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Uniform Decay for a Viscoelastic Wave Equation with Density and Time-Varying Delay in $\mathbb{R}^n$

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Abstract. A linear viscoelastic wave equation with density and a time-varying delay term in the internal feedback is considered. Under suitable assumptions on the relaxation function, we establish a decay result of solution for by using energy perturbation method in the space $\mathbb{R}^n$ $(n > 2)$. We extend a recent result in Feng [10].

1. Introduction and position of problem

It is well known that the PDEs with time delay have been much studied during the last years and their results is by now rather developed especially in the varying delay case, see [1], [7]–[9], [16]–[18], [21], and so on. In the classical theory of delayed wave equations, several main parts are joined in a fruitful way, it is very remarkable that the damped wave equation with varying delays occupies a similar position and arise in many applied problems.

In this paper, we consider the following wave equation with a time-varying delay term in the internal feedback:

$$\begin{cases} u_{tt} - \phi(x)\left(\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)ds\right) + \mu_1 u_t + \mu_2 u_t\left(x, t-\tau(t)\right) = 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x),\ x \in \mathbb{R}^n, \\ u_t(x,t) = f_0(x,t),\ x \in \mathbb{R}^n,\ t \in [-\tau(0), 0), \end{cases} \tag{1}$$

where $u_0(x)$, $u_1(x)$ and $f_0(x, t-\tau(0))$ are given initial data and the function g is the relaxation function. The function $\phi(x) := (\rho(x))^{-1}$ is the speed of sound at the point $x \in \mathbb{R}^n$ and the function $\rho(x)$ is the density. The constants μ_1 and μ_2 are two real numbers and the function $\tau(t)$ is the varying delay term.

We assume, on the time-delay functions, that there exist positive constants $\overline{\tau}_0$ and $\overline{\tau}$ such that

$$0 < \overline{\tau}_0 \le \tau(t) \le \overline{\tau}, \quad \forall t > 0, \tag{2}$$

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Moreover, we assume

$$\tau \in W^{2,\infty}([0,T]), \quad \forall T > 0, \tag{3}$$

$$\tau'(t) \leq d < 1, \quad \forall t > 0, \tag{4}$$

where d is the positive constant.

The relaxation function g satisfies the following assumptions:

(G1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0,\ 1 - \int_0^\infty g(s)\,ds = l > 0. \tag{5}$$

(G2) There exists a non-increasing differentiable function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_0^\infty \zeta(s)ds = +\infty,\ g'(t) \leq -\zeta(t)\,g(t), \quad \text{for } t \geq 0.. \tag{6}$$

The modified energy functional associate with problem (1) is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho} + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla_x u(t)\|_2^2 \\ &\quad + \frac{1}{2}(g \circ \nabla_x u)(t) + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho(x)\,e^{\lambda(s-t)} u_s^2(x,s)\,dxds, \end{aligned} \tag{7}$$

where and $\xi > 0$ will be chosen later, and the constant $\lambda > 0$, see [19], satisfies

$$\lambda < \frac{1}{\tau_1}\left|\log \frac{|\mu_2|}{\sqrt{1-d}}\right|.$$

and

$$(g \circ \nabla_x u)(t) = \int_0^t g(t-s)\,\|\nabla_x u(t) - \nabla_x u(s)\|^2\,ds, \tag{8}$$

For $\tau(t) = \tau_0$, system (1) has been investigated recently by many authors, where they showed the well-posedness and stabilities results in bounded/unbounded domains (see [1], [2], [3], [5], [7], and so on). Concerning the existence and uniqueness result, we refer the reader to read the existing works which is not our aim interesting here (see [10], Theorem 3.1). In the present work, we extend the result in [10] to time-varying delay.

The plan of the paper is as follows. The first section is devoted to introduce the problem. In Section 2, we give some preliminaries and our main results. In Section 3, we shall prove the stability of energy to the problem.

2. Preliminaries and main result

As in [11], [22], we introduce the weighted spaces $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $L^p_\rho(\mathbb{R}^n)$ for our system. First we assume the density $\rho(x) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions.

