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L-Classical d-Orthogonal Polynomial Sets of Sheffer Type

Youssèf Ben Cheikh[a], **Inès Gam**[b]

[a]*Faculté des Sciences de Monastir, Département de Mathématiques, 5019 Monastir, Tunisia*
[b]*Institut préparatoire aux études d'ingénieur de Monastir, 5019 Monastir, Tunisia*

Abstract. In this paper, we characterize L-classical d-orthogonal polynomial sets of Sheffer type where L being a lowering operator commutating with the derivative operator D and belonging to $\{D,\ e^D-1,\ sin(D)\}$. For the first case we state a $(d+1)$-order differential equation satisfied by the corresponding polynomials. We, also, show that, with these three lowering operators, all the orthogonal polynomial sets are classified as L-classical orthogonal polynomial sets.

1. Introduction

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. A polynomial sequence $\{P_n\}_{n\geq 0}$ is called a polynomial set (PS for short) if and only if $\deg P_n = n$ for all non-negative integer n. We denote by $< u, f >$ the effect of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Denote by $S(\mathcal{P})$ the set of polynomial sets $P = \{P_n\}_{n\geq 0}$, where $P_n \in \mathcal{P}$.

Definition 1.1. *[20, 24] Let $\{P_n\}_{n\geq 0}$ be in $S(\mathcal{P})$ and let d be an arbitrary positive integer. The polynomial sequence $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial set (d-OPS, for short) with respect to a d-dimensional functional $\mathcal{U} = {}^t(u_0, \cdots, u_{d-1})$ if it satisfies the following conditions:*

$$\begin{cases} \langle u_k, P_m P_n\rangle = 0, & m > dn + k \\ \langle u_k, P_n P_{dn+k}\rangle \neq 0, & n \geq 0 \end{cases}$$

for each integer k belonging to $\{0, 1, \ldots, d-1\}$.

For $d = 1$, we recover the well-known notion of orthogonality.

One of the important classes of PSs is the class of Sheffer A-type zero (which we shall hereafter call Sheffer type and note $\mathcal{SH}$).[25]

Definition 1.2. *A PS $P = \{P_n\}_{n\geq 0}$ is called of Sheffer type if it is generated by a function of the form*

$$G(x,t) = A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n, \qquad (1)$$

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Email addresses: youssef.bencheikh@planet.tn (Youssèf Ben Cheikh), gam_ines@yahoo.fr (Inès Gam)

where

$$A(t) = \sum_{n\geq 0} a_n t^n \quad \textit{and} \quad H(t) = \sum_{n\geq 1} h_n t^n$$

with

$$A(0) \neq 0,\ H(0) = 0 \textit{ and } H'(0) \neq 0,$$

we will denote a such polynomial set by $P(A, H)$.

Put $L_H(A) = \dfrac{A'}{AH'}$ the formal power series defined in terms of the logarithm derivation of A and the derivation of H.

An orthogonal polynomial set (OPS, for short) $\{P_n\}_{n\geq 0}$ in $\mathcal{P}$ is called $L_{q,w}$-classical if $\{L_{q,w}P_n\}_{n\geq 1}$ is also orthogonal, where $L_{q,w}$ denotes the Hahn operator given by [19]

$$L_{q,w}(f)(x) := \frac{f(qx + w) - f(x)}{(q - 1)x + w},\ (q \neq 0).$$

Particular interest is devoted to the derivative operator D ($w = 0$ and $q \to 1$), the finite difference operator Δ ($w = 1$ and $q = 1$), q-difference operator L_q ($w = 0$) and Dunkl operator $T_\mu = D + 2\mu L_{-1}$, $\mu > -1/2$. The literature on these topics is extremely vast. For a survey see for instance [1, 5].
This notion has been extended to the d-orthogonality by Douak and Maroni [16], who introduced the notion of classical d-OPSs which means that both $\{P_n\}_{n\geq 0}$ and its derivative $\{P'_{n+1}\}_{n\geq 0}$ are d-orthogonal. It is then significant to look for characteristic properties for $L_{q,w}$-classical d-OPSs as was done for the case $d = 1$. In this context, for the derivative operator D, Douak and Maroni [17] generalized the Pearson's equation for classical d-OPSs. The Sturm-Liouville equation is generalized for particular families of classical d-OPSs, some examples may be found in [2–4, 14, 15, 18, 21, 26]. Ben Cheikh and Ben Romdhane [2] gave some characteristic properties of the d-symmetric classical d-OPSs. Douak and Maroni [16], and later Boukhemis and Zerouki [14] quote some families of classical d-OPSs in the particular case $d = 2$. For the operator Δ, some examples of classical discrete d-OPSs of Sheffer type may be found in [8, 10, 11]. Some examples of L_q-classical d-OPSs are stated in [9, 22, 27]. Finally for the operator T_μ, Ben Cheikh and Gaied [6] studied the Dunkl-classical d-OPSs in the d-symmetric case.

Our contribution in this direction is to determine all classical $d-$OPSs of Sheffer type (Theorem 3.1), as well as $(d + 1)-$order differential equations satisfied by these polynomials. We also state two new characterizations of classical discrete d-OPSs of Sheffer type. We consider the operator $sin(D)$, to complete the classification of the OPSs of Sheffer type as L-classical polynomials, and we characterize all $sin(D)$-classical d-OPSs of Sheffer type. The cases $d = 2$ and $d = 3$ are specially carried out.

2. Main result

In this section, we state a general result that will have as applications the results of the next sections. To this end, we need to recall the following lemmas.

Lemma 2.1. *[7] Let* $P(A, H) = \{P_n\}_{n\geq 0}$ *be a Sheffer-type polynomial set.*
$\{P_n\}_{n\geq 0}$ *is a d-OPS if and only if*

$$\begin{cases} \dfrac{1}{H'(t)} \text{ is a polynomial of degree } \leq (d+1) \\ L_H(A) \text{ is a polynomial of degree } d. \end{cases}$$

Lemma 2.2. *[7] Let* $P(A, H)$ *be a d-OPS of Sheffer type. The polynomial set* $\mathcal{K}P = P(KA, H)$ *is a* d'*-OPS* ($d' > d$) *iff* $L_H(K)$ *is a polynomial of degree* d'.
$\mathcal{K}P$ *remains a d-OPS iff* $L_H(K)$ *is a polynomial of degree d having a leading coefficient different from that of* $-L_H(A)$, *or a polynomial of degree* $< d$.

