(k+1)\Delta t} E|z_1(s)|^2 ds$$
$$+ K((k+1)\Delta t - \tau + 1)\int_{\tau}^{(k+1)\Delta t} E|z_1(s-\tau)|^2 ds\}.$$

Case 1: If $0 \leq (k+1-N)\Delta t \leq \tau$, we have

$$E|x((k+1-N)\Delta t)|^2 \leq \sup_{r\in[0,\tau]} E|x(r)|^2.$$

Case 2: If $\tau \leq (k+1-N)\Delta t \leq k\Delta t$, we get

$$K(\theta\Delta t)^2 E|x((k+1-N)\Delta t)|^2 \leq K(T+1)\Delta t \sum_{j=0}^{k} E|x(j\Delta t)|^2.$$

Combining Case 1 and 2, we obtain

$$E|x((k+1)\Delta t)|^2 \leq 5\{ E|x(\tau)|^2 + K(\theta\Delta t)^2 E|x(\tau)|^2 + K(\theta\Delta t)^2 E|\xi(0)|^2 + K(\theta\Delta t)^2 \sup_{r\in[0,\tau]} E|x(r)|^2$$
$$+ K(\theta\Delta t)^2 E|x((k+1)\Delta t)|^2 + K(T+1)\Delta t \sum_{j=0}^{k} E|x(j\Delta t)|^2$$
$$+ 2K((k+1)\Delta t - \tau + 1)\int_{\tau}^{(k+1)\Delta t} E|z_1(s)|^2 ds$$
$$+ K((k+1)\Delta t - \tau + 1)\int_{0}^{\tau} E|z_1(s-\tau)|^2 ds\}.$$

According to $K(\theta\Delta t)^2 < \frac{1}{10}$, we have

$$\begin{aligned}
E|x((k+1)\Delta t)|^2 \leq & 11E|x(\tau)|^2 + E|\xi(0)|^2 + 10K(T+1)\Delta t \sum_{j=0}^{k} E|x(j\Delta t)|^2 \\
& + \sup_{r\in[0,\tau]} E|x(r)|^2 + 20K((k+1)\Delta t - \tau + 1)\int_{\tau}^{(k+1)\Delta t} E|z_1(s)|^2 ds \\
& + 10K((k+1)\Delta t - \tau + 1)\int_0^{\tau} E|z_1(s)|^2 ds \\
\leq & 11E|x(\tau)|^2 + E|\xi(0)|^2 + [10K(T+1)\tau + 1] \sup_{r\in[0,\tau]} E|x(r)|^2 \\
& + 30K(T+1)\Delta t \sum_{j=0}^{k} E|x(j\Delta t)|^2.
\end{aligned}$$

By the discrete Gronwall inequality, we hence obtain

$$\begin{aligned}
\sup_{\tau\leq(k+1)\Delta t\leq\tau+T} E|x((k+1)\Delta t)|^2 & \leq [13 + 10K(T+1)\tau] \sup_{r\in[0,\tau]} E|x(r)|^2 e^{30K(T+1)T} \\
& \leq C_0 \sup_{r\in[-\tau,0]} E|\xi(r)|^2.
\end{aligned}$$

where

$$C_0 = [13 + 10K(T+1)\tau][13 + 10K(\tau+1)\tau]e^{20K(\tau+1)\tau}e^{30K(T+1)T}.$$

Finally, we have

$$\begin{aligned}
\sup_{\tau\leq(k+1)\Delta t\leq\tau+T} E|x((k+1)\Delta t)|^p & \leq [\sup_{0\leq(k+1)\Delta t\leq\tau} E|x((k+1)\Delta t)|^2]^{\frac{p}{2}} \\
& \leq \tilde{H}(T,p,K) \sup_{r\in[0,\tau]} E|\xi(r)|^p.
\end{aligned}$$

where

$$\tilde{H}(T,p,K) = \{[13 + 10K(T+1)\tau][13 + 10K(\tau+1)\tau]\}^{\frac{p}{2}} \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{10pK(\tau+1)\tau} e^{15pK(T+1)T}.$$

Together with step 1 and 2, we get (11). Similarly, we can prove (12).

The proof is hence complete. □

Lemma 2.4. *Let* $(H1)$ *hold. Let* $p \in (0,1)$*. Then the continuous process* $\{x(t;0,\xi)\}$ *defined by the stochastic theta method* (10) *satisfies*

$$\sup_{-\tau\leq t\leq T} E|x(t;0,\xi)|^p \leq \bar{H}(T,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p \tag{13}$$

and

$$\sup_{u-\tau\leq t\leq u+T} E|x(t;u,\eta_u)|^p \leq \bar{H}(T,p,K) \sup_{r\in[u-\tau,u]} E|\eta(r)|^p, \tag{14}$$

where $T = m\Delta t$, $m = 0,1,2,\cdots$, *and*

$$\bar{H}(T,P,K) = 2^p[1 + K(T+1)(2T+\tau) + 3KT\tau]^{\frac{p}{2}}(C_0)^{\frac{p}{2}}.$$

Proof. Write $x(t) = x(t;0,\xi)$. By the condition (3) as well as the Hölder inequality and the property of the Itô isometry, we can show

$$\begin{aligned} E|x(t)|^2 \le & 4\{\ E|\xi(0)|^2 + (1-\theta)^2 Kt \int_0^t [E|z_1(s)|^2 + E|z_1(s-\tau)|^2]ds \\ & + \theta^2 Kt \int_0^t [E|z_2(s)|^2 + E|z_2(s-\tau)|^2]ds + K\int_0^t [E|z_1(s)|^2 + E|z_1(s-\tau)|^2]ds \\ \le & 4\{\ E|\xi(0)|^2 + 2K(t+1)\int_0^t E|z_1(s)|^2 ds + K(t+1)\tau C_0 \sup_{r\in[-\tau,0]} E|\xi(r)|^2 \\ & + 2Kt\Delta t \sum_{j=0}^{\lceil \frac{t}{\Delta t}\rceil \Delta t} E|x(j\Delta t)|^2 + Kt\tau C_0 \sup_{r\in[-\tau,0]} E|\xi(r)|^2]\}. \end{aligned}$$

where $\lceil x \rceil$ denotes the ceiling function(the smallest integer greater than or equal to x) By (11), we hence obtain

$$\sup_{0\le t\le T} E|x(t)|^2 \le C_0' \sup_{r\in[-\tau,0]} E|\xi(r)|^2], \tag{15}$$

where

$$C_0' = 4[1 + K(T+1)(2T+\tau) + 3KT\tau]C_0.$$

Finally, we have

$$\sup_{0\le t\le T} E|x(t)|^p \le [\sup_{0\le t\le T} E|x(t)|^2]^{\frac{p}{2}} \le \bar{H}(T,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p$$

as required, where

$$\bar{H}(T,P,K) = 2^p[1 + K(T+1)(2T+\tau) + 3KT\tau]^{\frac{p}{2}} C_0^{\frac{p}{2}}.$$

Moreover, for any $t \ge u \ge 0$, we write $x(t) = x(t;u,\eta_u)$ as the theta approximate solution of the SDDE (1) on $t \ge u-\tau$ with the initial data $\eta_u = \{\eta(r) : r \in [u-\tau,u]\}$. By the condition (3) as well as the Hölder inequality and the property of the Itô isometry, we can show

$$\begin{aligned} E|x(t)|^2 \le & 4\{\ E|\eta(u)|^2 + (1-\theta)^2 K(t-u) \int_u^t [E|z_1(s)|^2 + E|z_1(s-\tau)|^2]ds \\ & + \theta^2 K(t-u) \int_u^t [E|z_2(s)|^2 + E|z_2(s-\tau)|^2]ds + K\int_u^t [E|z_1(s)|^2 + E|z_1(s-\tau)|^2]ds\} \\ \le & 4\{\ E|\xi(0)|^2 + 2K(t-u+1)\int_0^t E|z_1(s)|^2 ds + Kt(t-u+1)\tau C_0 \sup_{r\in[-\tau,0]} E|\xi(r)|^p \\ & + 2K(t-u)\Delta t \sum_{j=\lceil \frac{u}{\Delta t}\rceil \Delta t}^{\lceil \frac{t}{\Delta t}\rceil \Delta t} E|x(j\Delta t)|^2 + K(t-u)\tau C_0 \sup_{r\in[-\tau,0]} E|\xi(r)|^p]\}. \end{aligned}$$

By (12), we hence obtain

$$\sup_{u\le t\le u+T} E|x(t)|^2 \le 4[1 + K(T+1)(2T+\tau) + 3KT\tau]C_0 \sup_{r\in[-\tau,u]} E|\eta(r)|^2 e^{8K(2T+1)T}.$$

Finally, we have

$$\sup_{u\le t\le u+T} E|x(t)|^p \le [\sup_{u\le t\le u+T} E|x(t)|^2]^{\frac{p}{2}} \le \bar{H}(T,p,K) \sup_{r\in[u-\tau,u]} E|\eta(r)|^p$$

as required.