(A) $\rho(x) > 0$, $\rho \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^{\frac{n}{2}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Now we define the weighted spaces $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and $L^p_\rho(\mathbb{R}^n)$, $(1 < p < \infty)$.

(1) The space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is defined to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to which norm

$$\mathcal{D}^{1,2}(\mathbb{R}^n) = \left\{u \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) : \nabla_x u \in L^2(\mathbb{R}^n)\right\},$$

equipped with the norm $\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla_x u|^2 \, dx$.

(2) We introduce the weighted space $L^2_\rho(\mathbb{R}^n)$ to be defined the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(u,v)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho uv dx,$$

and we know that $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space and $\|u\|^2_{L^2_\rho(\mathbb{R}^n)} = (u,u)_{L^2_\rho(\mathbb{R}^n)}$.

(3) If u is a measurable function on $\mathbb{R}^n$, we define

$$\|u\|^p_{L^p_\rho(\mathbb{R}^n)}(\mathbb{R}^n) = \left(\int_{\mathbb{R}^n} \rho \, |u|^p \, dx\right)^{\frac{1}{p}}, \text{ for } 1 < p < \infty,$$

and let $L^p_\rho(\mathbb{R}^n)$ consist of all u for which $\|u\|_{L^p_\rho(\mathbb{R}^n)} < \infty$.

We have the following Lemma.

Lemma 2.1. *[10], [11][23] Assume the function ρ satisfies (A), then for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$*

$$\|u\|_{L^q_\rho} \leq \|\rho\|_{L^s} \|\nabla_x u\| \text{ with } s = \frac{2n}{2n - qn + 2q} \text{ and } 2 \leq q \leq \frac{2n}{n-2}. \tag{1}$$

Remark 2.2. *For $q = 2$, we have*

$$\|u\|_{L^2_\rho} \leq \|\rho\|_{L^{\frac{n}{2}}} \|\nabla_x u\|. \tag{2}$$

If $\rho \in L^{\frac{n}{2}}(\mathbb{R}^n)$, we have

$$\|u\|_{L^2_\rho} \leq c_* \|\nabla_x u\|, \tag{3}$$

where $c_* > 0$ is a constant.

The main result of the present work is to establish a general decay rate of the energy, which is given by the following theorem.

Theorem 2.3. *Assume the assumptions (G1)-(G2) and $|\mu_2| < \sqrt{1-d}\mu_1$ hold. Let $U(0) = (u_0, u_1) \in \mathcal{D}^{1,2}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ and $f_0(x,t) \in L^2_\rho(\mathbb{R}^n \times (-\tau(0), 0))$, then there exist two constants $\beta > 0$ and $\gamma > 0$ such that the energy $E(t)$ defined by (7) satisfies*

$$E(t) \leq \beta e^{-\gamma \int_{t_0}^t \zeta(s) ds}, \ \forall t \geq 0. \tag{4}$$

for ant fixed $t_0 > 0$.

3. Proof of stability result

In this section, we show that problem (1), is uniformly exponentially stable using the multiplier technique. To achieve our goal, we need the following lemmas.

Lemma 3.1. *Under the assumptions of Theorem 2.3, the modified energy functional defined by (7) satisfies for any $t \geq 0$,*

$$\begin{aligned} E'(t) \leq & \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2}\right) \|u_t(t)\|^2_{L^2_\rho} \\ & + \left(\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2} e^{-\lambda \bar{\tau}} (1-d)\right) \int_{\mathbb{R}^n} \rho(x) u_t^2(t - \tau(t)) dx + \frac{1}{2}(g' \circ \nabla_x u)(t) \\ & - \frac{1}{2} g(t) \|\nabla_x u(t)\|_2^2 - \frac{\xi \lambda}{2} \int_{t-\tau(t)}^t \int_{\mathbb{R}^n} \rho(x) e^{-\lambda(t-s)} u_t^2(x,s) dx ds. \end{aligned} \tag{1}$$