Lemma 2.3. *[7] Let $P = P(A, H)$ be a d-OPS of Sheffer type, $\mathcal{K}P = P(KA, H)$ be a d'-OPS, $d' > d$, of Sheffer type and L be a lowering operator which commutates with the derivation operator D. If P is L-classical d-OPS then $\mathcal{K}P$ is L-classical d'-OPS.*

Theorem 2.1. *Let ψ be a formal power series satisfying*

$$\begin{cases} \left(\dfrac{\psi'}{\psi}\right)' = -F\left(\dfrac{\psi'}{\psi}\right), \\ \psi(0) = 0, \psi'(0) = 1, \end{cases}$$

where F is a monic polynomial of degree 2. Let $P(A, H)$ be a d-OPS of Sheffer type. $P(A, H)$ is $\psi(D)-$classical iff

$$H(t) = \left(\frac{\psi'}{\psi}\right)^{-1} \circ \frac{\pi(t)}{t},$$

where π is a polynomial of degree $\leq d + 1$ satisfying $\pi(0) \neq 0$ and $\pi - t\pi'$ divides $t^2F(\frac{\pi(t)}{t})$.

Proof. Let $P(A, H) = \{P_n\}_{n\geq 0}$ be a d-OPS of Sheffer type. $\left\{\dfrac{\psi(D)P_{n+1}}{n+1}\right\}_{n\geq 0}$ is generated by

$$\sum_{n=0}^{\infty} \frac{\psi(D)P_{n+1}(x)}{n+1}\frac{t^n}{n!} = \frac{1}{t}\psi(D)\left(\sum_{n=0}^{\infty} P_{n+1}(x)\frac{t^{n+1}}{(n+1)!}\right) = \frac{1}{t}\psi(D)(A(t)e^{xH(t)}) = \frac{\psi(H(t))}{t}A(t)e^{xH(t)},$$

which is the polynomial set of Sheffer type $P(KA, H)$, where $K(t) = \dfrac{\psi(H(t))}{t}$.
By Lemma 2.2, $P(KA, H)$ is a d-OPS iff $\dfrac{K'(t)}{K(t)H'(t)} = \dfrac{\psi' \circ H}{\psi \circ H}(t) - \dfrac{1}{tH'(t)}$ is a polynomial of degree d having a leading coefficient different from that of $-\dfrac{A'}{AH'}$, or a polynomial of degree $< d$. Since $R = \dfrac{1}{H'}$ is a polynomial of degree $\leq (d + 1)$ satisfying $R(0) \neq 0$, so $P(A, H)$ is $\psi(D)-$classical iff $\dfrac{\psi' \circ H}{\psi \circ H}(t) = \dfrac{\pi(t)}{t}$, where π is a polynomial of degree $\leq (d + 1)$ satisfying $\pi(0) \neq 0$. That is to say

$$H(t) = \left(\frac{\psi'}{\psi}\right)^{-1} \circ \frac{\pi(t)}{t}.$$

$P(A, H)$ is a d-OPS so by Lemma 2.1, $\dfrac{1}{H'(t)} = \dfrac{t^2F(\pi/t)}{\pi - t\pi'}$ is a polynomial, that is $\pi - t\pi'$ divides $t^2F(\pi/t)$.
Conversely, if $H(t) = \left(\dfrac{\psi'}{\psi}\right)^{-1} \circ \dfrac{\pi(t)}{t}$, where π is a polynomial of degree $\leq d + 1$ satisfying $\pi(0) \neq 0$ and $\pi - t\pi'$ divides $t^2F(\frac{\pi(t)}{t})$. Hence $\dfrac{1}{H'(t)} = \dfrac{t^2F(\pi/t)}{\pi - t\pi'}$ is a polynomial of leading coefficient $\pi(0)$. So, $\dfrac{\psi' \circ H}{\psi \circ H}(t) - \dfrac{1}{tH'(t)} = \dfrac{\pi(t) - 1/H'(t)}{t}$ is a polynomial of degree $\leq d$. □

This theorem provides three cases :

$$\begin{cases} (1)\ F(t) = (t - \alpha)^2,\ \alpha \in \mathbb{R} \\ (2)\ F(t) = (t - \alpha)(t - \beta),\ \alpha, \beta \in \mathbb{R} \\ (3)\ F(t) = (t - \alpha)(t - \overline{\alpha}),\ \alpha \in \mathbb{C} \end{cases}$$

Case (1) The resolution of the system

$$\begin{cases} \left(\dfrac{\psi'}{\psi}\right)' = -\left(\dfrac{\psi'}{\psi} - \alpha\right)^2, \\ \psi(0) = 0, \psi'(0) = 1, \end{cases}$$

leads to $\psi(t) = te^{\alpha t}$. For $\alpha = 0$, we have $\psi(D) = D$.

Case (2) The resolution of the system

$$\begin{cases} \left(\dfrac{\psi'}{\psi}\right)' = -\left(\dfrac{\psi'}{\psi} - \alpha\right)\left(\dfrac{\psi'}{\psi} - \beta\right), \\ \psi(0) = 0, \psi'(0) = 1, \end{cases}$$

leads to $\psi(t) = \dfrac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}$. For $\alpha = 1$ and $\beta = 0$, we have $\psi(D) = \Delta$.

Case (3) The resolution of the system

$$\begin{cases} \left(\dfrac{\psi'}{\psi}\right)' = -\left(\dfrac{\psi'}{\psi} - \alpha\right)\left(\dfrac{\psi'}{\psi} - \overline{\alpha}\right), \\ \psi(0) = 0, \psi'(0) = 1, \end{cases}$$

where $\alpha = a + ib$ $(b \neq 0)$, leads to $\psi(t) = e^{at}\dfrac{sin(bt)}{b}$. For $a = 0$ and $b = 1$ we have $\psi(D) = sin(D)$ which my be viewed as a central difference quotient operator since $sin(D) = \dfrac{1}{2i}(e^{iD} - e^{-iD})$.

For these three cases, $\psi(D)$ is a lowering operator belonging to $\{D,\ \Delta,\ sin(D)\}$. composed with a shift operator $e^{\alpha D}$. Since a shift operator preserves the d-orthogonality, we limit ourselves in the sequel to characterize L-classical d-OPS of Sheffer type where $L \in \{D,\ \Delta,\ sin(D)\}$.

3. Characterization of classical d-OPSs of Sheffer type

In this section, we consider the first case where $\psi(D) = D$ and we determine D-classical d-OPSs of Sheffer type. The particular case $d = 2$ was considered by Boukhemis [13]. He showed that the 2−OPSs of Hermite type and of Laguerre type are D-classical.

Theorem 3.1. *The only D−classical d-OPSs of Sheffer type are*

$$P(e^{\pi_{d+1}(t)}, at) \text{ and } P\left((1 - bt)^{\alpha} e^{\frac{\beta}{1-bt} + \pi_{d-1}(t)}, \frac{at}{1 - bt}\right),$$

where π_i is a polynomial of degree i; a, b are nonzero real constants and α, β are real numbers.