The proof is therefore complete. □

Definition 2.5. *Let $p > 0$. For a given step size $\Delta t > 0$, the numerical method is said to be exponentially stable in the pth moment on the SDDE* (1) *if there is a pair of positive constants γ and Q such that with initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0], \mathbf{R}^n)$.*

$$E|x(t;0,\xi)|^p \leq Q \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\gamma t}, \quad t \geq 0. \tag{16}$$

We see that (16) *is equivalent to*

$$E|x(t;r,\xi)|^p \leq Q \sup_{s\in[-\tau,0]} E|\xi(s)|^p e^{-\gamma(t-r)}, \quad t \geq r \geq 0. \tag{17}$$

Lemma 2.6. *Let* (H1) *hold. Let Δt be sufficiently small for $2K\Delta t < 1$. Then the solution of the SDDE* (1) *has the property*

$$\begin{aligned} &E|y(t;0,\xi) - y(k\Delta t;0,\xi)|^2 \vee E|y(t;0,\xi) - y((k+1)\Delta t;0,\xi)|^2 \leq \widehat{C}_T \Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2, \\ &E|y(t;u,\eta_u) - y(k\Delta t;u,\eta_u)|^2 \vee E|y(t;u,\eta_u) - y((k+1)\Delta t;u,\eta_u)|^2 \leq \widehat{C}_T \Delta t \sup_{r\in[u-\tau,u]} E|\eta(r)|^2 \end{aligned} \tag{18}$$

for all $0 \leq k\Delta t \leq t \leq (k+1)\Delta t \leq T, u \geq 0$, where

$$\widehat{C}_T = 6(2K+1)[1 + K(T+1)\tau](T+1)e^{6K(T+1)T}.$$

Proof. Write $y(t) = y(t;0,\xi)$, $y(k\Delta t) = y(k\Delta t;0,\xi)$. Noting

$$y(t) - y(k\Delta t) = \int_{k\Delta t}^{t} f(y(s), y(s-\tau))ds + \int_{k\Delta t}^{t} g(y(s), y(s-\tau))dw(s),$$

we can show easily that

$$\begin{aligned} E|y(t) - y(k\Delta t)|^2 \leq& 2K(\Delta t + 1) \int_{k\Delta t}^{t} E|y(s)|^2 ds + 2K(\Delta t + 1) \int_{k\Delta t}^{t} E|y(s-\tau)|^2 ds \\ \leq& 2(2K+1) \int_{k\Delta t}^{t} \sup_{k\Delta t - \tau \leq r \leq s} E|y(r)|^2 ds. \end{aligned}$$

By (6), we then have

$$\begin{aligned} E|y(t) - y(k\Delta t)|^2 \leq& (2K+1)\Delta t \cdot 3[1 + K(t+1)\tau]e^{6K(t+1)t} \sup_{r\in[-\tau,0]} E|\xi(r)|^2 \\ \leq& 6(2K+1)[1 + K(t+1)\tau]\Delta t e^{6K(T+1)T} \sup_{r\in[-\tau,0]} E|\xi(r)|^2, \end{aligned}$$

where $\widehat{C}_T = 6(2K+1)[1 + K(T+1)\tau]e^{6K(T+1)T}$.

Similarly, we can show

$$E|y(t) - y((k+1)\Delta t)|^2 \leq 6(2K+1)[1 + K(t+1)\tau]\Delta t e^{6K(T+1)T} \sup_{r\in[-\tau,0]} E|\xi(r)|^2.$$

Similarly, we can prove

$$E|y(t;u,\eta_u) - y(k\Delta t;u,\eta_u)|^2 \vee E|y(t;u,\eta_u) - y((k+1)\Delta t;u,\eta_u)|^2 \leq \widehat{C}_T \Delta t \sup_{r\in[u-\tau,u]} E|\eta(r)|^2.$$

The proof of this lemma is complete. □

Lemma 2.7. *Let* (*H*1) *hold. Let* Δt *be sufficiently small for* $3K\Delta t < 1$. *Then the solution of the SDDE* (1) *has the property*

$$\begin{aligned} E|x(t;0,\xi) - z_1(t;0,\xi)|^2 \vee E|x(t;0,\xi) - z_2(t;0,\xi)|^2 &\le \bar{C}_T\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2, \\ E|x(t;u,\eta_u) - z_1(t;u,\eta_u)|^2 \vee E|x(t;u,\eta_u) - z_2(t;u,\eta_u)|^2 &\le \bar{C}_T\Delta t \sup_{r\in[u-\tau,u]} E|\eta(r)|^2 \end{aligned} \tag{19}$$

for all $0 \le k\Delta t \le t \le (k+1)\Delta t \le T$, *where* $T = m\Delta t$, $m = 0, 1, 2, \cdots$, *and*

$$\bar{C}_T = 12(1 + 3K)[1 + K(T + 1)(2T + \tau) + 3KT\tau][13 + 10K(T + 1)\tau]^2 e^{40K(T+1)T}.$$

Proof. Write $x(t) = x(t; 0, \xi)$, $z_1(t) = z_1(t; 0, \xi)$, $z_2(t) = z_2(t; 0, \xi)$. Noting

$$\begin{aligned} x(t) - z_1(t) =&(1-\theta)\int_{k\Delta t}^{t} f(z_1(s), z_1(s-\tau))ds \\ &+ \theta\int_{k\Delta t}^{t} f(z_2(s), z_2(s-\tau))ds + \int_{k\Delta t}^{t} g(z_1(s), z_1(s-\tau))dw(s). \end{aligned}$$

By (15), we can show easily that

$$\begin{aligned} E|x(t) - z_1(t)|^2 \le& 3K[(1-\theta)^2\Delta t + 1]\int_{k\Delta t}^{t} E|z_1(s)|^2ds + 3K[(1-\theta)^2\Delta t + 1]\int_{k\Delta t}^{t} E|z_1(s-\tau)|^2ds \\ &+ 3K\theta^2\Delta t\int_{k\Delta t}^{t} E|z_2(s)|^2ds + 3K\theta^2\Delta t\int_{k\Delta t}^{t} E|z_2(s-\tau)|^2ds \\ \le& 6K[(1-\theta)^2\Delta t + \theta^2\Delta t + 1]C_0' \sup_{r\in[-\tau,0]} E|\xi(r)|^2 \int_{k\Delta t}^{(k+1)\Delta t} ds \\ &+ 3K[(1-\theta)^2\Delta t + \theta^2\Delta t + 1]\Delta t C_0' \sup_{r\in[-\tau,0]} E|\xi(r)|^2 \\ \le& 9K[(1-\theta)^2\Delta t + \theta^2\Delta t + 1]\Delta t C_0' \sup_{r\in[-\tau,0]} E|\xi(r)|^2 \\ \le& \bar{C}_T\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2, \end{aligned}$$

where

$$\bar{C}_T = 3(1 + 3K)C_0'.$$

Similarly, we can show

$$E|x(t) - z_2(t)|^2 \le \bar{C}_T\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2$$

and

$$E|x(t;u,\eta_u) - z_2(t;u,\eta_u)|^2 \vee E|x(t;u,\eta_u) - z_2(t;u,\eta_u)|^2 \le \bar{C}_T\Delta t \sup_{r\in[u-\tau,u]} E|\eta(r)|^2.$$

The proof of this lemma is complete. □

Lemma 2.8. *Let* (*H*1) *hold. Let* $p \in (0, 1)$ *and let* Δt *be sufficiently small for*

$$(6 \vee 3K)\Delta t < 1. \tag{20}$$

Then the stochastic theta method (10) *and the solution of the SDDE* (1) *satisfy*

$$\sup_{-\tau\le k\Delta t\le T} E|x(k\Delta t;0,\xi)-y(k\Delta t;0,\xi)|^p \le \widetilde{C}_T(\Delta t)^{\frac{p}{2}} \sup_{r\in[-\tau,0]} E|\xi(r)|^p,\ \ \forall T>0 \tag{21}$$

and for all $T > 0$,

$$\sup_{\bar{k}\Delta t\tau-\le k\Delta t\le \bar{k}\Delta t+T} E|x(k\Delta t;\bar{k}\Delta t,\eta_{\bar{k}\Delta t})-y(k\Delta t;\bar{k}\Delta t,\eta_{\bar{k}\Delta t})|^p \le \widetilde{C}_T(\Delta t)^{\frac{p}{2}} \sup_{r\in[\bar{k}\Delta t-\tau,\bar{k}\Delta t]} E|\eta(r)|^p, \tag{22}$$

where

$$C'_T=\{144K(2K+1)[1+K(T+1)\tau](2T+1)T(T+1)\}^{\frac{p}{2}}e^{21pK(T+1)T},$$

which is independent of Δt *and* $\xi\in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$.