Proof. Taking derivative of $E(t)$, we have

$$\begin{aligned}
E'(t) &= \int_{\mathbb{R}^n} \rho(x) u_t u_{tt} dx - \frac{1}{2} g(t) \|\nabla_x u\|_2^2 + \left(1 - \int_0^t g(s) ds\right) \int_{\mathbb{R}^n} \nabla_x u \nabla_x u_t dx \\
&+ \frac{1}{2}(g' \circ \nabla_x u) + \int_0^t g(t-s) \int_{\mathbb{R}^n} (\nabla_x u(t) - \nabla_x u(s)) \nabla_x u_t(t) dxds \\
&- \lambda \frac{\xi}{2} \int_{\mathbb{R}^n} \int_{t-\tau(t)}^t \rho(x) e^{\lambda(s-t)} u_s^2(x,s) dxds + \frac{\xi}{2} \int_{\mathbb{R}^n} \rho(x) u_t^2(x,t) dx \\
&- \frac{\xi}{2}(1 - \tau'(t)) e^{-\lambda \tau(t)} \int_{\mathbb{R}^n} \rho(x) u_t^2(x, t - \tau(t)) dx
\end{aligned}$$

By using equation (1) and integration by parts, we can easily get

$$\begin{aligned}
E'(t) &= \frac{1}{2}(g' \circ \nabla_x u) - \frac{1}{2} g(t) \|\nabla_x u\|_2^2 - \mu_1 \int_{\mathbb{R}^n} \rho(x) u_t^2 dx + \frac{\xi}{2} \|u_t\|_{L^2_\rho}^2 \\
&- \mu_2 \int_{\mathbb{R}^n} \rho(x) u_t u_t(x, t - \tau(t)) dx \\
&- \frac{\xi}{2}(1 - \tau'(t)) e^{-\lambda \tau(t)} \int_{\mathbb{R}^n} \rho(x) u_t^2(x, t - \tau(t)) dx \\
&- \lambda \frac{\xi}{2} \int_{\mathbb{R}^n} \int_{t-\tau(t)}^t \rho(x) e^{\lambda(s-t)} u_s^2(x,s) dxds.
\end{aligned}$$

By using Young's inequality, we can get

$$-\mu_2 \int_{\mathbb{R}^n} \rho u_t(t) \cdot u_t(t - \tau(t)) dx \leq \frac{|\mu_2|}{2\sqrt{1-d}} \|u_t\|_{L^2_\rho}^2 + \frac{|\mu_2|}{2} \sqrt{1-d} \int_{\mathbb{R}^n} \rho u_t^2(t - \tau(t)) dx,$$

which gives us (1). The proof is complete. □

Lemma 3.2. *Under the assumptions of Theorem 2.3, let (u, u_t) be the solution of problem (1). The functional $F_1(t)$ defined by*

$$\mathfrak{F}_1(t) = \int_{\mathbb{R}^n} \rho u u_t dx \tag{2}$$

satisfies that there exist a positive constants κ_1, κ_2 and κ_3 such that for any $t > 0$,

$$\mathfrak{F}_1'(t) \leq -\frac{l}{2} \|\nabla_x u(t)\|^2 + \kappa_1 \|u_t(t)\|_{L^2_\rho}^2 + \kappa_2 \int_{\mathbb{R}^n} \rho u_t^2(x, t - \tau(t)) dx + \kappa_3 (g \circ \nabla_x u)(t) \tag{3}$$

