Filomat 33:3 (2019), 825–836
https://doi.org/10.2298/FIL1903825C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Refined Bounds of the Quantum Quadratures Within the Class of Distance-Disturbed Convex Functions in Two Dimensions

Gabriela Cristescu[a], Muhammad Uzair Awan[b], Mihail Găianu[c]

[a]*Department of Mathematics and Computer Sciences, Äurel VlaicuÜniversity of Arad, Bd. Revolutiei, No. 77, 310130-Arad, Romania.*
[b]*GC University, Faisalabad, Pakistan*
[c]*West University of Timişoara, Faculty of Mathematics and Informatics, Bd. V.Pârvan, No.1, Timişoara, România*

Abstract. In this paper, we introduce the class of disturbed convex functions defined by means of distance perturbations in two dimensions on co-ordinates. Some quantum trapezoidal estimations are obtained for functions having two dimensional distance-disturbed convexity properties. Refined bounds of the quantum integrals of distance-disturbed convex functions on coordinates are deduced by using the rectangular finite elements technique. These approximations are as best as possible from the sharpness point of view. The sharpness of few results from the literature follows as consequence of the new results in this paper.

1. Introduction

The impressive range of applications of the convexity properties for sets and functions resulted in many attempts to define similar concepts, more general ones, that preserve as many properties as possible. A thorough discussion on generalized convexity properties for sets is in [4], counting more than one hundred of convexities with applications in discrete optimization, in picture processing and image understanding. The generalized convexities for sets are the framework for generalized convexities for functions. The idea of comparing the graph of the function with some other curve, altering somehow the straight-line segment, lead both to strengthening ([2], [21]) and to weakening ([3], [18], [8], [19]) the convexity.
Let us denote by $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$, respectively, the set of all real, rational, integer and natural numbers. Let us consider the numbers $a, b, c, d \in [0, +\infty)$ such as $a < b$ and $c < d$, and a rectangle $D = [a, b] \times [c, d]$. By $L_1(D)$ we denote the set of all Lebesque integrable functions over the rectangle D.
It is well known that a function $f : [a, b] \to \mathbb{R}$ is convex on $[a, b]$ if it is a solution of the following functional inequation:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \forall x, y \in [a, b], t \in [0, 1]. \tag{1}$$

The converse inequality defines the concave functions. The functions that make sharp the above inequality are called affine functions. It is well-known that a function is convex if and only if the following inequality,

2010 *Mathematics Subject Classification*. 26D15; 26A51
Keywords. Distance-disturbed convex functions, strong convexity, weak convexity, quantum integral, Hermite-Hadamard inequality, rectangular finite element
Received: 10 August 2016; Accepted: 25 June 2019
Communicated by Dragan S. Djordjević
Corresponding Author: Muhammad Uzair Awan
Email addresses: gabriela.cristescu@uav.ro (Gabriela Cristescu), awan.uzair@gmail.com (Muhammad Uzair Awan), mgaianu@info.uvt.ro (Mihail Găianu)

known as Hermite-Hadamard inequality (named after its authors, [9] and [14]):

$$f\left(\frac{p+q}{2}\right) \le \frac{1}{q-p}\int_p^q f(x)dx \le \frac{f(p)+f(q)}{2}, \tag{2}$$

holds whenever $p, q \in [a, b]$ and $p < q$. The reverse inequality characterizes the concave functions. The inequality is sharp. Each generalization of convexity for functions is accompanied by research on similar type of integral inequalities [6], [20].

Let $\sigma \in \mathbb{R}$ and X a linear normed space. Let $A \subseteq X$ be a convex set and suppose that $d : X \times X \to \mathbb{R}$ is a distance on X (it may be defined by means of the norm of the space). Consider a function $f : A \to \mathbb{R}$. The following inequation is frequent in the last two decades mathematical literature ([2], [10]), together with generalized versions ([1], [3], [11], [12], [13]):

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - \sigma t(1-t)d^2(x, y), \tag{3}$$

for all $x, y \in A$ and for all $t \in [0, 1]$.

Definition 1.1. *A function $f : A \to \mathbb{R}$ is said to be (σ, d)-disturbed convex (or distance-disturbed convex with modulus σ) if verifies (3) for all $x, y \in A$ and for all $t \in [0, 1]$.*

Remark 1.2. *If the inequality in (3) is strict then f is said to be strict (σ, d)-disturbed convex. The (σ, d)-disturbed convexity is an unifying concept that includes more types of convexities described by inequalities of type (3):*

- *If $\sigma > 0$ then a solution of inequation (3) is called strongly convex function (see, for example, [2]). The concept of strong convexity have been introduced by Polyak [21] in connection with solving some extremum problems with applications in mathematical economics and in optimization.*
- *If $\sigma = 0$ in (4), then inequation (3) reduces to the definition (1) of the convex functions.*
- *If $\sigma < 0$ then a solution of inequation (3) is called weak (σ, d)-convex. This concept unifies more types of weak convexity, as in [2] and [10], leading to the generalized weak convexity properties from [1], [3], [11], [12], [13].*

If $X = \mathbb{R}$ and $A = I \subseteq \mathbb{R}$ is an interval then (3) becomes

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - \sigma t(1-t)(x-y)^2, \tag{4}$$

for all $x, y \in A$ and for all $t \in [0, 1]$.