Proof. Let $P(A, H)$ be a d-OPS of Sheffer type. By Theorem 2.1, $P(A, H)$ is D−classical iff

$$H(t) = \frac{t}{\pi(t)},$$

where π is a polynomial of degree $\leq d + 1$ satisfying $\pi(0) \neq 0$ and $\pi - t\pi'$ divides π^2.
It is clear that constant polynomials do the job, so suppose that π is not constant.
Taking the factorization of π over $\mathbb{C}$

$$\pi(t) = c \prod_{k=1}^{r} (t - \alpha_k)^{m_k},$$

where α_k, $k = 1, \dots, r$ are nonzero complex numbers, m_k, $k = 1, \dots, r$ are positive integers and c is a nonzero real number. So the factorization of π' is of the form

$$\pi'(t) = \prod_{k=1}^{r} (t - \alpha_k)^{m_k - 1} Q(t),$$

where Q is a polynomial of degree $r - 1$, coprime with π. This gives

$$\pi(t) - t\pi'(t) = \prod_{k=1}^{r} (t - \alpha_k)^{m_k - 1} S(t),$$

where $S(t) = c \prod_{k=1}^{r} (t - \alpha_k) - tQ(t)$. We have $\deg(S) = \begin{cases} r & \text{if} \deg(\pi) > 1, \\ 0 & \text{if} \deg(\pi) = 1. \end{cases}$

If $\deg(\pi) > 1$, S is not constant, so let α be a root of S. Since S divides $\pi - t\pi'$ which divides π^2, there exists $i \in \{1, \dots, r\}$ such that $\alpha = \alpha_i$. So, α_i is a root of Q, which is impossible because Q and π are coprime. It follows that $\deg(\pi) = 1$.

We conclude that $H(t) = at$ or $H(t) = \dfrac{at}{1 - bt}$, where a and b are nonzero real constants.

- If $H(t) = at$, $\dfrac{A'}{AH'} = \dfrac{A'}{aA}$ is a polynomial of degree d iff $A(t) = e^{\pi_{d-1}(t)}$, where π_{d+1}a polynomial of degree $d + 1$.
- If $H(t) = \dfrac{at}{1 - bt}$, $\dfrac{A'}{AH'}$ must be a polynomial of degree d, that is $\dfrac{A'(t)}{A(t)} = \dfrac{T(t)}{(1 - bt)^2}$, where T is a polynomial of degree d. Taking the partial decomposition of this fraction, then its primitive, we obtain

$$A(t) = (1 - bt)^{\alpha} e^{\dfrac{\beta}{1 - bt} + \pi_{d-1}(t)},$$

where α, β are real constants and π_{d-1} is a polynomial of degree $d - 1$. □

Lemma 3.1. *Let* $\varphi(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_0 \neq 0$, *be a formal power series. We have*

$$\varphi(D)x = \left[x + \frac{\varphi'(D)}{\varphi(D)} \right] \varphi(D).$$

Theorem 3.2. *The classical d-OPSs of Sheffer type satisfy a* $(d + 1)$*-order differential equation of one of the forms*

$$(1) \qquad [D\pi_1(D) - 2xD + 2n]\, y = 0,$$

where π_1 *is a polynomial of degree d.*

$$(2) \qquad \left[-xD(1 - D)^d + D\pi_2(D) + n(1 - D)^{d-1} \right] y = 0,$$

where π_2 *is a polynomial of degree* $\leq d$ *satisfying* $\pi_2(1) \neq 0$.

Proof. The classical d-OPSs of Sheffer type given by Theorem 3.1 are related to Hermite and Laguerre polynomials by [[7], p.12]

$$P(e^{\pi_{d+1}(t)}, 2t) = \varphi_1(D)(H_n(x)), \quad P\left((1 - t)^{-\alpha - 1} e^{\frac{\beta}{1-t} + \pi_{d-1}(t)}, \frac{-t}{1 - t} \right) = \varphi_2(D)(L_n^{(\alpha)}(x)),$$

where $\varphi_1(t) = e^{\pi_{d+1}(\frac{t}{2})+(\frac{t}{2})^2}$, $\varphi_2(t) = e^{\beta(1-t)+\pi_{d-1}(\frac{-t}{1-t})}$, π_i is a polynomial of degree i.
Since Hermite polynomials $H_n(x)$ satisfy the Sturm-Liouville equation [12]

$$(D^2 - 2xD + 2n)y = 0.$$

Applying $\varphi_1(D)$ and using Lemma 3.1, we get

$$\left[-D\pi'_{d+1}(\frac{D}{2}) - 2xD + 2n\right]\varphi_1(D)(H_n) = 0.$$

So we obtain the first equation where $\pi_1(t) = -\pi'_{d+1}(\frac{t}{2})$.
On the other hand, Laguerre polynomials $L_n^{(\alpha)}(x)$ satisfy the equation [12]

$$(xD^2 + (\alpha + 1 - x)D + n)y = 0.$$

Now, applying $\varphi_2(D)$ and using Lemma 3.1, we get

$$\left[-xD(1 - D) + (\alpha + 1)D + n - D(1 - D)\frac{\varphi_2'(D)}{\varphi_2(D)}\right]\varphi_2(D)y = 0.$$

$$\text{Hence}\quad \left[(\beta - x)D(1 - D) + (\alpha + 1)D + \frac{D}{1 - D}\pi'_{d-1}(\frac{-D}{1 - D}) + n\right]\varphi_2(D)y = 0.$$

Taking the Taylor development of π'_{d-1} at the point 1, we obtain

$$\left[(\beta - x)D(1 - D) + (\alpha + 1)D + D\sum_{k=0}^{d-2}\frac{a_k}{(1 - D)^{k+1}} + n\right]\varphi_2(D)y = 0,$$

where $a_{d-2} \neq 0$. Applying $(1 - D)^{d-1}$, and using Lemma 3.1, we get

$$\left[(\beta - x)D(1 - D)^d + (\alpha + d)D(1 - D)^{d-1} + D\sum_{k=0}^{d-2}a_k(1 - D)^{d-2-k} + n(1 - D)^{d-1}\right]y = 0,$$

where $a_{d-2} \neq 0$. We obtain the second equation, where

$$\pi_2(t) = \beta(1 - t)^d + \alpha(1 - t)^{d-1} + \sum_{k=0}^{d-2}a_{d-2-k}(1 - t)^k,\ \ \pi_2(1) = a_{d-2} \neq 0. \quad \square$$

Since equations (1) and (2) are linear and homogeneous, multiplication of a solution by a constant again yields a solution. But such multiplication may destroy the property of being a Sheffer type set. We cannot therefore obtain a complete converse to Theorem 3.2. But we do have

Corollary 3.1. *If a set $\{P_n\}$ satisfies an equation of the forms* (1) *or* (2)*, then there exist nonzero constants c_n, so that $\{c_nP_n\}$ is a classical d-OPSs of Sheffer type.*

Proof. $P(e^{\pi_{d+1}(t)}, 2t)$ (resp. $P\left((1 - t)^{-\alpha-1}e^{\frac{\beta}{1-t}+\pi_{d-1}(t)}, \frac{-t}{1 - t}\right)$) satisfies equation (1) (resp. equation (2)). Since equation (1) (resp. (2)) has a polynomial solution, and this polynomial is unique to within an arbitrary multiplicative constant. Hence, c_n exists so that $P(e^{\pi_{d+1}(t)}, 2t) = \{c_nP_n\}$ (resp. $P\left((1 - t)^{-\alpha-1}e^{\frac{\beta}{1-t}+\pi_{d-1}(t)}, \frac{-t}{1 - t}\right) = \{c_nP_n\}$).