Proof. It is easy to get

$$\begin{aligned}
\mathfrak{F}_1'(t) &= \int_{\mathbb{R}^n} \rho u_t^2 dx + \int_{\mathbb{R}^n} u\Delta_x u dx - \int_{\mathbb{R}^n} u \int_0^t g(t-s)\Delta_x u(s) ds dx \\
&\quad -\mu_1 \int_{\mathbb{R}^n} \rho u u_t dx - \mu_2 \int_{\mathbb{R}^n} \rho u u_t \left(x, t-\tau(t)\right) dx \\
&= \int_{\mathbb{R}^n} \rho u_t^2 dx - \int_{\mathbb{R}^n} |\nabla_x u|^2 dx + \int_{\mathbb{R}^n} \nabla_x u(t) \int_0^t g(t-s)\nabla_x u(s) ds dx \\
&\quad -\mu_1 \int_{\mathbb{R}^n} \rho u u_t dx - \mu_2 \int_{\mathbb{R}^n} \rho u u_t \left(x, t-\tau(t)\right) dx \\
&= \int_{\mathbb{R}^n} \rho u_t^2 dx + \left(\int_0^t g(s) ds - 1\right) \|\nabla_x u\|^2 - \mu_1 \int_{\mathbb{R}^n} \rho u u_t dx \\
&\quad + \int_{\mathbb{R}^n} \nabla_x u(t) \int_0^t g(t-s) \left(\nabla_x u(s) - \nabla_x u(t)\right) ds dx \\
&\quad -\mu_2 \int_{\mathbb{R}^n} \rho u u_t \left(x, t-\tau(t)\right) dx
\end{aligned} \tag{4}$$

By using Young's and Hölder's inequalities, we arrive at for any $\varepsilon > 0$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla_x u(t) \int_0^t g(t-s) \left(\nabla_x u(s) - \nabla_x u(t)\right) ds dx \\
\leq\ & \varepsilon \int_{\mathbb{R}^n} |\nabla_x u|^2 dx + \frac{1}{4\varepsilon} \left(\int_0^t g(s) ds\right) (g \circ \nabla_x u) \\
\leq\ & \varepsilon \|\nabla_x u\|^2 + \frac{1-l}{4\varepsilon} (g \circ \nabla_x u).
\end{aligned} \tag{5}$$

By using the same calculations and (3)we have for any $\varepsilon > 0$

$$\left| -\mu_1 \int_{\mathbb{R}^n} \rho u u_t dx \right| \leq \mu_1 \varepsilon c_*^2 \|\nabla_x u\|^2 + \frac{\mu_1}{4\varepsilon} \|u_t\|_{L_\rho^2}^2 \tag{6}$$

$$\left| -\mu_2 \int_{\mathbb{R}^n} \rho u u_t \left(x, t-\tau(t)\right) \right| \leq \mu_2 \varepsilon c_*^2 \|\nabla_x u\|^2 + \frac{\mu_2}{4\varepsilon} \int_{\mathbb{R}^n} u_t^2 \left(x, t-\tau(t)\right) dx \tag{7}$$

Inserting (5)-(7) into (4), using Assumption (G1) and taking $\varepsilon > 0$ small enough, we can get (3) with

$$\kappa_1 = 1 + \frac{\mu_1}{4\varepsilon}, \kappa_2 = \frac{\mu_2}{4\varepsilon}, \kappa_3 = \frac{1-l}{4\varepsilon}.$$

The existence of viscoelastic term forces us to introduce the next Lemma.

Lemma 3.3. *Under the assumptions of Theorem 2.3, let (u, u_t) be the solution of problem(1). The functional $F_2(t)$ defined by*

$$\mathfrak{F}_2(t) = -\int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s) \left(u(t) - u(s)\right) ds dx \tag{8}$$

satisfies that there exists a positive constant κ_4 such that for any $\delta > 0$

$$\begin{aligned}
\mathfrak{F}_2'(t) &\leq \left(2\delta - \int_0^t g(s) ds\right) \|u_t(t)\|_{L_\rho^2}^2 + \left(\delta + 2\delta (1-l)^2\right) \|\nabla_x u(t)\|^2 + \kappa_4 \left(g \circ \nabla_x u\right)(t) \\
&\quad - \frac{g(0) c_*^2}{4\delta} \left(g' \circ \nabla_x u\right)(t) + \delta \int_{\mathbb{R}^n} \rho u_t^2 \left(x, t-\tau(t)\right) dx
\end{aligned} \tag{9}$$