In this paper we extend the properties of (σ, d)-disturbed convexity to real functions of two real variables, working on coordinates. Hermite-Hadamard type inequalities are deduced using the classic Lebesque integration in section 2. Few elements of quantum integral calculus are mentioned in Section 3 and some new auxiliary calculus identities are deduced in this context. The quantum calculus is used in mathematics, in particular in mathematical inequalities, since it provides good approximation of the calculus (see [7, 17]), without the use of limits. The quantum calculus has been initiated by Euler (1707-1783), but it formally developed in early twentieth century starting with the work of F. H. Jackson [15]. In quantum calculus, one establishes q-analogues of mathematical objects, which can be recaptured as $q \to 1$. Detailed presentation of quantum calculus are in [7], [17]. Section 3 contains estimates of the quantum q-integral of a function, which is (σ, d)-disturbed convex on co-ordinates. All the inequalities derived in Sections 2 and 3 are sharp. These quantum integral estimates have applications in many domains of mathematics (orthogonal polynomials, basic hypergeometric functions [17], [23]) science (quantum theory, mechanics, theory of relativity) and life (economics, statistics, information processing). We derive refined bounds of the quantum integrals of distance-disturbed convex functions on coordinates by splitting the integration domain into rectangular finite elements.

2. Distance-disturbed convexity on coordinates and Hermite-Hadamard type inequalities

In this paper we introduce and study the co-ordinated distance-disturbed convexity for two variables functions defined on rectangles. Some integral inequalities of Hermite-Hadamard type are derived for two variables functions having distance-disturbed convexities on co-ordinates. Let $A \subseteq \mathbb{R}^2$ be a convex set. By $L_1(A)$ one denotes the set of all Lebesque integrable over A. We consider functions $f : A \to \mathbb{R}$, restricting the research to particular rectangles $D = [a,b] \times [c,d] \subseteq A$. Denote by $A_x = \{y|(x,y) \in A\}$ and $A_y = \{x|(x,y) \in A\}$. Define, as usual, the partial mappings $f_y : A_y \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : A_x \to \mathbb{R}$, $f_x(v) = f(x,v)$. Let us chose the fixed numbers $\sigma_1, \sigma_2 \in \mathbb{R}$.

Definition 2.1. *A function $f : A \to \mathbb{R}$ is said to be co-ordinated distance-disturbed (σ_1, σ_2)-convex on A if*

$$\begin{aligned} & f(tx_1 + (1-t)x_2, \tau y_1 + (1-\tau)y_2) + \sigma_1 t(1-t)(x_1 - x_2)^2 + \sigma_2 \tau(1-\tau)(y_1 - y_2)^2 \\ & \le t\tau f(x_1, y_1) + t(1-\tau) f(x_1, y_2) + \tau(1-t) f(x_2, y_1) + (1-t)(1-\tau) f(x_2, y_2), \end{aligned} \tag{5}$$

whenever $(x_1, y_1), (x_2, y_2) \in A$ and $t, \tau \in [0,1]$.

The corresponding concept of co-ordinated distance-disturbed concavity is defined by reversing the inequality in (5).

Proposition 2.2. *A function $f : A \to \mathbb{R}$ is co-ordinated distance-disturbed (σ_1, σ_2)-convex on A if and only if the partial function f_y is distance-disturbed convex with modulus σ_1 for all $y \in A_x$ and f_x is distance-disturbed convex with modulus σ_2 for all $x \in A_y$.*

Proof. First, let us suppose that f is co-ordinated distance-disturbed (σ_1, σ_2)-convex on A. Let us consider two points $(x_1, y), (x_2, y) \in A$ and $t \in [0,1]$. By (5) we can write

$$\begin{aligned} & f_y(tx_1 + (1-t)x_2) \\ & = f(t(x_1, y) + (1-t)(x_2, y)) + \sigma_1 t(1-t)(x_1 - x_2)^2 \\ & \le t^2 f(x_1, y) + t(1-t) f(x_1, y) + t(1-t) f(x_2, y) + (1-t)^2 f(x_2, y) \\ & = t f(x_1, y) + (1-t) f(x_2, y) = t f_y(x_1) + (1-t) f_y(x_2), \end{aligned}$$

which means that f_y is distance-disturbed convex with modulus σ_1. One can prove the distance-disturbed convexity with modulus σ_2 of f_x in the same manner.
Now, let us suppose that the the partial function f_y is distance-disturbed convex with modulus σ_1 for all $y \in A_x$ and f_x is distance-disturbed convex with modulus σ_2 for all $x \in A_y$. Then, if $(x_1, y_1), (x_2, y_2) \in A$ and $t, \tau \in [0,1]$ one has

$$\begin{aligned} & f(tx_1 + (1-t)x_2, \tau y_1 + (1-\tau)y_2) + \sigma_1 t(1-t)(x_1 - x_2)^2 + \sigma_2 \tau(1-\tau)(y_1 - y_2)^2 \\ & = f_{\tau y_1 + (1-\tau)y_2}(tx_1 + (1-t)x_2) + \sigma_1 t(1-t)(x_1 - x_2)^2 + \sigma_2 \tau(1-\tau)(y_1 - y_2)^2 \\ & \le t f_{\tau y_1 + (1-\tau)y_2}(x_1) + (1-t) f_{t y_1 + (1-t)y_2}(x_2) + \sigma_2 \tau(1-\tau)(y_1 - y_2)^2 \\ & = t f(x_1, \tau y_1 + (1-\tau)y_2) + (1-t) f(x_2, t y_1 + (1-t)y_2) + \sigma_2 \tau(1-\tau)(y_1 - y_2)^2 \\ & \le t\tau f(x_1, y_1) + t(1-\tau) f(x_1, y_2) + \tau(1-t) f(x_2, y_1) + (1-t)(1-\tau) f(x_2, y_2), \end{aligned}$$

which means that f is co-ordinated distance-disturbed (σ_1, σ_2)-convex on A. □

Remark 2.3. *Suppose that $\sigma_1 = \sigma_2 = \sigma$. If f is co-ordinated distance-disturbed (σ_1, σ_2)-convex on A and d is the Euclidean distance in $\mathbb{R}^2$ then from (5) one obtains*

$$\begin{aligned} & f(t(x_1, y_1) + (1-t)(x_2, y_2)) + \sigma t(1-t) d^2((x_1, y_1), (x_2, y_2)) \\ & \le t^2 f(x_1, y_1) + t(1-t) f(x_1, y_2) + t(1-t) f(x_2, y_1) + (1-t)^2 f(x_2, y_2), \end{aligned} \tag{6}$$

whenever $(x_1, y_1), (x_2, y_2) \in A$ and $t \in [0,1]$.