4. Characterization of classical discrete d-OPSs of Sheffer type

In this section, we consider the second case where $\psi(D) = e^D - 1$ and we Characterize Δ-classical d-OPSs of Sheffer type. The particular case $d = 2$ was considered by Boukhemis [13]. He showed that the 2−OPSs of Charlier type and of Meixner type are Δ-classical.

Theorem 4.1. *Let $P(A, H)$ be a d-OPS of Sheffer type. Then the following statements are equivalent :*

(i) *$P(A, H)$ is $\Delta-$classical.*

(ii) $H(t) = \log\left(\dfrac{\pi(t)}{\pi(t) - t}\right)$, *where π is a real polynomial of degree $0 \le n \le d + 1$, satisfying $\pi(0) \ne 0$ and if $n \ge 2$ the number of real and complex roots of $\pi(\pi - t)$ is equal to n.*

(iii) $\dfrac{1}{H'(t)}$ *is equal to one of these polynomials :*

- $\dfrac{1}{H'(t)} = \pm(t - \alpha)$, *$\alpha$ is a nonzero real number,*
- $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha_1 - \alpha_2}(t - \alpha_1)(t - \alpha_2)$, *$\alpha_1, \alpha_2$ are nonzero real distinct numbers.*
- $\dfrac{1}{H'(t)} = \dfrac{1}{4\alpha}(t - \alpha)(t + \alpha)$, *$\alpha$ is a nonzero real number,*
- $\dfrac{1}{H'(t)} = c\prod_{k=1}^{p}(t - \alpha_k)$, $(3 \le p \le d + 1)$ *such that $\alpha_1, \cdots, \alpha_p$ are nonzero distinct complex numbers satisfying*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_p \\ \alpha_1^2 & \alpha_2^2 & \cdots & \cdots & \alpha_p^2 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \alpha_1^p & \alpha_2^p & \cdots & \cdots & \alpha_p^p \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_r \\ -m_{r+1} \\ \vdots \\ -m_p \end{pmatrix} = \frac{1}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{p}{(p-1)} \sum_{i=1}^{r} m_i\alpha_i \end{pmatrix}, \tag{1}$$

where the m_i's are positive integers such that $\sum_{i=1}^{r} m_i = \sum_{i=r+1}^{p} m_i = p$

Proof. (i)⇔(ii) Let $P(A, H)$ be a d-OPS of Sheffer type. By Theorem 2.1, $P(A, H)$ is $\Delta-$classical iff

$$H(t) = \log\left(\frac{\pi(t)}{\pi(t) - t}\right).$$

where π is a polynomial of degree $\le d + 1$ satisfying $\pi(0) \ne 0$ and $\pi - t\pi'$ divides $\pi(\pi - t)$.
It is clear that polynomials of degree ≤ 1 do the job, so suppose that $\deg \pi$ is an integer $n \ge 2$.
Taking the factorization of π and $\pi - t$ over $\mathbb{C}$

$$\pi(t) = c\prod_{k=1}^{r}(t - \alpha_k)^{m_k}, \;\; \pi(t) - t = c\prod_{k=1}^{r'}(t - \beta_k)^{m'_k},$$

where the α_k, β_k are nonzero numbers, the m_k, m'_k are positive integers and c is a nonzero real number. So

$$\pi(t) - t\pi'(t) = \prod_{k=1}^{r}(t - \alpha_k)^{m_k - 1} S_1(t),$$

$$\text{and} \quad \pi(t) - t\pi'(t) = (\pi(t) - t) - t(\pi'(t) - 1) = \prod_{k=1}^{r'} (t - \beta_k)^{m'_k - 1} S_2(t),$$

where S_1 and π are coprime, S_2 and $\pi - t$ are coprime.
Since π and $\pi - t$ are coprime, so the $\alpha_k s$ are different from the $\beta_k s$, we get

$$\pi(t) - t\pi'(t) = \prod_{k=1}^{r} (t - \alpha_k)^{m_k - 1} \prod_{k=1}^{r'} (t - \beta_k)^{m'_k - 1} S(t),$$

where S is coprime with π and $\pi - t$.
It follows that $\pi - t\pi'$ divides $\pi(\pi - t)$ iff S is a constant. That is

$$\pi - t\pi' = c(1 - n) \prod_{k=1}^{r} (t - \alpha_k)^{m_k - 1} \prod_{k=1}^{r'} (t - \beta_k)^{m'_k - 1}, \tag{2}$$

which is equivalent to $n = r + r'$.
(ii)⇒(iii) Suppose that $H(t) = \log\left(\frac{\pi(t)}{\pi(t) - t}\right)$, where π is a polynomial of degree $0 \leq n \leq d + 1$ such that $\pi(0) \neq 0$ and the number of real and complex roots of $\pi(\pi - t)$ is equal to n if $n \geq 2$.
- If $n = 0$, so $\frac{1}{H'(t)} = -(t - \alpha)$, α is a nonzero real number.
- If $n = 1$, $\frac{1}{H'(t)} = (t - \alpha)$ or $\frac{1}{H'(t)} = \frac{1}{\alpha_1 - \alpha_2}(t - \alpha_1)(t - \alpha_2)$, $\alpha, \alpha_1, \alpha_2$ are nonzero real numbers.
- If $n \geq 2$, define S_k, $k = 1, \ldots, n$ by the relations $S_k = \sum_{i=1}^{r} m_i \alpha_i^k$, where the $\alpha'_i s$ are the real and complex roots of multiplicity m_i of the polynomial $\pi(t)$ that will be noted by

$$\pi(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n. \tag{3}$$

Taking $\pi(t)$ of the form $\pi(t) = a_0 \prod_{k=1}^{r} (t - \alpha_k)^{m_k}$, we deduce that $\pi'(t) = \pi(t) \sum_{k=1}^{r} \frac{m_k}{t - \alpha_k}$.
Replacing $\frac{1}{t - \alpha_k}$ by its series expansion $\frac{1}{t - \alpha_k} = \frac{1}{t} + \frac{\alpha_k}{t^2} + \frac{\alpha_k^2}{t^3} + \cdots$, we get

$$\pi'(t) = \pi(t)\left[\frac{n}{t} + \frac{S_1}{t^2} + \frac{S_2}{t^3} + \cdots\right]. \tag{4}$$