Proof. We derive $\mathfrak{F}_2(t)$ and use (1) to obtain

$$\begin{aligned}
\mathfrak{F}_2'(t) &= -\int_{\mathbb{R}^n} \Delta_x u \int_0^t g(t-s)\,(u(t)-u\,(s))\,dsdx \\
&\quad + \int_{\mathbb{R}^n} \left(\int_0^t g(s)\Delta_x u(s)ds\right)\left(\int_0^t g(t-s)\,(u(t)-u\,(s))\,ds\right)dx \\
&\quad +\mu_1 \int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s)\,(u(t)-u\,(s))\,dsdx \\
&\quad +\mu_2 \int_{\mathbb{R}^n} \rho u_t\,(x,t-\tau\,(t)) \int_0^t g(t-s)\,(u(t)-u\,(s))\,dsdx \\
&\quad -\int_{\mathbb{R}^n} \rho u_t \int_0^t g'(t-s)\,(u(t)-u\,(s))\,dsdx - \int_0^t g(s)ds.\,\|u_t\|^2_{L^2_\rho} \qquad (10)
\end{aligned}$$

Using integration by parts, Young's inequality and Hölder's inequality, we have for any $\delta > 0$,

$$\begin{aligned}
&\left|-\int_{\mathbb{R}^n} \Delta_x u \int_0^t g(t-s)\,(\nabla_x u(s)-\nabla_x u\,(t))\,dsdx\right| \\
\leq\;& \delta\,\|\nabla_x u\|^2 + \frac{1-l}{4\delta}\,(g\circ\nabla_x u) \qquad (11)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \left(\int_0^t g(t-s)\,(u(t)-u\,(s))\,ds\right).\left(\int_0^t g(t-s)\Delta_x u(s)ds\right)dx\right| \\
=\;& \left|-\int_{\mathbb{R}^n} \left(\int_0^t g(t-s)\,(\nabla_x u(t)-\nabla_x u\,(s))\,ds\right).\left(\int_0^t g(s)\nabla_x u(s)ds\right)dx\right| \\
\leq\;& \delta \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)\nabla_x u(s)ds\right)^2 dx + \frac{1}{4\delta}\int_{\mathbb{R}^n} \left(\int_0^t g(t-s)\,(\nabla_x u(t)-\nabla_x u\,(s))\,ds\right)^2 dx \\
\leq\;& 2\delta \int_{\mathbb{R}^n} \left[\left(\int_0^t g(t-s)\,(\nabla_x u(t)-\nabla_x u\,(s))\,ds\right)^2 + \left(\int_0^t g(t-s)\nabla_x u(s)ds\right)^2\right]dx \\
&+\frac{1}{4\delta}\int_{\mathbb{R}^n} \left(\int_0^t g(t-s)\,(\nabla_x u(t)-\nabla_x u\,(s))\,ds\right)^2 dx \\
\leq\;& \left(2\delta+\frac{1}{4\delta}\right)\left(\int_0^t g(s)ds\right)(g\circ\nabla_x u) + 2\delta\left(\int_0^t g(s)ds\right)^2 \|\nabla_x u\|^2 \\
\leq\;& \left(2\delta+\frac{1}{4\delta}\right)(1-l)\,(g\circ\nabla_x u) + 2\delta\,(1-l)^2\,\|\nabla_x u\|^2 \qquad (12)
\end{aligned}$$

$$\begin{aligned}
&\left|\mu_1 \int_{\mathbb{R}^n} \rho u_t \int_0^t g(t-s)\,(u(t)-u\,(s))\,dsdx\right| \\
\leq\;& \delta\,\|u_t\|^2_{L^2_\rho} + \frac{c_*^2}{4\delta}\,(g\circ\nabla_x u) \qquad (13)
\end{aligned}$$

$$\begin{aligned}
&\left|\mu_2 \int_{\mathbb{R}^n} \rho u_t\,(x,t-\tau\,(t)) \int_0^t g(t-s)\,(u(t)-u\,(s))\,dsdx\right| \\
\leq\;& \delta \int_{\mathbb{R}^n} \rho u_t^2\,(x,t-\tau\,(t))\,dx + \frac{c_*^2}{4\delta}\,(g\circ\nabla_x u)
\end{aligned}$$

and

$$\begin{aligned} & \left| -\int_{\mathbb{R}^n} \rho u_t \int_0^t g'(t-s)\left(u(t)-u(s)\right) dsdx \right| \\ \leq & \ \delta \left\|u_t\right\|_{L^2_\rho}^2 - \frac{g(0)c_*^2}{4\delta}\left(g' \circ \nabla_x u\right) \end{aligned} \tag{14}$$

Combining (11)-(14) with (10), we can obtain (9) with

$$\kappa_4 = (1-l)\left[\left(2\delta + \frac{1}{4\delta}\right) + \frac{1}{4\delta}\right] + \frac{c_*^2}{2\delta}$$

The proof of the Lemma is complete.