Remark 2.4. *Definition 2.1 includes many possible cases of surfaces that have either some strong or some weak type of convexity on directions parallel to the coordinates axes, as follows:*

1. $\sigma_1 > 0$ *and* $\sigma_2 > 0$ *defines the (strong, strong)* (σ_1, σ_2)*-convex functions on coordinates;*
2. *(*$\sigma_1 > 0$ *and* $\sigma_2 < 0$*) (or (*$\sigma_1 < 0$ *and* $\sigma_2 > 0$*)) defines the (strong, weak)(or(weak, strong) respectively)* (σ_1, σ_2)*-convex functions on coordinates;*
3. $\sigma_1 < 0$ *and* $\sigma_2 < 0$ *defines the (weak, weak)* (σ_1, σ_2)*-convex functions on coordinates;*
4. *(*$\sigma_1 > 0$ *and* $\sigma_2 = 0$*) (or (*$\sigma_1 = 0$ *and* $\sigma_2 > 0$*)) defines the (strong, classic)(or(classic, strong) respectively)* (σ_1, σ_2)*-convex functions on coordinates;*
5. *(*$\sigma_1 < 0$ *and* $\sigma_2 = 0$*) (or (*$\sigma_1 = 0$ *and* $\sigma_2 < 0$*)) defines the (weak, classic)(or(classic, weak) respectively)* (σ_1, σ_2)*-convex functions on coordinates;*
6. $\sigma_1 = 0$ *and* $\sigma_2 = 0$ *defines the convex functions on coordinates as in [5].*

Proposition 2.5. *Suppose that* $\sigma_1 \in \mathbb{R}$ *and* $\sigma_2 \in \mathbb{R}$*. Let us consider two bounded functions* $u : [a, b] \to \mathbb{R}$ *and* $v : [c, d] \to \mathbb{R}$ *such as u is* (σ_1, d)*-disturbed convex and v is nonnegative and* (σ_2, d)*-disturbed convex. Then there are two real numbers* σ_3 *and* σ_4 *such that the function* $f : [a, b] \times [c, d] \to \mathbb{R}$ *defined by* $f(x, y) = u(x)v(y)$ *is co-ordinated distance-disturbed* (σ_3, σ_4)*-convex on* $D = [a, b] \times [c, d]$*.*

Proof. Denote $m_u = \inf\{u(x)|x \in [a, b]\}$, $m_v = \inf\{v(y)|y \in [c, d]\}$, $M_u = \sup\{u(x)|x \in [a, b]\}$ and $M_v = \sup\{v(y)|y \in [c, d]\}$. Since u is (σ_1, d)-disturbed convex and v is nonnegative then for $x_1, x_2 \in [a, b]$, $y \in [c, d]$ and $t \in [0, 1]$ one has

$$u(tx_1 + (1-t)x_2)v(y) + \sigma_1 t(1-t)(x_1 - x_2)^2 v(y) \le tu(x_1)v(y) + (1-t)u(x_2)v(y).$$

Let $y_1, y_2 \in [c, d]$, $\tau \in [0, 1]$ and take $y = \tau y_1 + (1-\tau)y_2$. Then, due to the (σ_2, d)-disturbed convexity of function v one has

$$\begin{aligned}
&u(tx_1 + (1-t)x_2)v(\tau y_1 + (1-\tau)y_2) + \sigma_1 t(1-t)(x_1 - x_2)^2 v(y) \\
&\le tu(x_1)[\tau v(y_1) + (1-\tau)v(y_2) - \sigma_2\tau(1-\tau)(y_1 - y_2)^2] \\
&\quad + (1-t)v(y_2)[\tau v(y_1) + (1-\tau)v(y_2) - \sigma_2\tau(1-\tau)(y_1 - y_2)^2] \\
&= t\tau u(x_1)v(y_1) + t(1-\tau)u(x_1)v(y_2) \\
&\quad + \tau(1-t)u(x_2)v(y_1) + (1-t)(1-\tau)u(x_2)v(y_2) - \sigma_2\tau(1-\tau)[tu(x_1) + (1-t)u(x_2)](y_1 - y_2)^2.
\end{aligned}$$

Now we analyze each case, according to Remark 2.4 as follows:

1. Suppose that $\sigma_1 > 0$ and $\sigma_2 > 0$. Than taking $\sigma_3 = \sigma_1 m_v$ and $\sigma_4 = \sigma_2 m_u$ one gets that function $f = uv$ is co-ordinated distance-disturbed (σ_3, σ_4)-convex on D.
2. Suppose that $\sigma_1 > 0$ and $\sigma_2 < 0$. Than taking $\sigma_3 = \sigma_1 m_v$ and $\sigma_4 = \sigma_2 M_u$ one gets that function $f = uv$ is co-ordinated distance-disturbed (σ_3, σ_4)-convex on D.
3. Suppose that $\sigma_1 < 0$ and $\sigma_2 < 0$. Than taking $\sigma_3 = \sigma_1 M_v$ and $\sigma_4 = \sigma_2 M_u$ one gets that function $f = uv$ is co-ordinated distance-disturbed (σ_3, σ_4)-convex on D.
4. In all the other existing cases one proceeds in the same manner and get the required result.