Substituting (3) in (4) gives

$$na_0 t^{n-1} + (n-1)a_1 t^{n-2} + \cdots + 2a_{n-2} t + a_{n-1} = (a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n)\left[\frac{n}{t} + \frac{S_1}{t^2} + \frac{S_2}{t^3} + \cdots\right].$$

Comparing coefficients of t^k on both sides, we obtain the Newton's identities [23]

$$\begin{cases} a_0 S_1 + a_1 = 0 \\ a_0 S_2 + a_1 S_1 + 2a_2 = 0 \\ a_0 S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0 \\ \vdots \\ a_0 S_{n-1} + a_1 S_{n-2} + \cdots + a_{n-2} S_1 + (n-1)a_{n-1} = 0 \\ a_0 S_n + a_1 S_{n-1} + \cdots + a_{n-1} S_1 + na_n = 0 \end{cases} \tag{5}$$

On the other hand, by hypothesis, the number of real and complex roots of $\pi - t$ is equal to $n - r$. So $\pi - t$ can be written in the forms

$$\pi(t) - t = a_0 t^n + a_1 t^{n-1} + \cdots + (a_{n-1} - 1)t + a_n = a_0 \prod_{k=r+1}^{n} (t - \alpha_k)^{m_k}.$$

We now apply the same reasoning, with S_k replaced by $T_k = \sum_{i=r+1}^{n} m_i \alpha_i^k$, to obtain the Newton's identities

$$\begin{cases} a_0 T_1 + a_1 = 0 \\ a_0 T_2 + a_1 T_1 + 2a_2 = 0 \\ a_0 T_3 + a_1 T_2 + a_2 T_1 + 3a_3 = 0 \\ \vdots \\ a_0 T_{n-1} + a_1 T_{n-2} + \cdots + a_{n-2} T_1 + (n-1)(a_{n-1} - 1) = 0 \\ a_0 T_n + a_1 T_{n-1} + \cdots + (a_{n-1} - 1) T_1 + n a_n = 0 \end{cases} \tag{6}$$

The resolution of the systems (5) and (6) leads to two cases

- If $n = 2$: $S_k = 2\alpha_1^k$ and $T_k = 2\alpha_2^k$ satisfy $\begin{cases} S_1 = T_1 - \frac{1}{a_0} \\ S_2 = T_2 + \frac{a_1}{a_0^2} - \frac{T_1}{a_0} \end{cases}$.

Replacing a_1 by $1 - a_0 T_1$, we get $\begin{cases} \alpha_1 = -\frac{1}{4a_0} \\ \alpha_2 = \frac{1}{4a_0} \end{cases}$. So $\dfrac{1}{H'(t)} = \dfrac{1}{4\alpha}(t - \alpha)(t + \alpha)$.

- If $n \geq 3$: $\begin{cases} S_k = T_k, \ \forall 1 \leq k \leq n-2 \\ S_{n-1} = T_{n-1} - \frac{n-1}{a_0} \\ S_n = T_n + \frac{(n-1)a_1}{a_0^2} - \frac{T_1}{a_0} \end{cases}$. Replacing a_1 by $-a_0 T_1$, we get $\begin{cases} S_k = T_k, \ \forall 1 \leq k \leq n-2 \\ S_{n-1} = T_{n-1} - \frac{n-1}{a_0} \\ S_n = T_n - \frac{n}{a_0} T_1 \end{cases}$.

On the other hand, we have $H(t) = \log(\frac{\pi(t)}{\pi(t) - t})$, hence $\dfrac{1}{H'} = \dfrac{\pi(\pi - t)}{\pi - t\pi'}$. Analysis similar to that in the proof of $(i) \Leftrightarrow (ii)$ shows that $\pi(t) - t\pi'(t) = a_0(1-n) \prod_{k=1}^{n} (t - \alpha_k)^{m_k - 1}$. It follows that

$$\frac{1}{H'(t)} = \frac{a_0}{1-n} \prod_{k=1}^{n} (t - \alpha_k) = c \prod_{k=1}^{n} (t - \alpha_k),$$

where $\alpha_1, \ldots, \alpha_n$ satisfy the equations

$$\begin{cases} S_k = T_k, \ \forall 1 \leq k \leq n-2 \\ S_{n-1} = T_{n-1} + \frac{1}{c} \\ S_n = T_n + \frac{n}{(n-1)c} S_1. \end{cases} \tag{7}$$

which is equivalent to (1).

(iii)⇒(ii)

- If $\dfrac{1}{H'(t)} = -(t - \alpha)$, since $H(0) = 0$, it follows that $H(t) = \log(\frac{\alpha}{\alpha - t}) = \log(\frac{\pi}{\pi - t})$, $\pi = \alpha$.
- If $\dfrac{1}{H'(t)} = (t - \alpha)$, so $H(t) = \log(\dfrac{t - \alpha}{-\alpha}) = \log(\dfrac{\pi}{\pi - t})$, $\pi = t - \alpha$.
- If $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha_1 - \alpha_2}(t - \alpha_1)(t - \alpha_2)$, so $H(t) = \log(\dfrac{\alpha_2(t - \alpha_1)}{\alpha_1(t - \alpha_2)}) = \log(\dfrac{\pi}{\pi - t})$, $\pi = \frac{\alpha_2}{\alpha_2 - \alpha_1}(t - \alpha_1)$.
- If $\dfrac{1}{H'(t)} = \dfrac{1}{4\alpha}(t - \alpha)(t + \alpha)$, so $H(t) = \log(\dfrac{t - \alpha}{t + \alpha})^2 = \log(\dfrac{\pi}{\pi - t})$, $\pi = -\frac{1}{4\alpha}(t - \alpha)^2$.
- If $\dfrac{1}{H'(t)} = c \prod_{k=1}^{n} (t - \alpha_k)$, $(3 \leq n \leq d + 1)$ such that $\alpha_1, \cdots, \alpha_n$ are nonzero distinct complex numbers satisfying

(1). Let $\pi(t) = c(1-n)\prod_{k=1}^{r}(t-\alpha_k)^{m_k}$. If we take the notation (3), we get (5), and hence we deduce (6) from (7). Newton's identities given by (6), implies that $\pi(t)-t = c(1-n)\prod_{k=r+1}^{p}(t-\alpha_k)^{m_k}$. So the number of roots of $\pi(\pi-t)$ is equal to p, it follows by the same method as in (2), that $\pi - t\pi' = c(1-p)^2\prod_{k=1}^{r}(t-\alpha_k)^{m_k-1}\prod_{k=r+1}^{p}(t-\beta_k)^{m_k-1}$. Hence, $\dfrac{\pi(\pi-t)}{\pi-t\pi'} = c\prod_{k=1}^{p}(t-\alpha_k) = \dfrac{1}{H'}$. It follows, that $H(t) = \log(\dfrac{\pi(t)}{\pi(t)-t})$, where π is a polynomial of degree $2 \le p \le d+1$, such that the number of roots of $\pi(\pi-t)$ is equal to p. □

Examples.