Proof. Write $x(k\Delta t)=x(k\Delta t;0,\xi)$, $y(k\Delta t)=y(k\Delta t;0,\xi)$. It follows from (1) and (10) that for any $0\le(k+1)\Delta t\le T$,

$$\begin{aligned} x((k+1)\Delta t)-y((k+1)\Delta t) &= \int_0^{(k+1)\Delta t}(1-\theta)[f(z_1(s),z_1(s-\tau))-f(y(s),y(s-\tau))]ds\\ &+\int_0^{(k+1)\Delta t}\theta[f(z_2(s),z_2(s-\tau))-f(y(s),y(s-\tau))]ds\\ &+\int_0^{(k+1)\Delta t}[g(z_1(s),z_1(s-\tau))-g(y(s),y(s-\tau))]dw(s). \end{aligned}$$

Define

$$\begin{aligned} y_1(t) &= \sum_{k=-N}^{\infty} y(k\Delta t)\mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t),\\ y_2(t) &= \sum_{k=-N}^{\infty} y((k+1)\Delta t)\mathbf{1}_{[k\Delta t,(k+1)\Delta t)}(t). \end{aligned} \tag{23}$$

Then

$$\begin{aligned} x((k+1)\Delta t)-y((k+1)\Delta t) &= \int_0^{(k+1)\Delta t}(1-\theta)[f(z_1(s),z_1(s-\tau))-f(y_1(s),y_1(s-\tau))]ds\\ &+\int_0^{(k+1)\Delta t}(1-\theta)[f(y_1(s),y_1(s-\tau))-f(y(s),y(s-\tau))]ds\\ &+\int_0^{(k+1)\Delta t}\theta[f(z_2(s),z_2(s-\tau))-f(y_2(s),y_2(s-\tau))]ds\\ &+\int_0^{(k+1)\Delta t}\theta[f(y_2(s),y_2(s-\tau))-f(y(s),y(s-\tau))]ds\\ &+\int_0^{(k+1)\Delta t}[g(z_1(s),z_1(s-\tau))-g(y_1(s),y_1(s-\tau))]dw(s)\\ &+\int_0^{(k+1)\Delta t}[g(y_1(s),y_1(s-\tau))-g(y(s),y(s-\tau))]dw(s). \end{aligned} \tag{24}$$

But

$$\begin{aligned}
&\int_0^{(k+1)\Delta t} [f(z_2(s), z_2(s-\tau)) - f(y_2(s), y_2(s-\tau))]ds \\
&= \int_0^{k\Delta t} [f(z_2(s), z_2(s-\tau)) - f(y_2(s), y_2(s-\tau))]ds + \int_{k\Delta t}^{(k+1)\Delta t} [f(z_2(s), z_2(s-\tau)) - f(y_2(s), y_2(s-\tau))]ds \\
&= \int_{\Delta t}^{(k+1)\Delta t} [f(z_1(s), z_1(s-\tau)) - f(y_1(s), y_1(s-\tau))]ds \\
&\quad + [f(x((k+1)\Delta t), x((k+1-N)\Delta t) - f(y((k+1)\Delta t), y((k+1-N)\Delta t)]\Delta t.
\end{aligned} \tag{25}$$

Hence

$$\begin{aligned}
&x((k+1)\Delta t) - y((k+1)\Delta t) \\
=&\theta[f(x((k+1)\Delta t), x((k+1-N)\Delta t)) - f(y((k+1)\Delta t), y((k+1-N)\Delta t)]\Delta t \\
&+ \int_0^{(k+1)\Delta t} [f(z_1(s), z_1(s-\tau)) - f(y_1(s), y_1(s-\tau))]ds \\
&+ \int_0^{(k+1)\Delta t} (1-\theta)[f(y_1(s), y_1(s-\tau)) - f(y(s), y(s-\tau))]ds \\
&+ \int_0^{(k+1)\Delta t} \theta[f(y_2(s), y_2(s-\tau)) - f(y(s), y(s-\tau))]ds \\
&+ \int_0^{(k+1)\Delta t} [g(z_1(s), z_1(s-\tau)) - g(y_1(s), y_1(s-\tau))]dw(s) \\
&+ \int_0^{(k+1)\Delta t} [g(y_1(s), y_1(s-\tau)) - g(y(s), y(s-\tau))]dw(s).
\end{aligned} \tag{26}$$

This, together with Lemma 2.4, implies

$$\begin{aligned}
&E|x((k+1)\Delta t) - y((k+1)\Delta t)|^2 \\
\leq&6K(\theta\Delta t)^2 E|(x((k+1)\Delta t) - y((k+1)\Delta t)|^2 + 6K(T+1)\int_0^{(k+1)\Delta t} E|y_1(s) - y(s)|^2 ds \\
&+ 6K(\theta\Delta t)^2 E|(x((k+1-N)\Delta t) - y((k+1-N)\Delta t)|^2 + 6K(T+1)\int_0^{(k+1)\Delta t} E|z_1(s) - y_1(s)|^2 ds \\
&+ 6K(T+1)\int_0^{(k+1)\Delta t} E|z_1(s-\tau) - y_1(s-\tau)|^2 ds + 6KT\int_0^{(k+1)\Delta t} E|y_2(s) - y(s)|^2 ds \\
&+ 6K(T+1)\int_0^{(k+1)\Delta t} E|y_1(s-\tau) - y(s-\tau)|^2 ds \\
&+ 6KT\int_0^{(k+1)\Delta t} E|y_2(s-\tau) - y(s-\tau)|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&\le 6K(\theta\Delta t)^2 E|x((k+1)\Delta t) - y((k+1)\Delta t)|^2 + 6K(\theta\Delta t)^2 E|x(|k+1-N|\Delta t) - y(|k+1-N|\Delta t)|^2 \\
&\quad + 6K(\theta\Delta t)^2 E|\xi(-|k+1-N|\Delta t) - \xi(-|k+1-N|\Delta t)|^2 \\
&\quad + 12K(T+1)\int_0^{(k+1)\Delta t} E|z_1(s) - y_1(s)|^2 ds + 6K(T+1)\int_{-\tau}^0 E|z_1(s) - y_1(s)|^2 \\
&\quad + 12K(2T+1)\widehat{C}_T\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2 ds + 6KT\int_{-\tau}^0 E|y_2(s) - y(s)|^2 \qquad (27)\\
&\le 6K(\theta\Delta t)^2 E|x((k+1)\Delta t) - y((k+1)\Delta t)|^2 + 6K(T+1)\int_0^{(k+1)\Delta t} E|z_1(s) - y_1(s)|^2 ds \\
&\quad + 12K(2T+1)\widehat{C}_T\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2 ds + 12K(T+1)\int_0^{(k+1)\Delta t} E|z_1(s) - y_1(s)|^2 ds.
\end{aligned}
$$

However, by (20), it is easy to see

$$6K(\theta\Delta t)^2 \le 2(3K\Delta t)\Delta t \le 2\Delta t \le \frac{1}{2}.$$

So

$$
\begin{aligned}
E|x((k+1)\Delta t) - y((k+1)\Delta t)|^2 \le& 24K(2T+1)\widehat{C}_T\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2 \\
&+ 36K(T+1)\Delta t\sum_{j=0}^{k} E|x(j\Delta t) - y(j\Delta t)|^2. \qquad (28)
\end{aligned}
$$

This, by the discrete Gronwall inequality, yields

$$
\begin{aligned}
E|x((k+1)\Delta t) - y((k+1)\Delta t)|^2 \le& 24K(2T+1)\widehat{C}_T\Delta t e^{36K(T+1)T} \sup_{r\in[-\tau,0]} E|\xi(r)|^2 \\
\le& \widetilde{C}_T\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2.
\end{aligned}
$$

where

$$\widetilde{C}_T = 24K(2T+1)\widehat{C}_T\Delta t e^{36K(T+1)T},$$

$\forall\ 0 \le t_{k+1} \le T$. Finally, the required assertion (21) follows as

$$E|x((k+1)\Delta t) - y((k+1)\Delta t)|^p \le (E|x((k+1)\Delta t) - y((k+1)\Delta t)|^2)^{\frac{p}{2}}.$$