□

Remark 2.6. *The previous property stays valid if the hypothesis changes by requiring that function* $u : [a, b] \to \mathbb{R}$ *is a non-negative, bounded and* (σ_1, d)*-disturbed convex and function v is bounded and* (σ_2, d)*-disturbed convex.*

Theorem 2.7. *If function* $f : A \to \mathbb{R}$ *is co-ordinated distance-disturbed* (σ_1, σ_2)*-convex on A and* $f \in L_1(A)$ *then*

$$\begin{aligned}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{1}{12}\left[\sigma_1(b-a)^2 + \sigma_2(d-c)^2\right] \\
&\le \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y)dxdy \\
&\le \frac{1}{4}\left[f(a, c) + f(a, d) + f(b, c) + f(b, d)\right] - \frac{1}{6}\left[\sigma_1(b-a)^2 + \sigma_2(d-c)^2\right], \qquad (7)
\end{aligned}$$

whenever $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$ *and* $D = [a, b] \times [c, d] \subseteq A$*.*

Proof. Let us remark that

$$f(x,y) = f_x(y) = f_y(x), \forall x, y \in A. \tag{8}$$

To prove the lower bound, one start by using Proposition 2.2, which states that the partial function f_y is distance-disturbed convex with modulus σ_1 under the hypothesis of the theorem. It means that, for every $y \in [c,d]$, the following inequality

$$f_y\left(\frac{x_1+x_2}{2}\right) + \frac{\sigma_1}{4}(x_1-x_2)^2 \le \frac{f_y(x_1)+f_y(x_2)}{2}$$

holds whenever $x_1, x_2 \in [a,b]$. Consider $x_1 = ta + (1-t)b$ and $x_2 = (1-t)a + tb$, implying that $x_1 + x_2 = a + b$, $x_1 - x_2 = (1-2t)(b-a)$ and

$$f_y\left(\frac{a+b}{2}\right) + \frac{\sigma_1(b-a)^2}{4}(1-2t)^2 \le \frac{f_y(ta+(1-t)b) + f_y((1-t)a+tb)}{2}. \tag{9}$$

Integrate this inequality with respect to t over $[0,1]$ and get

$$f_y\left(\frac{a+b}{2}\right) + \frac{\sigma_1(b-a)^2}{12} \le \frac{1}{b-a}\int_a^b f_y(x)dx. \tag{10}$$

In a similar manner one deduces

$$f_x\left(\frac{c+d}{2}\right) + \frac{\sigma_2(d-c)^2}{12} \le \frac{1}{d-c}\int_c^d f_x(y)dy.$$

Taking $x = \frac{a+b}{2}$ and using (8) one obtains

$$f_{\frac{a+b}{2}}\left(\frac{c+d}{2}\right) + \frac{\sigma_2(d-c)^2}{12} \le \frac{1}{d-c}\int_c^d f_{\frac{a+b}{2}}(y)dy$$

$$= \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy = \frac{1}{d-c}\int_c^d f_y\left(\frac{a+b}{2}\right)dy.$$

Introducing the evaluation from (10) under the right side integral one gets

$$\begin{aligned}
&f_{\frac{a+b}{2}}\left(\frac{c+d}{2}\right) + \frac{\sigma_2(d-c)^2}{12} \le \frac{1}{d-c}\int_c^d f_y\left(\frac{a+b}{2}\right)dy \\
&\le \frac{1}{d-c}\int_c^d \left[\frac{1}{b-a}\int_a^b f(x,y)dx - \frac{\sigma_1(b-a)^2}{12}\right]dy \\
&= \frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy - \frac{\sigma_1(b-a)^2}{12}.
\end{aligned}$$

After arranging the last inequality the proof of the lower bound of (7) is complete.
To prove the upper bound of (7), one starts by the distance-disturbed convexity of f_y and write the following inequalities:

$$f_y(ta+(1-t)b) + \sigma_1 t(1-t)(b-a)^2 \le tf_y(a) + (1-t)f_y(b),$$

$$f_y((1-t)a+tb) + \sigma_1 t(1-t)(b-a)^2 \le (1-t)f_y(a) + tf_y(b).$$

The sum of these two inequalities is:

$$f_y(ta+(1-t)b) + f_y((1-t)a+tb) + 2\sigma_1 t(1-t)(b-a)^2 \le f_y(a) + f_y(b). \tag{11}$$

Integrating both sides of this inequality with respect to t over $[0,1]$ and changing the variables as usual one gets

$$\frac{1}{b-a}\int_a^b f_y(x)dx+\frac{\sigma_1(b-a)^2}{6}\le\frac{f_y(a)+f_y(b)}{2}. \tag{12}$$

In a similar manner one obtains

$$\frac{1}{d-c}\int_c^d f_x(y)dy+\frac{\sigma_2(d-c)^2}{6}\le\frac{f_x(c)+f_x(d)}{2}. \tag{13}$$

Integrate (12) with respect to y over $[c,d]$:

$$\frac{1}{b-a}\int_c^d\int_a^b f_y(x)dxdy+\frac{\sigma_1(b-a)^2(d-c)}{6}\le\frac{1}{2}\int_c^d\left[f_y(a)+f_y(b)\right]dy.$$

By (8) this inequality becomes

$$\frac{1}{b-a}\int_c^d\int_a^b f(x,y)dxdy+\frac{\sigma_1(b-a)^2(d-c)}{6}\le\frac{1}{2}\int_c^d\left[f_y(a)+f_y(b)\right]dy. \tag{14}$$