1. $d = 1$:
   - $\dfrac{1}{H'(t)} = \pm(t-\alpha)$, α is a nonzero real number : Charlier polynomials.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha_1-\alpha_2}(t-\alpha_1)(t-\alpha_2)$, α_1,α_2 are nonzero distinct real numbers : Meixner polynomials.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{4\alpha}(t-\alpha)(t+\alpha)$, α is a nonzero real number : Meixner polynomials.
2. $d = 2$:
   - $\dfrac{1}{H'(t)} = \pm(t-\alpha)$, α is a nonzero real number : 2-OPS of Charlier type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha_1-\alpha_2}(t-\alpha_1)(t-\alpha_2)$, α_1,α_2 are nonzero distinct real numbers : 2-OPS of Meixner type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{4\alpha}(t-\alpha)(t+\alpha)$, α is a nonzero real number : 2-OPS of Meixner type.
   - $\dfrac{1}{H'(t)} = c(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)$, $\alpha_1,\alpha_2,\alpha_3$ are nonzero distinct complex numbers satisfying one of these two equations

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2\\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3\end{pmatrix}\begin{pmatrix}3\\-1\\-2\end{pmatrix} = \begin{pmatrix}0\\1/c\\9\alpha_1/2c\end{pmatrix};\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2\\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3\end{pmatrix}\begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}0\\1/c\\9\alpha_3/2c\end{pmatrix}$$

   which we can solve using Maple for example to get

$$\frac{1}{H'(t)} = \pm\frac{1}{54\alpha^2}(t-\alpha)(t+2\alpha)(t+8\alpha).$$

3. $d = 3$:
   - $\dfrac{1}{H'(t)} = \pm(t-\alpha)$: 3-OPS of Charlier type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha_1-\alpha_2}(t-\alpha_1)(t-\alpha_2)$, α_1,α_2 are nonzero distinct real numbers : 3-OPS of Meixner type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{4\alpha}(t-\alpha)(t+\alpha)$, α is a nonzero real number : 3-OPS of Meixner type.
   - $\dfrac{1}{H'(t)} = \pm\dfrac{1}{54\alpha^2}(t-\alpha)(t+2\alpha)(t+8\alpha)$.
   - $\dfrac{1}{H'(t)} = c(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)(t-\alpha_4)$, $\alpha_1,\ldots,\alpha_4$ are nonzero distinct complex numbers satisfying one of these equations

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2\\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3\\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4\end{pmatrix}\begin{pmatrix}4\\-2\\-1\\-1\end{pmatrix} = \pm\begin{pmatrix}0\\0\\1/c\\\frac{16\alpha_1}{3c}\end{pmatrix};\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2\\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3\\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4\end{pmatrix}\begin{pmatrix}3\\1\\-2\\-2\end{pmatrix} = \pm\begin{pmatrix}0\\0\\1/c\\\frac{4(3\alpha_1+\alpha_2)}{3c}\end{pmatrix};$$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/c \\ \frac{4(3\alpha_1+\alpha_2)}{3c} \end{pmatrix};\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 \\ \alpha_1^4 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/c \\ \frac{8(\alpha_1+\alpha_2)}{3c} \end{pmatrix};$$

which gives the solutions

$$\frac{1}{H'(t)} = \pm\frac{1}{2^8 3\alpha^3}(t-\alpha)(t+3\alpha)[t+(7+4\sqrt{2}i)\alpha][t+(7-4\sqrt{2}i)\alpha];$$

$$\frac{1}{H'(t)} = \pm\frac{1}{2^6 3\alpha^3}(t-\alpha)(t-9\alpha)[t-(3-2\sqrt{3})\alpha][t-(3+2\sqrt{3})\alpha];$$

$$\frac{1}{H'(t)} = \frac{1}{2^4 3\alpha^3}(t-\alpha)(t+\alpha)(t-3\alpha)(t+3\alpha).$$

5. Characterization of *sinD*−classical *d*-OPSs of Sheffer type

In this section, we consider the third case where $\psi(D) = sinD$ and we Characterize *sinD*-classical *d*-OPSs of Sheffer type.

Theorem 5.1. *Let $P(A, H)$ be a d-OPS of Sheffer type. Then the following statements are equivalent :*

(i) *$P(A, H)$ is sinD−classical.*

(ii) *$H(t) = tan^{-1}\left(\frac{t}{\pi(t)}\right)$, where π is a real polynomial of degree $0 \le n \le d+1$, satisfying $\pi(0) \ne 0$ and if $n \ge 2$, the number of roots of $\pi - it$ is equal to $\frac{n}{2}$.*

(iii) *$\frac{1}{H'(t)}$ is equal to one of these polynomials :*

- $\frac{1}{H'(t)} = \frac{1}{\alpha}(t^2+\alpha^2)$, *$\alpha$ is a nonzero real number.*
- $\frac{1}{H'(t)} = \frac{1}{Im(\alpha)}(t-\alpha)(t-\overline{\alpha})$, $\alpha \in \mathbb{C}\setminus\mathbb{R}$.
- $\frac{1}{H'(t)} = \frac{1}{2\alpha}(t^2+\alpha^2)$, *$\alpha$ is a nonzero real number.*
- $\frac{1}{H'(t)} = c\prod_{k=1}^{r}(t-\alpha_k)(t-\overline{\alpha}_k)$, $(2 \le r \le (d+1)/2)$, $\alpha_1, \cdots, \alpha_r \in \mathbb{C}\setminus\mathbb{R}$ *satisfying*