Define the Lyapunov functional

$$\mathfrak{L}(t) = E(t) + N_1 \mathfrak{F}_1(t) + N_2 \mathfrak{F}_2(t) \tag{15}$$

where, N_1 and N_2 are positive constants that will be fixed later.

Lemma 3.4. *For $N_1 > 0$ and $N_2 > 0$ small enough, we have*

$$\frac{1}{2}E(t) \leq \mathfrak{L}(t) \leq 2E(t) \tag{16}$$

Proof. By using Hölder's inequality, Young's inequality and making use of the above Lemmas, and (3) we obtain for any $\delta > 0$

$$\begin{aligned} |\mathfrak{L}(t) - E(t)| &\leq N_1 \int_{\mathbb{R}^n} \left|\rho u u_t dx\right| + N_2 \int_{\mathbb{R}^n} \left|\rho u_t \int_0^t g(t-s)\left(u(t)-u(s)\right) dsdx\right| \\ &\leq N_1 \left(\delta \left\|u_t\right\|_{L^2_\rho}^2 + \frac{c_*^2}{4\delta}\left\|\nabla_x u\right\|_{L^2_\rho}^2\right) + N_2 \left(\delta \left\|u_t\right\|_{L^2_\rho}^2 + \frac{c_*^2}{4\delta}(1-l)\left(g \circ \nabla_x u\right)\right) \\ &\leq \delta\left(N_1 + N_2\right)\left\|u_t\right\|_{L^2_\rho}^2 + \frac{N_1 c_*^2}{4\delta}\left\|\nabla_x u\right\|_{L^2_\rho}^2 + \frac{N_2 c_*^2}{4\delta}(1-l)\left(g \circ \nabla_x u\right) \end{aligned}$$

which implies us there exists a positive constant $\varepsilon > 0$ such that

$$|\mathfrak{L}(t) - E(t)| \leq \varepsilon E(t), \tag{17}$$

or

$$(1-\varepsilon)E(t) \leq \mathfrak{L}(t) \leq (1+\varepsilon)E(t) \tag{18}$$

when we choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough. The proof is complete.

Proof of Theorem 2.3. For any fixed $t_0 > 0$, we know that for any $t \geq t_0$,

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds := g_0. \tag{19}$$

Now we derive (15) and using (9), (3) and (1)

$$\begin{aligned}\mathfrak{L}'(t) \leq\ & \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} + N_1\kappa_1 + N_2(2\delta - g_0)\right)\|u_t\|^2_{L^2_\rho} \\ & + \left(\frac{1}{2} - N_2\frac{g(0)c_*^2}{4\delta}\right)(g' \circ \nabla_x u) + \left(N_2\left(\delta + 2\delta(1-l)^2\right) - N_1\frac{l}{2}\right)\|\nabla_x u\|^2 \\ & + \left(\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}e^{-\lambda\bar{\tau}}(1-d) + N_1\kappa_2 + N_2\delta\right)\int_{\mathbb{R}^n}\rho(x)\,u_t^2(x, t-\tau(t))\,dx \\ & - \frac{\lambda\xi}{2}\int_{\mathbb{R}^n}\int_{t-\tau(t)}^{t}\rho(x)\,e^{\lambda(s-t)}u_s^2(x,s)\,dxds \\ & + (N_1\kappa_3 + N_2\kappa_4)(g \circ \nabla_x u). \end{aligned} \tag{20}$$