Now, from (13), by (8), we can write

$$\int_c^d f_a(y)dy\le(d-c)\frac{f_a(c)+f_a(d)}{2}-\frac{\sigma_2(d-c)^3}{6},$$

$$\int_c^d f_b(y)dy\le(d-c)\frac{f_b(c)+f_b(d)}{2}-\frac{\sigma_2(d-c)^3}{6}.$$

Now we use these inequalities, introducing them into the right side of (14) to increase it. Then we use (8) once again, writing

$$\begin{aligned}&\frac{1}{b-a}\int_c^d\int_a^b f(x,y)dxdy+\frac{\sigma_1(b-a)^2(d-c)}{6}\\&\le\frac{d-c}{2}\left[\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{2}-\frac{2\sigma_2(d-c)^2}{6}\right]dy.\end{aligned}$$

Dividing this inequality by $(d-c)$ and conveniently arranging it one completes the proof of the theorem. □

Remark 2.8. *The inequality (7) is sharp. To prove that one can consider the following example. Take* $\sigma_1=\sigma_2=0$, $[a,b]=[c,d]=[0,1]$ *so* $D=[0,1]^2$ *and consider* $f:D\to\mathbb{R}$ *defined by* $f(x,y)=xy$. *In this case* f *is co-ordinated distance-disturbed* $(0,0)$ *convex and continuous on* D. *In this case (7) becomes* $\frac{1}{4}\le\frac{1}{4}\le\frac{1}{4}$, *which proves the sharpness of inequality (7).*

Remark 2.9. *If we take* $\sigma_1=\sigma_2=0$ *in (7) then inequality (7) becomes*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4},$$

which is Theorem 1 from [5]. The previous remark shows that this inequality is sharp too.

3. Quantum integral inequalities for co-ordinated distance-disturbed convex functions

We now recall some preliminary details of quantum calculus on finite intervals. Let $J = [a, b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. The q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a, b]$ is defined as follows.

Definition 3.1 ([22, 23]). *Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f on J at x is defined as*

$$ {}_a\mathrm{D}_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{15}$$

It is obvious that ${}_a\mathrm{D}_q f(a) = \lim\limits_{x\to a} {}_a\mathrm{D}_q f(x)$.
A function f is q-differentiable on J if ${}_aD_q f(x)$ exists for all $x \in J$. Also if $a = 0$, then ${}_0\mathrm{D}_q f = \mathrm{D}_q f$, where D_q is the q-derivative of the function f [7, 17] defined as

$$\mathrm{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Definition 3.2 ([22, 23]). *Let $f : J \to \mathbb{R}$ be a continuous function. A second-order q-derivative on J, which is denoted as ${}_a\mathrm{D}_q^2 f$, provided ${}_a\mathrm{D}_q f$ is q-differentiable on J is defined as ${}_a\mathrm{D}_q^2 f = {}_a\mathrm{D}_q({}_a\mathrm{D}_q f) : J \to \mathbb{R}$. Similarly, higher order q-derivative on J is defined by ${}_a\mathrm{D}_q^n f =: J \to \mathbb{R}$.*

The q-integral is defined as follows:

Definition 3.3 ([22, 23]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on I is defined as*

$$\int_a^x f(t)_a\mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{16}$$

for $x \in J$.

This concept can be viewed as the Riemann type of q-integral (or q-antiderivative [17]). If $a = 0$ in (1.3), then we have the classical q-integral, that is

$$\int_0^x f(t)_0\mathrm{d}_q t = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in [0, \infty).$$

The quantum calculus is didactically presented in [7] and [17]. Useful quantum differentiation and integration formulas are derived in [22] and [23]. The following properties are necessary tools in the proofs of the main results in this section.

Theorem 3.4 ([22, 23]). *Let $f : I \to \mathbb{R}$ be a continuous function, then*

1. ${}_a\mathrm{D}_q \int_a^x f(t)\, {}_a\mathrm{d}_q t = f(x)$
2. $\int_c^x {}_a\mathrm{D}_q f(t)_a\mathrm{d}_q t = f(x) - f(c)$ *for $x \in (c, x)$.*

Lemma 3.5. *If $j \in \mathbb{N}$, $0 < q < 1$, $a, b \in \mathbb{R}$, $a < b$ then*

$$\int_a^b x^j\, {}_a\mathrm{d}_q x = (1-q)(b-a)\sum_{k=0}^{j} \frac{1}{1-q^{j-k+1}} \mathrm{C}_j^k (b-a)^{j-k} a^k, \tag{17}$$

where by C_j^k we denote the combinations of k elements taken out of a set of j elements.

Proof. We use Newton's binomial formula

$$(u+v)^j = \sum_{k=0}^{j} C_j^k u^{j-k} v^k,$$

to compute the required q-integral as follows:

$$\begin{aligned}\int_a^b x^j \, {}_a\mathrm{d}_q x &= (1-q)(b-a)\sum_{n=0}^{\infty} q^n \left[q^n b + (1-q^n)a\right]^j \\ &= (1-q)(b-a)\sum_{n=0}^{\infty} q^n \left[q^n (b-a) + a\right]^j \\ &= (1-q)(b-a)\sum_{n=0}^{\infty} q^n \left[\sum_{k=0}^{j} C_j^k q^{n(j-k)} (b-a)^{j-k} a^k\right],\end{aligned}$$

which is convergent to (17). □

Corollary 3.6. *If* $0 < q < 1$, $a, b \in \mathbb{R}$, $a < b$ *then*

$$\begin{aligned}&\int_a^b {}_a\mathrm{d}_q x = b-a, \\ &\int_a^b x \, {}_a\mathrm{d}_q x = (1-q)(b-a)\left(\frac{b-a}{1-q^2} + \frac{a}{1-q}\right), \\ &\int_a^b x^2 \, {}_a\mathrm{d}_q x = (1-q)(b-a)\left(\frac{(b-a)^2}{1-q^3} + \frac{2a(b-a)}{1-q^2} + \frac{a^2}{1-q}\right).\end{aligned}$$

Suppose now that $0 < q < 1$.