Similarly, we can prove for all $T > 0$,

$$\sup_{\bar{k}\Delta t\le k\Delta t\le \bar{k}\Delta t+T} E|x(k\Delta t; \bar{k}\Delta t, \eta_{\bar{k}\Delta t}) - y(k\Delta t; \bar{k}\Delta t, \eta_{\bar{k}\Delta t})|^p \le C'_T(\Delta t)^{\frac{p}{2}} \sup_{r\in[\bar{k}\Delta t-\tau, \bar{k}\Delta t]} E|\eta(r)|^p.$$

This proof is hence complete. □

Lemma 2.9. *Let* (H1) *hold. Let* $p \in (0,1)$ *and let* Δt *be sufficiently small for*

$$(6 \vee 3K)\Delta t < 1. \qquad (29)$$

Then the stochastic theta method (10) *and the solution of the SDDE* (1) *satisfy*

$$\sup_{-\tau\le t\le T} E|x(t; 0, \xi) - y(t; 0, \xi)|^p \le C_T(\Delta t)^{\frac{p}{2}} \sup_{r\in[-\tau,0]} E|\xi(r)|^p, \forall T > 0, \qquad (30)$$

$$\sup_{u-\tau\le t\le u+T} E|x(t;u,\eta_u)-y(t;u,\eta_u)|^p \le C_T(\Delta t)^{\frac{p}{2}} \sup_{r\in[u-\tau,u]} E|\eta(r)|^p, \forall T>0, \tag{31}$$

where

$$C_T = [3(\bar{C}_T + \widehat{C}_T + \widetilde{C}_T)]^{\frac{p}{2}},$$

which is independent of Δt *and* $\xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$.

Proof. Write $x(t)=x(t;0,\xi)$, $y(t)=y(t;0,\xi)$, $z_1(t)=x(k\Delta t)=x(k\Delta t;0,\xi)$, $y(k\Delta t)=y(k\Delta t;0,\xi)$. It is easily to show

$$\begin{aligned}E|x(t)-y(t)|^2 \le& 3E|x(t)-x(k\Delta t)|^2+3E|y(t)-y(k\Delta t)|^2+3E|x(k\Delta t)-y(k\Delta t)|^2\\ \le& 3(\bar{C}_T+\widehat{C}_T+\widetilde{C}_T))\Delta t \sup_{r\in[-\tau,0]} E|\xi(r)|^2.\end{aligned}$$

Similarly, we can prove

$$\sup_{u\le t\le u+T} E|x(t;u,\eta_u)-y(t;u,\eta_u)|^p \le C_T(\Delta t)^{\frac{p}{2}} \sup_{r\in[u-\tau,u]} E|\eta(r)|^p, \ \ \forall T>0.$$

where

$$C_T = [3(\bar{C}_T + \widehat{C}_T + \widetilde{C}_T)]^{\frac{p}{2}}.$$

This proof is hence complete. □

Remark 2.10. *It is useful to point out that condition* (20) *implies* $K(\theta\Delta t)^2 < \frac{1}{10}$, *which is the condition required of Lemma 2.3. In fact, if* $6 > 3K$, *namely,* $2 \ge K$, *then* (29) *means* $6\Delta t < 1$. *Hence*

$$10K(\theta\Delta t)^2 \le 20(\Delta t)^2 < 1.$$

But if $2 < K$, *then* (20) *means* $3K\Delta t < 1$. *Thus*

$$10K(\theta\Delta t)^2 < 18K(\Delta t)^2 < 9K^2(\Delta t)^2 < 1.$$

This is, we always have $K(\theta\Delta t)^2 < \frac{1}{10}$ *if* (20) *holds.*

2.3. The theta method shares the pth moment stability with the SDDE.

The following theorem gives a positive answer to question (P1) from section 1.

Theorem 2.11. *Under (H1), assume that the SDDE* (1) *is pth moment exponentially stable. Then there exists a* $\Delta t^\star > 0$ *such that for every* $0 < \Delta t < \Delta t^\star$ *the theta method is pth moment exponentially stable on the SDDE* (1) *with rate constant* $\gamma = \frac{1}{2}\lambda$ *and growth constant* $Q = \bar{H}(T+\tau,p,K)e^{\frac{1}{2}\lambda(T+\tau)}$, *where* $T = \lceil 8\tau + \frac{4\log(2^pM)}{\lambda} \rceil$ *and* $\bar{H}(T+\tau,p,K)$ *is given by Lemma 2.3.*

(Please note that both γ and Q are independent of Δt.)

Proof. Without any loss of generality, we let $\Delta t < 1$. We divide the whole proof into 2 steps.

Step 1. By the definition of T, we observe that

$$2^pMe^{-\lambda(T-2\tau)} \le e^{-\frac{3}{4}\lambda T}. \tag{32}$$

By the elementary inequality

$$(a+b)^p \le (2(a\vee b))^p \le 2^p(a^p \vee b^p) \le 2^p(a^p+b^p) \ \ \forall a,b \ge 0, \tag{33}$$

in particular, for $t \geq \tau$, write $x(t) = x(t;\tau, x_\tau)$ and $y(t) = y(t;\tau, x_\tau)$, we have

$$E|x(t)|^p \leq 2^p(E|x(t) - y(t)|^p + E|y(t)|^p) \quad \forall t \geq \tau. \tag{34}$$

Using conditions (17) and (8),

$$E|y(t)|^p \leq Me^{-\lambda(t-\tau)} \sup_{r\in[0,\tau]} E|x(r)|^p, t \geq \tau$$

and

$$\sup_{t\in[T-\tau,2T-\tau]} E|x(t) - y(t)|^p \leq \sup_{t\in[\tau,2T-\tau]} E|x(t) - y(t)|^p \leq C_{2T-2\tau}(\Delta t)^{\frac{p}{2}} \sup_{r\in[0,\tau]} E|x(r)|^p,$$

we then have

$$\sup_{t\in[T-\tau,2T-\tau]} E|x(t)|^p \leq \sup_{r\in[0,\tau]} E|x(r)|^p 2^p(C_{2T-2\tau}(\Delta t)^{\frac{p}{2}} + Me^{-\lambda(T-2\tau)}). \tag{35}$$

This, together with (32), yields

$$\sup_{t\in[T-\tau,2T-\tau]} E|x(t)|^p \leq \sup_{r\in[0,\tau]} E|x(r)|^p(2^p C_{2T-2\tau}(\Delta t)^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda T}).$$

Choose $\Delta t^\star \in (0, 1)$ sufficiently small for

$$2^p C_{2T-2\tau}(\Delta t^\star)^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda T} \leq e^{-\frac{1}{2}\lambda T}.$$

Then, for every $0 < \Delta t \leq \Delta t^\star$,

$$\sup_{t\in[T-\tau,2T-\tau]} E|x(t)|^p \leq \sup_{r\in[0,\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda T} \leq \sup_{r\in[-\tau,T+\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda T}. \tag{36}$$

Moreover, by condition (11),

$$E|x(t)|^p \leq \bar{H}(T + \tau, p, K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p, -\tau \leq t \leq T + \tau. \tag{37}$$

Hence

$$E|x(t)|^p \leq Q \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\frac{1}{2}\lambda t}, -\tau \leq t \leq T + \tau, \tag{38}$$

where $Q = \bar{H}(T + \tau, p, K)e^{\frac{1}{2}\lambda(T+\tau)}$.