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_r & \overline{\alpha}_1 & \cdots & \overline{\alpha}_r \\ \alpha_1^2 & \cdots & \alpha_r^2 & \overline{\alpha}_1^2 & \cdots & \overline{\alpha}_r^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1^{2r} & \cdots & \alpha_r^{2r} & \overline{\alpha}_1^{2r} & \cdots & \overline{\alpha}_r^{2r} \end{pmatrix}\begin{pmatrix} m_1 \\ \vdots \\ m_r \\ -m_1 \\ \vdots \\ -m_r \end{pmatrix} = -\frac{2i}{c}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \frac{2r}{2r-1}\sum_{i=1}^{r} m_i\alpha_i \end{pmatrix}, \qquad (8)$$

where the m_i's are positive integers such that $\sum_{i=1}^{r} m_i = 2r$

Proof. $(i) \Leftrightarrow (ii)$ Let $P(A,H)$ be a d-OPS of Sheffer type. By Theorem 2.1, $P(A,H)$ is $sinD$−classical iff

$$H(t) = tan^{-1}\left(\frac{t}{\pi(t)}\right).$$

where π is a polynomial of degree $\leq d+1$ satisfying $\pi(0) \neq 0$ and $\pi - t\pi'$ divides $\pi^2 + t^2$. It is clear that polynomials of degree ≤ 1 do the job, so suppose that $\deg \pi$ is an integer $n \geq 2$.
Taking the factorization of $\pi - it$ and $\pi + it$ over $\mathbb{C}$

$$\pi(t) - it = c\prod_{k=1}^{r}(t-\alpha_k)^{m_k}, \quad \pi(t) + it = c\prod_{k=1}^{r}(t-\overline{\alpha}_k)^{m_k},$$

where $\alpha_1, \ldots, \alpha_r \in \mathbb{C} \setminus \mathbb{R}$, $m_1, \ldots, m_r$ are positive integers and c is a nonzero real number. So

$$\pi(t) - t\pi'(t) = (\pi(t) - it) - t(\pi(t) - it)' = \prod_{k=1}^{r}(t-\alpha_k)^{m_k-1}S_1(t),$$

$$and \quad \pi(t) - t\pi'(t) = (\pi(t) + it) - t(\pi(t) + it)' = \prod_{k=1}^{r}(t-\overline{\alpha}_k)^{m_k-1}\overline{S_1}(t),$$

where S_1 and $\pi - it$ are coprime, $\overline{S_1}$ and $\pi + it$ are coprime. So,

$$\pi(t) - t\pi'(t) = \prod_{k=1}^{r}(t-\alpha_k)^{m_k-1}(t-\overline{\alpha}_k)^{m_k-1}S(t),$$

where S is coprime with $\pi - it$ and $\pi + it$.
It follows that $\pi - t\pi'$ divides $\pi^2 + t^2$ iff S is a constant. That is

$$\pi - t\pi' = c(1-p)\prod_{k=1}^{r}(t-\alpha_k)^{m_k-1}(t-\overline{\alpha}_k)^{m_k-1}, \tag{9}$$

which is equivalent to $n = 2r$.

$(ii) \Rightarrow (iii)$ $H(t) = tan^{-1}\left(\frac{t}{\pi(t)}\right)$, where π is a polynomial of degree $0 \leq n \leq d+1$, satisfying $\pi(0) \neq 0$ and if $n \geq 2$, the number r of roots of $\pi - it$ is equal to $\frac{n}{2}$.
- If $n = 0$, so $\frac{1}{H'(t)} = \frac{1}{\alpha}(t^2 + \alpha^2)$, α is a nonzero real number.
- If $n = 1$, so $\frac{1}{H'(t)} = \frac{1}{Im(\alpha)}(t-\alpha)(t-\overline{\alpha})$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$.
- If $n = 2r \geq 2$, denote by $\pi(t) - it = a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n = a_0\prod_{k=1}^{r}(t-\alpha_k)^{m_k}$.

So, $\pi(t) + it = a_0t^n + a_1t^{n-1} + \cdots + (a_{n-1} + 2i)t + a_n = a_0\prod_{k=1}^{r}(t-\overline{\alpha}_k)^{m_k}$.

Define S_k, $k = 1, \ldots, n$ by the relations $S_k = \sum_{i=1}^{r} m_i\alpha_i^k$. Analysis similar to that in the proof of Theorem 4.1

gives the Newton's identities

$$\begin{cases} a_0S_1 + a_1 = 0 \\ a_0S_2 + a_1S_1 + 2a_2 = 0 \\ a_0S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0 \\ \vdots \\ a_0S_{n-1} + \cdots + a_{n-2}S_1 + (n-1)a_{n-1} = 0 \\ a_0S_n + a_1S_{n-1} + \cdots + a_{n-1}S_1 + na_n = 0 \end{cases} ; \begin{cases} a_0\overline{S}_1 + a_1 = 0 \\ a_0\overline{S}_2 + a_1\overline{S}_1 + 2a_2 = 0 \\ a_0\overline{S}_3 + a_1\overline{S}_2 + a_2\overline{S}_1 + 3a_3 = 0 \\ \vdots \\ a_0\overline{S}_{n-1} + \cdots + a_{n-2}\overline{S}_1 + (n-1)(a_{n-1}+2i) = 0 \\ a_0\overline{S}_n + a_1\overline{S}_{n-1} + \cdots + (a_{n-1}+2i)\overline{S}_1 + na_n = 0 \end{cases}$$

The resolution of these systems leads to two cases

- If $n = 2$: $S_k = 2\alpha_1^k,\ k = 1, 2$ satisfy $\begin{cases} S_1 = \overline{S}_1 + \frac{2i}{a_0} \\ S_2 = \overline{S}_2 - \frac{2ia_1}{a_0^2} + \frac{2i}{a_0}\overline{S}_1 \end{cases}$.

Replacing a_1 by $-2i - a_0\overline{S}_1$, we get $\alpha_1 = \dfrac{i}{2a_0}$. So $\dfrac{1}{H'(t)} = \dfrac{1}{2\alpha}(t - i\alpha)(t + i\alpha)$.

- If $n \geq 3$: $\begin{cases} S_k = \overline{S}_k,\ \forall 1 \leq k \leq n-2 \\ S_{n-1} = \overline{S}_{n-1} + \frac{2i(n-1)}{a_0} \\ S_n = \overline{S}_n - \frac{2i(n-1)a_1}{a_0^2} + \frac{2i}{a_0}S_1 \end{cases}$. Replacing a_1 by $-a_0S_1$, we get $\begin{cases} S_k = \overline{S}_k,\ \forall 1 \leq k \leq n-2 \\ S_{n-1} = \overline{S}_{n-1} + \frac{2i(n-1)}{a_0} \\ S_n = \overline{S}_n + \frac{2in}{a_0}S_1. \end{cases}$

On the other hand, we have $H(t) = \tan^{-1}(\frac{t}{\pi(t)})$, hence $\dfrac{1}{H'} = \dfrac{\pi^2 + t^2}{\pi - t\pi'}$. Analysis similar to that in the proof of $(i) \Leftrightarrow (ii)$ shows that $\pi(t) - t\pi'(t) = a_0(1-n)\prod_{k=1}^{r}(t-\alpha_k)^{m_k-1}(t-\overline{\alpha}_k)^{m_k-1}$.