Theorem 3.7. *If function* $f : A \to \mathbb{R}$ *is co-ordinated distance-disturbed* (σ_1, σ_2)*-convex on* A*, if* f *is* q*-integrable both with respect to* x *and with respect to* y *then*

$$\begin{aligned}&f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{Q(q)}{4}\left[\sigma_1(b-a)^2 + \sigma_2(d-c)^2\right] \\ &\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y) \, {}_a\mathrm{d}_q x \, {}_c\mathrm{d}_q y \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} - \frac{1-Q(q)}{4}\left\{\sigma_1(b-a)^2 + \sigma_2(d-c)^2\right\}, \qquad (18)\end{aligned}$$

whenever $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$ *and* $D = [a,b] \times [c,d] \subseteq A$*. Here*

$$Q(q) = 1 - \frac{4}{1+q} + \frac{4}{1+q+q^2}, \qquad (19)$$

for $q \in (0,1)$.

Proof. In order to prove the lower bound we start as in the proof of Theorem 2.7 and q-integrate inequality (9) over $[0,1]$ with respect to t, getting

$$f_y\left(\frac{a+b}{2}\right) + \frac{\sigma_1(b-a)^2}{4}\left(1 - \frac{4}{1+q} + \frac{4}{1+q+q^2}\right) \leq \frac{1}{b-a}\int_a^b f_y(x) \, {}_a\mathrm{d}_q x. \qquad (20)$$

In a similar manner one deduces

$$f_x\left(\frac{c+d}{2}\right)+\frac{\sigma_2(d-c)^2}{4}\left(1-\frac{4}{1+q}+\frac{4}{1+q+q^2}\right)\le\frac{1}{d-c}\int_c^d f_x(y)\,{}_c\mathrm{d}_q y.$$

As above, taking $x=\frac{a+b}{2}$ and using (8) one gets

$$\begin{aligned}&f_{\frac{a+b}{2}}\left(\frac{c+d}{2}\right)+\frac{\sigma_2(d-c)^2}{4}Q(q)\\&\le\frac{1}{d-c}\int_c^d f_{\frac{a+b}{2}}(y)\,{}_c\mathrm{d}_q y\\&=\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\,{}_c\mathrm{d}_q y=\frac{1}{d-c}\int_c^d f_y\left(\frac{a+b}{2}\right)\,{}_c\mathrm{d}_q y.\end{aligned}$$

Introduce the evaluation from (20) under the right side integral and compute as follows

$$\begin{aligned}&f_{\frac{a+b}{2}}\left(\frac{c+d}{2}\right)+\frac{\sigma_2(d-c)^2}{4}Q(q)\\&\le\frac{1}{d-c}\int_c^d\left[\frac{1}{b-a}\int_a^b f(x,y)\,{}_a\mathrm{d}_q x-\frac{\sigma_1(b-a)^2}{4}Q(q)\right]{}_c\mathrm{d}_q y\\&=\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,{}_a\mathrm{d}_q x\,{}_c\mathrm{d}_q y-\frac{\sigma_1(b-a)^2}{4}Q(q).\end{aligned}$$

After arranging the last inequality and using (8), the proof of the lower bound of (18) is complete.
To prove the upper bound of (18), we start again as in the proof of Theorem 2.7 and q-integrate inequality (9) over $[0,1]$ with respect to t. After the common change of variable by using (19) one deduces

$$\frac{2}{b-a}\int_a^b f_y(x)\,{}_a\mathrm{d}_q x+2(b-a)^2\sigma_1\int_0^1(t-t^2)\,{}_0\mathrm{d}_q t\le f_y(a)+f_y(b),$$

$$\frac{1}{b-a}\int_a^b f_y(x)\,{}_a\mathrm{d}_q x+\sigma_1(b-a)^2\frac{1-Q(q)}{4}\le\frac{f_y(a)+f_y(b)}{2}. \tag{21}$$

In a similar manner one obtains

$$\frac{1}{d-c}\int_c^d f_x(y)\,{}_c\mathrm{d}_q y+\sigma_2(d-c)^2\frac{1-Q(q)}{4}\le\frac{f_x(c)+f_x(d)}{2}. \tag{22}$$

Again, q-integrate (21) with respect to y over $[c,d]$, getting

$$\begin{aligned}&\frac{1}{b-a}\int_c^d\int_a^b f(x,y)\,{}_a\mathrm{d}_q x\,{}_c\mathrm{d}_q y+\sigma_1(b-a)^2(d-c)\frac{1-Q(q)}{4}\\&\le\frac{1}{2}\int_c^d\left[f_y(a)+f_y(b)\right]{}_c\mathrm{d}_q y.\end{aligned} \tag{23}$$

Due to (22) and (8), we can write

$$\int_c^d f_y(a)\,{}_c\mathrm{d}_q y=\int_c^d f_a(y)\,{}_c\mathrm{d}_q y\le(d-c)\frac{f_a(c)+f_a(d)}{2}-\frac{\sigma_2(d-c)^3[1-Q(q)]}{4},$$

$$\int_c^d f_y(b)\,{}_c\mathrm{d}_q y=\int_c^d f_b(y)\,{}_c\mathrm{d}_q y\le(d-c)\frac{f_b(c)+f_b(d)}{2}-\frac{\sigma_2(d-c)^3[1-Q(q)]}{4}.$$

Using these two estimates in (23) and dividing by $d-c$ we complete the proof of the right side of (18). □

If we take $\sigma_1 = \sigma_2 = 0$ in (18), one deduces the following new quantum integral inequality for classically co-ordinated convex functions.