Step 2. Let us now consider the approximate solutions on $t \geq T + \tau$, we write $x(t) = x(t; T + \tau, x_{T+\tau})$. The process $\{x(t; T + \tau, x_{T+\tau})\}$ can be regarded as the process which is produced by the theta method applied to the SDDE (1) on $t \geq T + \tau$ with the initial data $x_{T+\tau} = \{x(r), r \in [T, T + \tau]\}$. On the other hand, let $\bar{y}(t) = y(t; T + \tau, x_{T+\tau})$ be the unique solution of the SDDE (1) with the initial data $x_{T+\tau}$. The condition (31) implies that

$$\sup_{2T-\tau\leq t\leq 3T-\tau} E|x(t) - \bar{y}(t)|^p \leq \sup_{T+\tau\leq t\leq 3T-\tau} E|x(t) - \bar{y}(t)|^p \leq C_{2T-2\tau} \sup_{r\in[T,T+\tau]} E|x(r)|^p(\Delta t)^{\frac{p}{2}}. \tag{39}$$

Moreover, by (8) (more precisely, by its equivalent form (11)), we have

$$E|\bar{y}(t)|^p \leq M \sup_{r\in[T,T+\tau]} E|x(r)|^p e^{-\lambda(t-T-\tau)}. \tag{40}$$

Using (39), (40), we can show, in the same way as we did in Step 1, that

$$\begin{aligned}\sup_{2T-\tau\le t\le 3T-\tau} E|x(t)|^p &\le \sup_{r\in[T,T+\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda T}\\ &\le \sup_{r\in[T-\tau,2T-\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda T}\end{aligned} \tag{41}$$

and

$$\begin{aligned}\sup_{2T-\tau\le t\le 3T-\tau} E|x(t)|^p &\le \sup_{r\in[T-\tau,2T-\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda T}\\ &\le \sup_{-\tau\le t\le T+\tau} E|x(t)|^p e^{-\frac{1}{2}\lambda 2T}\\ &\le \bar{H}(T+\tau,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\frac{1}{2}\lambda 2T}\\ &\le Q \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\frac{1}{2}\lambda t}.\end{aligned} \tag{42}$$

where $Q = \bar{H}(T+\tau,p,K)e^{\frac{1}{2}\lambda(T+\tau)}$.

Repeating this procedure, we can show that for any nonnegative integer i,

$$\sup_{r\in[iT-\tau,(i+1)T-\tau]} E|x(r)|^p \le \sup_{r\in[(i-1)T-\tau,iT-\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda T}. \tag{43}$$

Consequently,

$$\begin{aligned}\sup_{r\in[iT-\tau,(i+1)T-\tau]} E|x(r)|^p &\le \sup_{r\in[(i-1)T-\tau,iT-\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda T}\\ &\le \cdots \le \sup_{r\in[-\tau,T-\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda iT} \le \sup_{r\in[-\tau,T+\tau]} E|x(r)|^p e^{-\frac{1}{2}\lambda iT}\end{aligned} \tag{44}$$

and then

$$\begin{aligned}\sup_{t\in[iT-\tau,(i+1)T-\tau]} E|x(t)|^p &\le \bar{H}(T+\tau,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\frac{1}{2}\lambda iT}\\ &\le Q \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\frac{1}{2}\lambda t}.\end{aligned}$$

This is, the numerical method is exponentially stable in the *pth* moment on the SDDE (1) with rate constant $\gamma = \frac{1}{2}\lambda$ and growth constant

$$Q = \bar{H}(T+\tau,p,K)e^{\frac{1}{2}\lambda(T+\tau)}.$$

The proof is hence complete. □

The next theorem gives a positive answer to question (P2) from section 1.

Theorem 2.12. *Under (H1), assume that for a step size $\Delta t > 0$, the theta method is pth moment exponentially stable with rate constant γ and growth constant Q. If Δt satisfies*

$$2^p C_{2T-2\tau}(\Delta t)^{\frac{p}{2}} + e^{-\frac{3}{4}\gamma T} \le e^{-\frac{1}{2}\gamma T}, \tag{45}$$

where $T = \bar{k}\Delta t$ and $\bar{k}$ is the smallest integer which is no less than $8N + \frac{4\log(2^pQ)}{\gamma\Delta t}$, then the SDDE (1) is pth moment exponentially stable with rate constant $\lambda = \frac{1}{2}\gamma$ and growth constant $M = H(T+\tau,p,K)e^{\frac{1}{2}\gamma(T+\tau)}$, where $H(T,p,K)$ is given in (7).

Proof. It is easy to see from $8N + 4\log(2^p Q)/(\gamma\Delta t) \le \bar{k}$ that

$$2^p Q e^{-\gamma(\bar{k}\Delta t - 2\tau)} \le e^{-\frac{3}{4}\gamma\bar{k}\Delta t},$$

namely,

$$2^p Q e^{-\gamma(T-2\tau)} \le e^{-\frac{3}{4}\gamma T}. \tag{46}$$

As $T = \bar{k}\Delta t$, for $t \in [T-\tau, 2T-\tau]$ and $T-\tau \ge \tau$, write $y(t) = y(t;\tau, y_\tau)$ and $x(t) = x(t;\tau, y_\tau)$. By the elementary inequality (33), we have

$$E|y(t)|^p \le 2^p(E|x(t) - y(t)|^p + E|x(t)|^p). \tag{47}$$

Using (30) and (16), we obtain

$$\sup_{t\in[T-\tau,2T-\tau]} E|y(t) - x(t)|^p \le \sup_{t\in[\tau,2T-\tau]} E|y(t) - x(t)|^p \le C_{2T-2\tau}(\Delta t)^{\frac{p}{2}} \sup_{r\in[0,\tau]} E|y(r)|^p,$$

$$E|x(t)|^p \le Q \sup_{r\in[0,\tau]} E|y(r)|^p e^{-\gamma(t-\tau)}, t \ge \tau,$$

we then have

$$\begin{aligned}\sup_{t\in[T-\tau,2T-\tau]} E|y(t)|^p \le& 2^p \sup_{r\in[0,\tau]} E|y(r)|^p [C_{2T-\tau}(\Delta t)^{\frac{p}{2}} + Qe^{-\gamma(t-\tau)}]\\ \le& 2^p \sup_{r\in[0,\tau]} E|y(r)|^p [C_{2T-2\tau}(\Delta t)^{\frac{p}{2}} + Qe^{-\gamma(T-2\tau)}].\end{aligned}$$

Choose $\Delta t^\star \in (0,1)$ sufficiently small for

$$2^p C_{2T-2\tau}(\Delta t^\star)^{\frac{p}{2}} + Qe^{-\frac{3}{4}\gamma T} \le e^{-\frac{1}{2}\gamma T}.$$

Then, for every $0 < \Delta t \le \Delta t^\star$, this, together with (46) and (45), yields

$$\begin{aligned}\sup_{t\in[T-\tau,2T-\tau]} E|y(t)|^p \le& \sup_{r\in[0,\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma T}\\ \le& \sup_{r\in[-\tau,T-\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma T}\\ \le& \sup_{r\in[-\tau,T+\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma T}.\end{aligned} \tag{48}$$

Moreover, it follows from (5) that

$$\sup_{t\in[-\tau,T+\tau]} E|y(t)|^p \le H(T+\tau, p, K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p.$$

Hence

$$\sup_{t\in[-\tau,T-\tau]} E|y(t)|^p \le \sup_{t\in[-\tau,T+\tau]} E|y(t)|^p \le M \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\frac{1}{2}\gamma t}, \tag{49}$$

where $M = H(T+\tau, p, K)e^{\frac{1}{2}\gamma(T+\tau)}$.

Let us now consider the solution $y(t) = y(t; T+\tau, y_{T+\tau})$ on $t \ge T+\tau$. As explained before, this can be regarded as the solution of the SDDE (1) with the initial data $y_{T+\tau}$. Moreover, let $\{\bar{x}(t; T+\tau, y_{T+\tau})\}$ be the process which is produced by the theta method applied to the SDDE (1) on $t \ge T$ with the initial data $y_{T+\tau}$. By (18) ,

$$\sup_{2T-\tau\le t\le 3T-\tau} E|\bar{x}(t) - y(t)|^p \le \sup_{T+\tau\le t\le 3T-\tau} E|\bar{x}(t) - y(t)|^p \le C_{2T-2\tau} \sup_{r\in[T,T+\tau]} E|y(r)|^p (\Delta t)^{\frac{p}{2}}. \tag{50}$$

Also, (31) implies

$$E|\bar{x}(t)|^p \leq M \sup_{r\in[T,T+\tau]} E|y(r)|^p e^{-\lambda(t-T-\tau)}, t \geq T+\tau. \tag{51}$$

Using (50) and (51), we can show, in the same way that (35) and (36) were obtained, that

$$\sup_{2T-\tau\leq t\leq 3T-\tau} E|y(t)|^p \leq \sup_{r\in[T,T+\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma T} \leq \sup_{r\in[T-\tau,2T-\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma T}. \tag{52}$$

Repeating the procedure,we can show that for any nonnegative integer i,

$$\sup_{iT-\tau\leq t\leq (i+1)T-\tau} E|y(t)|^p \leq \sup_{r\in[(i-1)T-\tau,iT-\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma T}. \tag{53}$$

Consequently,

$$\begin{aligned}\sup_{r\in[iT-\tau,(i+1)T-\tau]} E|y(t)|^p &\leq \sup_{r\in[(i-1)T-\tau,iT-\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma T}\\ &\leq\cdots\leq \sup_{r\in[-\tau,T-\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma iT} \leq \sup_{r\in[-\tau,T+\tau]} E|y(r)|^p e^{-\frac{1}{2}\gamma iT}\end{aligned} \tag{54}$$

and then

$$\begin{aligned}\sup_{r\in[iT-\tau,(i+1)T-\tau]} E|y(t)|^p \leq& H(T+\tau,p,K) \sup_{r\in[-\tau,0]} E|y(r)|^p e^{-\frac{1}{2}\gamma iT}\\ \leq& H(T+\tau,p,K)e^{\frac{1}{2}\gamma(T+\tau)}e^{-\frac{1}{2}[\gamma(i+1)T+\tau]} \sup_{r\in[-\tau,0]} E|\xi(r)|^p\\ \leq& M \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\frac{1}{2}\gamma t},\end{aligned}$$

where

$$M = H(T+\tau,p,K)e^{\frac{1}{2}\gamma(T+\tau)}.$$

The proof is completed. □

Theorem 2.11 and 2.12 lead to the following theorem.