It follows that

$$\frac{1}{H'(t)} = \frac{a_0}{1-n}\prod_{k=1}^{r}(t-\alpha_k)(t-\overline{\alpha}_k) = c\prod_{k=1}^{r}(t-\alpha_k)(t-\overline{\alpha}_k),$$

where $\alpha_1, \ldots, \alpha_n$ satisfy the equations

$$\begin{cases} S_k = \overline{S}_k,\ \forall 1 \leq k \leq n-2 \\ S_{n-1} = \overline{S}_{n-1} - \frac{2i}{c} \\ S_n = \overline{S}_n - \frac{2in}{(n-1)c}S_1. \end{cases}$$

which is equivalent to (8).

$(iii) \Rightarrow (ii)$ - If $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha}(t^2 + \alpha^2)$, since $H(0) = 0$, it follows that $H(t) = \tan^{-1}(\frac{t}{\pi})$, $\pi = \alpha$.

- If $\dfrac{1}{H'(t)} = \dfrac{1}{Im(\alpha)}(t-\alpha)(t-\overline{\alpha})$, $\alpha = x + iy \in \mathbb{C} \setminus \mathbb{R}$. So $H(t) = \tan^{-1}(\dfrac{t}{\pi})$, $\pi = -\frac{x}{y}t + \frac{x^2+y^2}{y}$.

- If $\dfrac{1}{H'(t)} = \dfrac{1}{2\alpha}(t^2 + \alpha^2)$, so $H(t) = \tan^{-1}(\frac{t}{\pi})$, $\pi = \frac{-1}{2\alpha}(t^2 - \alpha^2)$.

- If $\dfrac{1}{H'(t)} = c\prod_{k=1}^{r}(t-\alpha_k)(t-\overline{\alpha}_k)$, $(2 \leq r \leq (d+1)/2)$, $\alpha_1, \cdots, \alpha_r \in \mathbb{C}\backslash\mathbb{R}$ satisfying (9). Let π be the polynomial defined by $\pi(t) - it = c(1-p)\prod_{k=1}^{r}(t-\alpha_k)^{m_k}$. So the degree of π is equal to $\sum_{i=1}^{r} m_i = 2r$. It follows by the same method as in (9), that $\pi(t) - t\pi'(t) = c(1-p)^2\prod_{k=1}^{r}(t-\alpha_k)^{m_k-1}(t-\overline{\alpha}_k)^{m_k-1}$. Hence, $\dfrac{\pi^2+t^2}{\pi - t\pi'} = c\prod_{k=1}^{r}(t-\alpha_k)(t-\overline{\alpha}_k) = \dfrac{1}{H'}$.

It follows that $H(t) = tan^{-1}\left(\dfrac{t}{\pi(t)}\right)$, where π is a polynomial of degree $n = 2r$ satisfying $\pi(0) \neq 0$ and the number of roots of $\pi - it$ is equal to $n/2$. □

Examples.

1. $d = 1$:
   - $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha}(t^2 + \alpha^2)$, α is a nonzero real number : Meixner-Pollaczek polynomials.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{Im(\alpha)}(t - \alpha)(t - \overline{\alpha})$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$: Meixner-Pollaczek polynomials.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{2\alpha}(t^2 + \alpha^2)$, α is a nonzero real number : Meixner-Pollaczek polynomials.
2. $d = 2$:
   - $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha}(t^2 + \alpha^2)$, α is a nonzero real number : 2-OPS of Meixner-Pollaczek type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{Im(\alpha)}(t - \alpha)(t - \overline{\alpha})$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$: 2-OPS of Meixner-Pollaczek type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{2\alpha}(t^2 + \alpha^2)$, α is a nonzero real number : 2-OPS of Meixner-Pollaczek type.
3. $d = 3$:
   - $\dfrac{1}{H'(t)} = \dfrac{1}{\alpha}(t^2 + \alpha^2)$, α is a nonzero real number : 3-OPS of Meixner-Pollaczek type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{Im(\alpha)}(t - \alpha)(t - \overline{\alpha})$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$: 3-OPS of Meixner-Pollaczek type.
   - $\dfrac{1}{H'(t)} = \dfrac{1}{2\alpha}(t^2 + \alpha^2)$, α is a nonzero real number : 3-OPS of Meixner-Pollaczek type.
   - $\dfrac{1}{H'(t)} = c(t - \alpha_1)(t - \overline{\alpha}_1)(t - \alpha_2)(t - \overline{\alpha}_2)$, α_1, α_2 are distinct numbers in $\mathbb{C} \setminus \mathbb{R}$ satisfying one of these equations

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \overline{\alpha}_1 & \overline{\alpha}_2 \\ \alpha_1^2 & \alpha_2^2 & \overline{\alpha}_1^2 & \overline{\alpha}_2^2 \\ \alpha_1^3 & \alpha_2^3 & \overline{\alpha}_1^3 & \overline{\alpha}_2^3 \\ \alpha_1^4 & \alpha_2^4 & \overline{\alpha}_1^4 & \overline{\alpha}_2^4 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2i/c \\ -\frac{16i(\alpha_1+\alpha_2)}{3c} \end{pmatrix}; \begin{pmatrix} \alpha_1 & \alpha_2 & \overline{\alpha}_1 & \overline{\alpha}_2 \\ \alpha_1^2 & \alpha_2^2 & \overline{\alpha}_1^2 & \overline{\alpha}_2^2 \\ \alpha_1^3 & \alpha_2^3 & \overline{\alpha}_1^3 & \overline{\alpha}_2^3 \\ \alpha_1^4 & \alpha_2^4 & \overline{\alpha}_1^4 & \overline{\alpha}_2^4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2i/c \\ -\frac{8i(3\alpha_1+\alpha_2)}{3c} \end{pmatrix};$$

which yields

$$\frac{1}{H'(t)} = \frac{1}{24\alpha^3}(t^2 + 9\alpha^2)(t^2 + \alpha^2).$$

6. Concluding remarks

Remark 6.1. *It's well known that all classical OPS satisfy a second order differential equation. A natural question arises:*
Do all classical d-OPSs satisfy a $(d + 1)$-order differential equation ?
The answer is affirmative for all known classical d-OPS (See for instance [2–4, 8, 14, 15, 18, 21]). In this paper, we provide a further case for which the answer of this question is also affirmative.

Remark 6.2. *In this paper, we obtain all the lowering operator L used to classify the OPSs of Sheffer type as L-classical. It's of interest to generalize this result to d-OPSs of Sheffer type.*

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