Theorem 3.8. *If function $f : A \to \mathbb{R}$ is co-ordinated convex on A, if f is q-integrable both with respect to x and with respect to y then*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)\, {}_a\mathrm{d}_q x\, {}_c\mathrm{d}_q y$$
$$\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \tag{24}$$

whenever $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$ and $D = [a,b] \times [c,d] \subseteq A$.

Remark 3.9. *If $q \to 1$ and $f \in L_1(A)$ then (18) becomes (7) and (24) becomes Theorem 1 from [5].*

Remark 3.10. *The previous remark, together with the sharpness results from Section 2 of this paper, shows that both inequality (18) and (24) are sharp as $q \to 1$.*

4. Refined bounds of quadratures and q-quadratures within the class of the distance-disturbed convex functions on co-ordinates

In this section we refine the inequalities (7) and (18) by splitting the domain $D = [a,b] \times [c,d]$ in finite elements and evaluating the integrals on the domain by using their partial boundaries on each finite element.

Theorem 4.1. *If function $f : A \to \mathbb{R}$ is co-ordinated distance-disturbed (σ_1, σ_2)-convex on A and $f \in L_1(A)$ then*

$$\sum_{k=1}^{n}\sum_{l=1}^{m} f\left(\frac{(2n-2k-1)a+(2k+1)b}{2n}, \frac{(2m-2l-1)c+(2l+1)d}{2m}\right) + \frac{mn}{12}\left[\sigma_1\left(\frac{b-a}{n}\right)^2 + \sigma_2\left(\frac{d-c}{m}\right)^2\right]$$
$$\leq \frac{mn}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)dxdy$$
$$\leq \frac{1}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] + \frac{1}{2}\sum_{l=1}^{m-1}\left[f\left(a, \frac{(m-l)c+ld}{m}\right) + f\left(b, \frac{(m-l)c+ld}{m}\right)\right]$$
$$+ \frac{1}{2}\sum_{k=1}^{n-1}\left[f\left(\frac{(n-k)a+kb}{n}, c\right) + f\left(\frac{(n-k)a+kb}{n}, d\right)\right] + \sum_{k=1}^{n-1}\sum_{l=1}^{m-1} f\left(\frac{(n-k)a+kb}{n}, \frac{(m-l)c+ld}{m}\right)$$
$$- \frac{mn}{6}\left[\sigma_1\left(\frac{b-a}{n}\right)^2 + \sigma_2\left(\frac{d-c}{m}\right)^2\right], \tag{25}$$

whenever $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$ and $D = [a,b] \times [c,d] \subseteq A$.

Proof. Let us split the rectangle D into rectangular finite elements as follows:

$$D = [a,b] \times [c,d] = \bigcup_{k=0}^{n-1}\bigcup_{l=0}^{m-1}\left[a + \frac{k(b-a)}{n}, a + \frac{(k+1)(b-a)}{n}\right]\left[c + \frac{l(d-c)}{m}, c + \frac{(l+1)(d-c)}{m}\right]. \tag{26}$$

Consider each rectangle of (26) and write the corresponding inequality (7). For example, let us chose the rectangle

$$\left[\frac{(n-k)a+kb}{n}, \frac{(n-k-1)a+(k+1)b}{n}\right] \times \left[\frac{(m-l)c+ld}{m}, \frac{(n-l-1)c+(l+1)d}{m}\right].$$

Inequality (7) becomes on this rectangle:

$$f\left(\frac{(2n-2k-1)a+(2k+1)b}{2n}, \frac{(2m-2l-1)c+(2l+1)d}{2m}\right)+\frac{1}{12}\left[\sigma_1\left(\frac{b-a}{n}\right)^2+\sigma_2\left(\frac{d-c}{m}\right)^2\right]$$
$$\leq \frac{mn}{(b-a)(d-c)}\int_{\frac{(m-l)c+ld}{m}}^{\frac{(n-l-1)c+(l+1)d}{m}}\int_{\frac{(n-k)a+kb}{n}}^{\frac{(n-k-1)a+(k+1)b}{n}} f(x,y)dxdy$$
$$\leq \frac{1}{4}f\left(\frac{(n-k)a+kb}{n}, \frac{(m-l)c+ld}{m}\right)+\frac{1}{4}f\left(\frac{(n-k)a+kb}{n}, \frac{(n-l-1)c+(l+1)d}{m}\right)$$
$$+\frac{1}{4}f\left(\frac{(n-k-1)a+(k+1)b}{n}, \frac{(m-l)c+ld}{m}\right)+\frac{1}{4}f\left(\frac{(n-k-1)a+(k+1)b}{n}, \frac{(n-l-1)c+(l+1)d}{m}\right)$$
$$-\frac{1}{6}\left[\sigma_1\left(\frac{b-a}{n}\right)^2+\sigma_2\left(\frac{d-c}{m}\right)^2\right]. \tag{27}$$

Computing the sum over $k \in \{0, 1, ..., n-1\}$ and $l \in \{0, 1, ..., m-1\}$ of all the inequalities, side by side, one gets (25). □