Theorem 2.13. *The SDDE* (1) *is exponentially stable in the pth moment if and only if the theta method is exponentially stable in the pth moment with rate constant γ, growth constant Q, step size Δt, and global error constant C_T for $T = \bar{k}\Delta t$ satisfying* (32)*, where $\bar{k}$ is the smallest integer which is no less than* $8N + 4\log(2^pQ)/(\gamma\Delta t)$.

Proof. The "if" part of the theorem follows from Theorem 2.12 directly. To prove the "only if" part, suppose the SDDE (1) is exponentially stable in the *pth* moment with rate constant λ and growth constant M. Theorem 2.11 shows that there is a $\Delta t^\star > 0$, the theta method is exponentially stable in the *pth* moment with rate constant $\gamma = \frac{1}{2}\lambda$ and growth constant $Q = 2^p(C_{2T-2\tau} + M)e^{\frac{1}{2}\lambda(T+\tau)}$, where $T = 8\tau + \frac{(4\log(2^pM))}{\lambda}$. Noting that both of these constants are independent of Δt, it follows that we may reduce Δt if necessary until (45) becomes satisfied. □

Corollary 2.14. *Assume that the SDDE* (1) *is pth moment exponentially stable and satisfies* (8)*. Let $\epsilon \in (0,1)$. Then the theta method is pth moment exponentially stable on the SDDE* (1) *with rate constant $\gamma = (1-\epsilon)\lambda$ and growth constant $Q = 2^p(C_{2T-2\tau} + M)e^{(1-\epsilon)\lambda(T+\tau)}$, where $T = 2\log(2^pM)/(\epsilon\lambda) + (2-\epsilon)/\epsilon$.*

(Please note that both γ and Q are independent of Δt.)

The proof is similar to that of Theorem 2.11 and so is omitted. □

Corollary 2.15. *Let $\epsilon \in (0,1)$. Assume that for a step size $\Delta t > 0$,the theta method is pth moment exponentially stable with rate constant γ, growth constant Q. If Δt satisfies*

$$2^p C_{2T-2\tau}(\Delta t)^{\frac{p}{2}} + e^{-(1-0.5\epsilon)\gamma T} \leq e^{-(1-\epsilon)\gamma T}, \tag{55}$$

where $T = t_{\bar{k}}$ and $\bar{k}$ is the smallest integer which is not less than $8N + \frac{2\log(2^p Q)}{\epsilon\gamma\Delta t}$, then the SDDE (1) *is the pth moment exponentially stable with rate constant γ and growth constant $M = H(T+\tau, p, K)e^{(1-\epsilon)\gamma(T+\tau)}$, where $H(T+\tau, p, K)$ is given in* (7).

The proof is similar to that of Theorem 2.12 and so is omitted. □

2.4. Almost sure exponential stability

Definition 2.16. *The trivial solution of the SDDE* (1) *is said to be almost surely exponentially stable if there exists a constant $\alpha > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|y(t;0,\xi)|) \leq -\alpha \quad a.s.$$

for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0], \mathbf{R}^n)$.

Definition 2.17. *The numerical solution $\{x(t;0,\xi)\}$ is said to be almost surely exponentially stable if there exists a constant $\alpha > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t;0,\xi)|) \leq -\alpha \quad a.s.$$

for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0], \mathbf{R}^n)$.

Our paper is mainly concerned with the almost sure exponential stability of both exact and numerical approximations with the objective of finding positive answers to problems (P1) and (P2) in the section 1. It is therefore time to relate the *pth* moment exponential stability to the almost sure exponential stability.

Theorem 2.18. *Assume that* (*H*1) *holds and Δt satisfies $2(2K)^{\frac{p}{2}}[(\Delta t)^p + C_p(\Delta t)^{\frac{p}{2}}] \leq \frac{1}{2}$. Let $p \in (0,1)$. Assume that the SDDE* (1) *is pth moment exponentially stable and satisfies* (8). *Then the solution of the SDDE* (1) *satisfies*

$$\lim_{t\to\infty} \frac{1}{t}\log(|y(t;0,\xi)|) \leq -\frac{\lambda}{p} \quad a.s. \tag{56}$$

for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0], \mathbf{R}^n)$. That is, the SDDE (1) *is also almost surely exponentially stable.*

Proof. We write $y(t) = y(t;0,\xi)$. Let $\varepsilon \in (0, \frac{\lambda}{2})$ be arbitrary, we have

$$E|y(t)|^p \leq M \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-(\lambda-\varepsilon)t}, \quad t \geq 0. \tag{57}$$

Noting that for any $a, b, c, \geq 0$,

$$(a+b+c)^p \leq [3(a \vee b \vee c)]^p = 3^p(a^p \vee b^p \vee c^p) \leq 3^p(a^p + b^p + c^p),$$

we have

$$\begin{aligned} E|y(t)|^p \leq & 3^p E|y((k-1)\Delta t)|^p + 3^p E|\int_{(k-1)\Delta t}^t f(y(s), y(s-\tau))ds|^p \\ & + 3^p E|\int_{(k-1)\Delta t}^t g(y(s), y(s-\tau))dw(s)|^p \end{aligned} \tag{58}$$

for all $(k-1)\Delta t \le t \le k\Delta t$.

It is easy to see that

$$E|y((k-1)\Delta t)|^p \le M \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-(\lambda-\varepsilon)(k-1)\Delta t}. \tag{59}$$

By Hölder inequality and (H1), we have

$$\begin{aligned}
E|\int_{(k-1)\Delta t}^{t} f(y(s),y(s-\tau))ds|^p \le& (E|\int_{(k-1)\Delta t}^{t} f(y(s),y(s-\tau))ds|^2)^{\frac{p}{2}}\\
\le&[t-(k-1)\Delta t]^{\frac{p}{2}}(\int_{(k-1)\Delta t}^{t} E|f(y(s),y(s-\tau))|^2 ds)^{\frac{p}{2}}\\
\le&[\Delta t]^{\frac{p}{2}}K^{\frac{p}{2}}[\int_{(k-1)\Delta t}^{t}(E|y(s)|^2+E|y(s-\tau)|^2)ds]^{\frac{p}{2}}\\
\le&[\Delta t]^{\frac{p}{2}}K^{\frac{p}{2}}2^{\frac{p}{2}}[(\int_{(k-1)\Delta t}^{t}E|y(s)|^2ds)^{\frac{p}{2}}+(\int_{(k-1)\Delta t}^{t}E|y(s-\tau)|^2ds)^{\frac{p}{2}}]\\
\le&2[\Delta t]^{\frac{p}{2}}K^{\frac{p}{2}}2^{\frac{p}{2}}[\Delta t]^{\frac{p}{2}}\sup_{r\in[(k-1)\Delta t-\tau,t]}E|y(r)|^p\\
\le&2(2K)^{\frac{p}{2}}[\Delta t]^p M\sup_{r\in[-\tau,0]}E|\xi(r)|^p e^{-(\lambda-\varepsilon)[(k-1)\Delta t-\tau]}.
\end{aligned} \tag{60}$$

By Itô isometry, the Burkholder-Davis-Gundy inequality and (H1), we have

$$\begin{aligned}
E|\int_{(k-1)\Delta t}^{t} g(y(s),y(s-\tau))dw(s)|^p =&E(|\int_{(k-1)\Delta t}^{t} g(y(s),y(s-\tau))dw(s)|^2)^{\frac{p}{2}}\\
\le&C_p(\int_{(k-1)\Delta t}^{t}E|g(y(s),y(s-\tau))|^2ds)^{\frac{p}{2}}\\
\le&C_pK^{\frac{p}{2}}[\int_{(k-1)\Delta t}^{t}(E|y(s)|^2+E|y(s-\tau)|^2)ds]^{\frac{p}{2}}\\
\le&C_pK^{\frac{p}{2}}2^{\frac{p}{2}}[(\int_{(k-1)\Delta t}^{t}E|y(s)|^2ds)^{\frac{p}{2}}+(\int_{(k-1)\Delta t}^{t}E|y(s-\tau)|^2ds)^{\frac{p}{2}}]\\
\le&2C_pK^{\frac{p}{2}}2^{\frac{p}{2}}[\Delta t]^{\frac{p}{2}}\sup_{r\in[(k-1)\Delta t-\tau,t]}E|y(r)|^p\\
\le&2C_p(2K)^{\frac{p}{2}}[\Delta t]^{\frac{p}{2}}M\sup_{r\in[-\tau,0]}E|\xi(r)|^p e^{-(\lambda-\varepsilon)[(k-1)\Delta t-\tau]},
\end{aligned} \tag{61}$$

where C_p is a positive constant dependent of p only. Let Δt satisfies $2(2K)^{\frac{p}{2}}[(\Delta t)^p + C_p(\Delta t)^{\frac{p}{2}}] \le \frac{1}{2}$.

