Generalized Weighted Composition Operators from the Bloch-Type Spaces to the Weighted Zygmund Spaces

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Abstract. The boundedness and compactness of generalized weighted composition operators from Bloch-type spaces and little Bloch-type spaces into weighted Zygmund spaces on the unit disc are characterized, in this paper.

1. Introduction

Let $D$ be the open unit disc in the complex plane $\mathbb{C}$, $H(D)$ the class of all analytic functions on $D$, and $H^\infty = H^\infty(D)$ the space of bounded analytic functions on $D$ with the norm $\|f\|_\infty = \sup_{z \in D} |f(z)|$. For $0 < \alpha < \infty$, a function $f \in H(D)$ is said to be in the Bloch-type spaces $B^\alpha = B^\alpha(D)$, if

$$b_\alpha(f) = \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$ 

The space $B^\alpha$ becomes a Banach space under the norm $\|f\|_{B^\alpha} = |f(0)| + b_\alpha(f)$. The little Bloch-type space $B^\alpha_0$, is a subspace of $B^\alpha$, consisting of all $f \in H(D)$ such that

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$ 

When $\alpha = 1$, $B^1 = B$ is the well-known Bloch space, while $B^1_0 = B_0$ is the well-known little Bloch space. For more information on Bloch-type spaces see [12, 13].

Every positive and continuous function on $D$ is called a weight. Let $\mu(z)$ be a weight. The weighted Zygmund space $Z_\mu = Z_\mu(D)$ is the space of all analytic functions $f$ on $D$ such that

$$b_{W_\mu}(f) = \sup_{z \in D} \mu(z) |f''(z)| < \infty.$$ 

The space $Z_\mu$ becomes a Banach space with the following norm

$$\|f\|_{Z_\mu} = |f(0)| + |f'(0)| + b_{W_\mu}(f).$$
When \( \mu(z) = (1 - |z|^2) \), \( \mathcal{Z}_\mu = \mathcal{Z} \) is the well-known Zygmund space. More information on the Zygmund-type space on the unit disc or the unit ball, can be found in [9, 14, 17].

Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( u \in H(\mathbb{D}) \). The weighted composition operator \( uC_\varphi \), which induced by \( \varphi \) and \( u \), is defined as follows

\[
(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.
\]

If \( u(z) \equiv 1 \), then the weighted composition operator is reduced to the composition operator, usually denoted by \( C_\varphi \), while for \( \varphi(z) = z \), it is reduced to the multiplication operator, usually denoted by \( M_u \).

Let \( D \) be the differentiation operator and \( n \) be a nonnegative integer. Write

\[
Df = f', \quad D^n f = f^{(n)}, \quad f \in H(\mathbb{D}).
\]

The generalized weighted composition operator, denoted by \( D_n^{\varphi,n} \), is defined as follows (see [15, 16, 18, 20])

\[
(D_n^{\varphi,n} f)(z) = u(z)f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.
\]

When \( n = 0 \), then \( D_n^{\varphi,n} \) becomes the weighted composition operator. If \( n = 0 \) and \( u(z) = 1 \), then \( D_n^{\varphi,n} = C_\varphi \).

If \( n = 1 \) and \( u(z) = \varphi'(z) \), then \( D_n^{\varphi,n} = DC_\varphi \). If \( n = 1 \) and \( u(z) = 1 \), then \( D_n^{\varphi,n} = C_\varphi D \).

The operators \( DC_\varphi \) and \( C_\varphi D \) were studied in [2, 5, 6, 8, 11].

Stević in [10] has found some characterizations for boundedness and compactness of generalized weighted composition operators \( D_n^{\varphi,n} \) from \( H^\infty \) and Bloch space to \( n \)th weighted-type spaces on the unit disc. In addition, Zhu in [19, 21] has found some characterizations for boundedness and compactness of operator \( D_n^{\varphi,n} \) from \( \mathcal{B} \) to \( H^\infty \) and \( \mathcal{B}^a \) to \( \mathcal{B}^b \).

Li and Stević in [7] provide some results for boundedness and compactness of \( D_n^{\varphi,n} \) from \( \mathcal{B}^a \) to \( H^\infty \). In this paper, inspired by previous works, we attempt to study boundedness and compactness of generalized weighted composition operators from \( \mathcal{B}^a \) to \( \mathcal{Z}_\mu \).

Throughout this paper, \( C \) is used to denote a positive constant which may differ from one occurrence to the other. We say that \( A \leq B \) if there exists a constant \( C \) such that \( A \leq CB \). The symbol \( A \approx B \) means that \( A \leq B \leq A \).

\section{Boundedness of \( D_n^{\varphi,n} : \mathcal{B}^a(\mathcal{B}^2_0) \rightarrow \mathcal{Z}_\mu \)}

In this section, we give some characterizations for boundedness of generalized weighted composition operators from the Bloch-type spaces into the weighted Zygmund spaces.