Theorem 4.2. *If function $f : A \to \mathbb{R}$ is co-ordinated distance-disturbed (σ_1, σ_2)-convex on A, if f is q-integrable both with respect to x and with respect to y then*

$$\sum_{k=1}^{n}\sum_{l=1}^{m} f\left(\frac{(2n-2k-1)a+(2k+1)b}{2n}, \frac{(2m-2l-1)c+(2l+1)d}{2m}\right)+\frac{mnQ(q)}{4}\left[\sigma_1\left(\frac{b-a}{n}\right)^2+\sigma_2\left(\frac{d-c}{m}\right)^2\right]$$
$$\leq \frac{mn}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,{}_a\mathrm{d}_q x\,{}_c\mathrm{d}_q y$$
$$\leq \frac{1}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]+\frac{1}{2}\sum_{l=1}^{m-1}\left[f\left(a, \frac{(m-l)c+ld}{m}\right)+f\left(b, \frac{(m-l)c+ld}{m}\right)\right]$$
$$+\frac{1}{2}\sum_{k=1}^{n-1}\left[f\left(\frac{(n-k)a+kb}{n}, c\right)+f\left(\frac{(n-k)a+kb}{n}, d\right)\right]+\sum_{k=1}^{n-1}\sum_{l=1}^{m-1} f\left(\frac{(n-k)a+kb}{n}, \frac{(m-l)c+ld}{m}\right)$$
$$-\frac{mn(1-Q(q))}{6}\left[\sigma_1\left(\frac{b-a}{n}\right)^2+\sigma_2\left(\frac{d-c}{m}\right)^2\right], \tag{28}$$

whenever $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$ and $D = [a, b] \times [c, d] \subseteq A$. Here $Q(q)$ defined by (19) for $q \in (0, 1)$.

Proof. One proceeds as in the proof of Theorem 4.1, by splitting the rectangle D as in (26) and applying (18) on each finite element. The required inequality (28) is obtained by computing the sum of all the inequalities on finite elements over $k \in \{0, 1, ..., n-1\}$ and $l \in \{0, 1, ..., m-1\}$. □

Remark 4.3. *Inequality (28) becomes (25) as $q \to 1$.*

5. Conclusion

The concept of distance-disturbed convexity is an unifying one, including both strengthened and weakened types of generalized convexity that exist in nowadays mathematical literature. The inequalities derived in this paper are useful tools whenever quantum integral approximations involving functions having some type of distance-disturbed convexity are discussed (see, for example the context of [23]). The technique developed in this paper is useful in deriving quantum integral estimates in case of other types of generalized convexity, which may be approached on co-ordinates [20]. The strong case of generalized convexities, as in [1], may be approached in two dimensions in the same manner in quantum calculus. The paper provides mathematical tools for quantum integral estimates involving two dimensions weakened convexity properties, as [3], [8], [18], [19]. The representation of the domain by finite elements in order to refine the computation is a fruitful technique, opening new directions of research in quantum mathematics. The quantum computation and the quantum inequalities in the plane is a fertile tool, with impact on the development of quantum physics, information processing techniques and all the domains of science and life in which these new fields of research apply.

Literatur

[1] H. Angulo, J. Giménez, A.M. Moros, K. Nikodem, On strongly h-convex functions, Ann. Funct. Anal. 2(2)(2011), 85–91.
[2] A. Azócar, J. Giménez, K. Nikodem, J.L. Sánchez, On strongly midconvex functions, Opuscula Math. 31(1) (2011), 15–26.
[3] P. Burai, A. Házy, T. Juhász, On approximately Breckner s-convex functions, Control and Cybernetics 40(1)(2011), 91–99.
[4] G. Cristescu, L. Lupşa, Non-connected Convexities and Applications. Kluwer Academic Publishers, Dordrecht, Holland, 2002.
[5] S.S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese J. Math. 5(2001), 775–788.
[6] S. S. Dragomir, C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications. Victoria University, Australia (2000).
[7] T. Ernst, A Comprehensive Treatment of q-Calculus, Springer Basel Heidelberg New York Dordrecht London, 2012.
[8] R. Ger, Almost approximately convex functions, Math. Slovaca 38(1)(1988), 61–78.
[9] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math. Pures Appl., 58(1893), 171–215.
[10] A. Házy, Zs. Páles, Approximately midconvex functions, Bulletin London Math. Soc. 36 (2004), 339-350.
[11] A. Házy, Zs. Páles, On approximately t-convex functions, Publ. Math. Debrecen 66(3-4) (2005), 489-501, Dedicated to the 75th birthday of Professor Heinz König.
[12] A. Házy, On approximately t-convexity, Math. Ineq. and Appl. 8 (3) (2005), 389-402.
[13] A. Házy, On the stability of t-convex functions, Aequationes Math. 74 (2007), 210-218.
[14] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis 3(1883), 82.
[15] F. H. Jackson, On a q-definite integrals, Quarterly J. Pure Appl. Math. 41, 193-203 (1910).
[16] J.L.W.V. Jensen, Sur les fonctions convexes et les inégalités entre les valeurs moyennes, Acta. Math. 30(1906), 175–193.
[17] V. Kak, P. Cheung, Quantum Calculus, Springer Verlag, 2002.
[18] M. Kuczma, Almost convex functions, Colloquium Math. 21 (1970), 279–284.
[19] C. T. Ng, K. Nikodem, On approximately convex functions, Proc. Amer. Math. Soc. 118(1)(1993), 103–108.
[20] M.A. Noor, K.I. Noor, M.U. Awan, Some quantum integral inequalities via preinvex functions, Appl. Math. Comput. 269(2015), 242–251.
[21] B.T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Soviet. Math. Dokl. 7(1966), 72–75.
[22] J. Tariboon, S.K. Ntouyas, Quantum integral inequalities on finite intervals, J. Inequal. App. (2014), 2014:121.
[23] J. Tariboon, S.K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, Adv. Differ. Equ. (2013), 2013:282.