Substituting (59)-(61) into (58), yields that

$$E|y(t)|^p \le LM \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-(\lambda-\varepsilon)(k-1)\Delta t},$$

where $L = 3^p\{1 + e^{\lambda\tau}\}$.

Hence, by Chebyshev inequality, we have

$$\begin{aligned}
P\{\omega : |y(t)| > e^{\frac{-(\lambda-2\varepsilon)(k-1)\Delta t}{p}}\} \le& \frac{LM\sup_{r\in[-\tau,0]}E|\xi(r)|^p e^{-(\lambda-\varepsilon)(k-1)\Delta t}}{e^{\frac{-(\lambda-2\varepsilon)(k-1)\Delta t}{p}}}\\
\le& LM\sup_{r\in[-\tau,0]}E|\xi(r)|^p e^{-\varepsilon(k-1)\Delta t}.
\end{aligned}$$

In view of the well-know Borel Cantelli lemma, we see that for almost all $\omega \in \Omega$

$$|y(t)| \le e^{\frac{-(\lambda-2\varepsilon)(k-1)\Delta t}{p}} \tag{62}$$

hold for all but finitely many k. Hence there exists a $k_0(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, for which (62) holds whenever $k \ge k_0(\omega)$. Consequently, for almost all $\omega \in \Omega$

$$\frac{1}{t}\log|y(t)| \le \frac{-(\lambda-2\varepsilon)(k-1)\Delta t}{pt} \le \frac{-(\lambda-2\varepsilon)(k-1)}{pk},$$

if $(k-1)\Delta t \le t \le k\Delta t$ and $k \ge k_0(\omega)$. Therefore

$$\lim_{t\to\infty}\frac{1}{t}\log|y(t)| \le -\frac{\lambda-2\varepsilon}{p} \quad a.s.$$

and the required (56) follows by letting $\varepsilon \to 0$. The proof is completed. □

The following theorem is an analogue for the numerical solutions.

Theorem 2.19. *Assume that the theta method is pth moment exponentially stable satisfies* (16) *and* Δt *satisfies* $2(2K)^{\frac{p}{2}}\{[(1-\theta)^{\frac{p}{2}}+\theta^{\frac{p}{2}}](\Delta t)^p + C_p(\Delta t)^{\frac{p}{2}}\} \le \frac{1}{2}$. *Then the theta method satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t;0,\xi)|) \le -\frac{\gamma}{p} \quad a.s. \tag{63}$$

for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$. *That is, the theta method is also almost surely exponentially stable.*

Proof. We write $x(t) = x(t;0,\xi)$. Let $\varepsilon \in (0,\frac{\gamma}{2})$ be arbitrary, we have

$$E|x(t)|^p \le Q\sup_{r\in[-\tau,0]}E|\xi(r)|^p e^{-(\gamma-\varepsilon)t}, t \ge 0. \tag{64}$$

Noting that for any $a, b, c, \ge 0$,

$$(a+b+c+d)^p \le [4(a\vee b\vee c\vee d)]^p = 4^p(a^p\vee b^p\vee c^p\vee d^p) \le 4^p(a^p+b^p+c^p+d^p),$$

we have

$$\begin{aligned} E|x(t)|^p \le & 4^pE|x((k-1)\Delta t)|^p + 4^pE|\int_{(k-1)\Delta t}^t (1-\theta)f(z_1(s),z_1(s-\tau))ds|^p \\ & + 4^pE|\int_{(k-1)\Delta t}^t \theta f(z_2(s),z_2(s-\tau))ds|^p + 4^pE|\int_{(k-1)\Delta t}^t g(z_1(s),z_1(s-\tau))dw(s)|^p \end{aligned} \tag{65}$$

for all $(k-1)\Delta t \le t \le k\Delta t$.

It is obvious that

$$E|x((k-1)\Delta t)|^p \le Q\sup_{r\in[-\tau,0]}E|\xi(r)|^p e^{-(\gamma-\varepsilon)(k-1)\Delta t}. \tag{66}$$

By Hölder inequality and (H1), we have

$$
\begin{aligned}
&E|\int_{(k-1)\Delta t}^{t}(1-\theta)f(z_1(s),z_1(s-\tau))ds|^p\\
&\le (1-\theta)^{\frac{p}{2}}(E|\int_{(k-1)\Delta t}^{t}f(z_1(s),z_1(s-\tau))ds|^2)^{\frac{p}{2}}\\
&\le (1-\theta)^{\frac{p}{2}}[t-(k-1)\Delta t]^{\frac{p}{2}}(\int_{(k-1)\Delta t}^{t}E|f(z_1(s),z_1(s-\tau))|^2ds)^{\frac{p}{2}}\\
&\le (1-\theta)^{\frac{p}{2}}[\Delta t]^{\frac{p}{2}}K^{\frac{p}{2}}[\int_{(k-1)\Delta t}^{t}(E|z_1(s)|^2+E|z_1(s-\tau)|^2)ds]^{\frac{p}{2}}\\
&\le [2K(1-\theta)\Delta t]^{\frac{p}{2}}[(\int_{(k-1)\Delta t}^{t}E|z_1(s)|^2ds)^{\frac{p}{2}}+(\int_{(k-1)\Delta t}^{t}E|z_1(s-\tau)|^2ds)^{\frac{p}{2}}]\\
&\le 2[2K(1-\theta)]^{\frac{p}{2}}[\Delta t]^p\sup_{r\in[(k-1)\Delta t-\tau,t]}E|x(r)|^p\\
&\le 2[2K(1-\theta)]^{\frac{p}{2}}[\Delta t]^pM\sup_{r\in[-\tau,0]}E|\xi(r)|^pe^{-(\gamma-\varepsilon)[(k-1)\Delta t-\tau]},
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
E|\int_{(k-1)\Delta t}^{t}(1-\theta)f(z_1(s),z_1(s-\tau))ds|^p &\le \theta^{\frac{p}{2}}(E|\int_{(k-1)\Delta t}^{t}f(z_2(s),z_2(s-\tau))ds|^2)^{\frac{p}{2}}\\
&\le \theta^{\frac{p}{2}}[t-(k-1)\Delta t]^{\frac{p}{2}}(\int_{(k-1)\Delta t}^{t}E|f(z_2(s),z_2(s-\tau))|^2ds)^{\frac{p}{2}}\\
&\le [K\theta\Delta t]^{\frac{p}{2}}[\int_{(k-1)\Delta t}^{t}(E|z_2(s)|^2+E|z_2(s-\tau)|^2)ds]^{\frac{p}{2}}\\
&\le [2K\theta\Delta t]^{\frac{p}{2}}[(\int_{(k-1)\Delta t}^{t}E|z_2(s)|^2ds)^{\frac{p}{2}}+(\int_{(k-1)\Delta t}^{t}E|z_2(s-\tau)|^2ds)^{\frac{p}{2}}]\\
&\le 2[2K\theta]^{\frac{p}{2}}[\Delta t]^p\sup_{r\in[(k-1)\Delta t-\tau,k\Delta t]}E|x(r)|^p\\
&\le 2[2K\theta]^{\frac{p}{2}}[\Delta t]^pM\sup_{r\in[-\tau,0]}E|\xi(r)|^pe^{-(\gamma-\varepsilon)[(k-1)\Delta t-\tau]}.
\end{aligned}
\tag{68}
$$