We can easily get that $e^{\lambda\tau_1}$ goes to 1 as $\lambda \to 0^+$. Noting the continuity of the set of real numbers, we can take λ so small that there exists a positive constant ξ such that

$$\frac{e^{\lambda\tau_1}|\mu_2|}{\sqrt{1-d}} < \xi < \mu_1. \tag{21}$$

From (21) we infer that

$$\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} < 0, \tag{22}$$

and

$$\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2e^{\lambda\bar{\tau}}}(1-d) < 0. \tag{23}$$

We can choose $0 < \delta < \frac{g_0}{2}$ such that $(2\delta - g_0) < 0$. For any fixed $\delta < 0$, we at last choose N_2 and N_1 small enough so that

$$N_2 < \min\left\{\frac{2\delta}{g(0)c_*^2}, \frac{1}{\delta}\left(-\frac{|\mu_2|}{2}\sqrt{1-d} + \frac{\xi}{2}e^{-\lambda\bar{\tau}}(1-d)\right)\right\}, \tag{24}$$

and

$$\frac{2N_2}{l}\left(\delta + 2\delta(1-l)^2\right) < N_1 < \min\left\{\frac{N_2}{\kappa_1}(g_0 - 2\delta), -\frac{|\mu_2|}{2\kappa_2}\sqrt{1-d} + \frac{\xi}{2\kappa_2}e^{-\lambda\bar{\tau}}(1-d) - \frac{N_2\delta}{\kappa_2}\right\}, \tag{25}$$

which gives us

$$\frac{1}{2} - N_2\frac{g(0)c_*^2}{4\delta} > 0,\ \frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}e^{-\lambda\bar{\tau}}(1-d) + N_2\delta < 0, \tag{26}$$

$$N_1\kappa_1 + N_2(2\delta - g_0) < 0,\ N_2\left(\delta + 2\delta(1-l)^2\right) - N_1\frac{l}{2} < 0, \tag{27}$$

and

$$\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}e^{-\lambda\bar{\tau}}(1-d) + N_1\kappa_2 + N_2\delta < 0. \tag{28}$$

At this point it follows that there exist two positive constants γ_1 and γ_2 such that for any $t \geq t_0$,

$$\mathfrak{L}'(t) \leq -\gamma_1 E(t) + \gamma_2 (g \circ \nabla_x u). \tag{29}$$

We multiply (29) by $\zeta(t)$ which is $\zeta'(t) \leq 0$ and from (G2), we use

$$\zeta(t)(g \circ \nabla_x u) \leq -(g' \circ \nabla_x u) \leq -2E(t)$$

we obtain

$$\begin{aligned} \zeta(t)\mathfrak{L}'(t) &\leq -\gamma_1\zeta(t)E(t) + \gamma_2\zeta(t)(g \circ \nabla_x u) \\ &\leq -\gamma_1\zeta(t)E(t) - 2\gamma_2 E'(t). \end{aligned} \tag{30}$$

which implies

$$\zeta(t)\mathfrak{L}'(t) + 2\gamma_2 E'(t) \leq -\gamma_1\zeta(t)E(t). \tag{31}$$

We note $\mathcal{E}(t)$ such that

$$\mathcal{E}(t) = \zeta(t)\mathfrak{L}'(t) + 2\gamma_2 E'(t),$$

then $\mathcal{E}(t)$ is equivalent to the modified energy $E(t)$ by using (20), which implies there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq \mathcal{E}(t) \leq \beta_2 E(t). \tag{32}$$

By using (31) and (32), we infer that for any $t \geq t_0$,

$$\mathcal{E}'(t) \leq -\gamma_1\zeta(t)E(t) \leq -\frac{\gamma_1}{\beta_2}\zeta(t)\mathcal{E}(t), \tag{33}$$

we get

$$\mathcal{E}(t) \leq \mathcal{E}(t_0)e^{-\frac{\gamma_1}{\beta_2}\int_{t_0}^{t}\zeta(s)ds},$$

which implies

$$E(t) \leq \frac{\beta_2}{\beta_1}E(t_0)e^{-\frac{\gamma_1}{\beta_2}\int_{t_0}^{t}\zeta(s)ds}. \tag{34}$$

By renaming the constants, and by the continuity and boundedness of $E(t)$. This completes the proof of Theorem 2.3. □

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