By Itô isometry, the Burkholder-Davis-Gundy inequality and (H1), we have

$$
\begin{aligned}
E|\int_{(k-1)\Delta t}^{t}g(z_1(s),z_1(s-\tau))dw(s)|^p &=E(|\int_{(k-1)\Delta t}^{t}g(z_1(s),z_1(s-\tau))ds|^2)^{\frac{p}{2}}\\
&\le C_p(\int_{(k-1)\Delta t}^{t}E|g(z_1(s),z_1(s-\tau))|^2ds)^{\frac{p}{2}}\\
&\le C_pK^{\frac{p}{2}}[\int_{(k-1)\Delta t}^{t}(E|z_1(s)|^2+E|z_1(s-\tau)|^2)ds]^{\frac{p}{2}}\\
&\le C_pK^{\frac{p}{2}}2^{\frac{p}{2}}[(\int_{(k-1)\Delta t}^{t}E|z_1(s)|^2ds)^{\frac{p}{2}}+(\int_{(k-1)\Delta t}^{t}E|z_1(s-\tau)|^2ds)^{\frac{p}{2}}]\\
&\le 2C_pK^{\frac{p}{2}}2^{\frac{p}{2}}[\Delta t]^{\frac{p}{2}}\sup_{r\in[(k-1)\Delta t-\tau,t]}E|x(r)|^p\\
&\le 2C_p[2K\Delta t]^{\frac{p}{2}}M\sup_{r\in[-\tau,0]}E|\xi(r)|^pe^{-(\gamma-\varepsilon)[(k-1)\Delta t-\tau]},
\end{aligned}
\tag{69}
$$

where C_p is a positive constant dependent of p only. Let Δt satisfies $2(2K)^{\frac{p}{2}}\{[(1-\theta)^{\frac{p}{2}}+\theta^{\frac{p}{2}}](\Delta t)^p+C_p(\Delta t)^{\frac{p}{2}}\}\le\frac{1}{2}$.

Substituting (66)-(69) into (65), yields that

$$E|x(t)|^p \leq LM \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-(\gamma-\varepsilon)(k-1)\Delta t}.$$

where $L = 4^p\{1 + e^{\lambda\tau}\}$.

Hence, by Chebyshev inequality, we have

$$P\{\omega : |x(t)| > e^{\frac{-(\gamma-2\varepsilon)(k-1)\Delta t}{p}}\} \leq \frac{LM \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-(\gamma-\varepsilon)(k-1)\Delta t}}{e^{\frac{-(\gamma-2\varepsilon)(k-1)\Delta t}{p}}} \leq LM \sup_{r\in[-\tau,0]} E|\xi(r)|^p e^{-\varepsilon(k-1)\Delta t}.$$

In view of the well-know Borel Cantelli lemma, we see that for almost all $\omega \in \Omega$

$$|x(t)| \leq e^{\frac{-(\gamma-2\varepsilon)(k-1)\Delta t}{p}} \tag{70}$$

hold for all but finitely many k. Hence there exists a $k_0(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, for which (70) holds whenever $k \geq k_0(\omega)$. Consequently, for almost all $\omega \in \Omega$

$$\frac{1}{t}\log|x(t)| \leq \frac{-(\gamma-2\varepsilon)(k-1)\Delta t}{pt} \leq \frac{-(\gamma-2\varepsilon)(k-1)}{pk},$$

if $(k-1)\Delta t \leq t \leq k\Delta t$ and $k \geq k_0(\omega)$. Therefore

$$\lim_{t\to\infty} \frac{1}{t}\log|x(t)| \leq -\frac{\gamma-2\varepsilon}{p} \quad a.s.$$

and the required (63) follows by letting $\varepsilon \to 0$. The proof is completed. □

2.5. Conclusions

In this paper, we show that, under the standing (H1),

$$(A) \Leftarrow (B) \Leftrightarrow (C) \Rightarrow (D),$$

where

(A) denote the almost sure exponential stability of the SDDE (1) under global Lipschitz condition,

(B) denote the *pth* moment exponential stability of the SDDE (1) ($p \in (0,1)$ is sufficient small) under global Lipschitz condition,

(C) denote the *pth* moment exponential stability of the stochastic theta method for a sufficient small step size under global Lipschitz condition,

(D) denote the almost sure exponential stability of the stochastic theta method for a sufficient small step size under global Lipschitz condition.

3. Improved result.

3.1. pth moment stability.

In this section we shall replace the global Lipschitz condition with a more general condition. As a standing Hypothesis we assume that the coefficients f and g of the SDDE (1) are sufficiently smooth so that it has the unique solution $y(t;0,\xi)$ for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$. Moreover, we will write the numerical approximate as $x(t;0,\xi)$.

As pointed out in the previous section, we recall that the proof of Theorem 2.11 uses only properties Lemma 2.4 and Lemma 2.9 rather than hypothesis (H1) itself while the proof of Theorem 2.12 makes use of Lemma 2.1 and Lemma 2.9. This leads to the following definition then a improved result.

Assumption 3.1. *Let $p \in (0,1)$. For all sufficiently small Δt the SDDE* (1) *and the corresponding numerical method with initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$ satisfies*

$$\sup_{-\tau\le t\le T} E|y(t;0,\xi)|^p \le h(T,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p \tag{71}$$

and

$$\sup_{-\tau\le t\le T} E|x(t;0,\xi)|^p \le \bar{h}(T,p,K) \sup_{r\in[-\tau,0]} E|\xi(r)|^p \quad \forall T \ge 0, \tag{72}$$

where $h(t,p,K), \bar{h}(T,p,K)$ are independent of Δt.

Assumption 3.2. *The numerical method* (10) *and the solution of the SDDE* (1) *satisfy*

$$\sup_{-\tau\le t\le T} E|x(t;0,\xi) - y(t;0,\xi)|^p \le c_T\beta(\Delta t) \sup_{r\in[-\tau,0]} E|\xi(r)|^p \tag{73}$$

and

$$\sup_{\bar{k}\Delta t-\tau\le t\le \bar{k}\Delta t+T} E|x(t;\bar{k}\Delta t,\eta_{\bar{k}\Delta t}) - y(t;\bar{k}\Delta t,\eta_{\bar{k}\Delta t})|^p \le c_T\beta(\Delta t) \sup_{r\in[\bar{k}\Delta t-\tau,\bar{k}\Delta t]} E|\eta(r)|^p \tag{74}$$

for $T \ge 0$, where c_T depends on T but not on initial data and Δt and $\beta : \mathbf{R}^+ \to \mathbf{R}^+$ is a strictly increasing continuous function with $\beta(0) = 0$.

Theorem 3.3. *Suppose that the numerical method satisfied Assumption 3.1 and 3.2. Then the SDDE* (1) *is exponentially stable in the pth moment if and only if the numerical method is exponentially stable in the pth moment with rate constant γ, growth constant Q, step size Δt, and global error constant c_T for $T = \bar{k}\Delta t$ satisfying* (45)*, where $\bar{k}$ is the smallest integer which is no less than $8N + 4\log(2^pQ)/(\gamma\Delta t)$.*

The proof is similar to Theorem 2.13. □

3.2. Almost sure exponential stability.

Theorem 3.4. *Let Assumption* (71) *hold. Let $p \in (0,1)$. Assume that the SDDE* (1) *is pth moment exponentially stable, then the solution of the SDDE* (1) *satisfies*

$$\lim_{t\to\infty} \frac{1}{t}\log(|y(t;0,\xi)|) \le -\frac{\lambda}{p} \ \ a.s.$$

for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$. That is, the SDDE (1) *is also almost surely exponentially stable.*

The proof is similar to Theorem 2.18. □

The following theorem is an analogue for the numerical solutions.

Theorem 3.5. *Let Assumption* (72) *hold. Let $p \in (0,1)$. Assume that the numerical method is pth moment exponentially stable, then the theta method satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t;0,\xi)|) \le -\frac{\gamma}{p} a.s. \tag{75}$$

for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau,0],\mathbf{R}^n)$. That is, the numerical method is also almost surely exponentially stable.

The proof is similar to Theorem 2.19. □

3.3. Conclusions.

In this paper, we show that, under the standing Assumption (71) and (72) hold,

$$(A) \Leftarrow (B) \Leftrightarrow (C) \Rightarrow (D),$$

where

(A) denote the almost sure exponential stability of the SDDE (1),
(B) denote the *pth* moment exponential stability of the SDDE (1) ($p \in (0,1)$ is sufficient small),
(C) denote the *pth* moment exponential stability of the numerical method for a sufficient small step size,
(D) denote the almost sure exponential stability of the numerical method for a sufficient small step size.

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