For \( a \in \mathbb{D} \) and \( 0 < a < \infty \), set

\[
f_a(z) = \frac{1 - |a|^2}{(1 - az)^2}, \quad h_a(z) = \frac{(1 - |a|^2)^2}{(1 - az)^{3+1}}, \quad g_a(z) = \frac{(1 - |a|^2)^3}{(1 - az)^{5+2}}, \quad z \in \mathbb{D}.
\]

We have

\[
f_a^{(n)}(z) = \frac{(1 - |a|^2)a^n}{(1 - az)^{n+1}} \prod_{j=0}^{n-1} (\alpha + j),
\]

\[
h_a^{(n)}(z) = \frac{(1 - |a|^2)^2a^n}{(1 - az)^{n+1}} \prod_{j=1}^{n} (\alpha + j),
\]

\[
g_a^{(n)}(z) = \frac{(1 - |a|^2)^3a^n}{(1 - az)^{n+2}} \prod_{j=2}^{n+1} (\alpha + j).
\]
By using \(f_a\), \(h_a\) and \(g_a\), for any \(n \in \mathbb{N}\) we define \(m_{n,a}\), \(l_{n,a}\) and \(k_{n,a}\) as follows

\[
m_{n,a}(z) = f_a(z) - \frac{\alpha(2\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} h_a(z) + \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} g_a(z),
\]

\[
l_{n,a}(z) = f_a(z) - \frac{2\alpha}{\alpha + n + 1} h_a(z) + \frac{\alpha(\alpha + 1)}{(\alpha + n + 1)(\alpha + n + 2)} g_a(z),
\]

\[
k_{n,a}(z) = f_a(z) - \frac{2\alpha}{\alpha + n} h_a(z) + \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 1)} g_a(z).
\]

**Lemma 2.1.** For any \(a \in \mathbb{D}\) and \(n \in \mathbb{N}\), \(m_{n,a}^{(n)}(a) = m_{n,a}^{(n+2)}(a) = 0\) and

\[
m_{n,a}^{(n+1)}(a) = -\frac{\alpha^{n+1}}{(\alpha + n + 2)(1 - |a|^2)^{\alpha + n}} \prod_{j=0}^{n-1} (\alpha + j).
\]

**Proof.**

\[
m_{n,a}^{(n)}(a) = \frac{\alpha^n}{(1 - |a|^2)^{\alpha + n - 1}} \prod_{j=0}^{n-1} (\alpha + j) - \frac{\alpha^n}{(1 - |a|^2)^{\alpha + n - 1}} \frac{\alpha(2\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} \prod_{j=1}^{n} (\alpha + j)
\]

\[+ \frac{\alpha^n}{(1 - |a|^2)^{\alpha + n - 1}} \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} \prod_{j=2}^{n+1} (\alpha + j)
\]

\[= \frac{\alpha^n}{(1 - |a|^2)^{\alpha + n - 1}} \prod_{j=0}^{n-1} (\alpha + j)(1 - \frac{2\alpha + 2n + 3}{\alpha + n + 2} + \frac{\alpha + n + 1}{\alpha + n + 2})
\]

\[= 0,
\]

\[
m_{n,a}^{(n+1)}(a) = \frac{\alpha^{n+1}}{(1 - |a|^2)^{\alpha + n}} \prod_{j=0}^{n} (\alpha + j) - \frac{\alpha^{n+1}}{(1 - |a|^2)^{\alpha + n}} \frac{\alpha(2\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} \prod_{j=1}^{n+1} (\alpha + j)
\]

\[+ \frac{\alpha^{n+1}}{(1 - |a|^2)^{\alpha + n}} \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} \prod_{j=2}^{n+2} (\alpha + j)
\]

\[= \frac{\alpha^{n+1}}{(1 - |a|^2)^{\alpha + n}} \prod_{j=0}^{n-1} (\alpha + j)(\alpha + n - \frac{(2\alpha + 2n + 3)(\alpha + n + 1)}{\alpha + n + 2} + \alpha + n + 1)
\]

\[= -\frac{\alpha^{n+1}}{(\alpha + n + 2)(1 - |a|^2)^{\alpha + n}} \prod_{j=0}^{n-1} (\alpha + j)
\]

and

\[
m_{n,a}^{(n+2)}(a) = \frac{\alpha^{n+2}}{(1 - |a|^2)^{\alpha + n + 1}} \prod_{j=0}^{n+1} (\alpha + j) - \frac{\alpha^{n+2}}{(1 - |a|^2)^{\alpha + n + 1}} \frac{\alpha(2\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} \prod_{j=1}^{n+2} (\alpha + j)
\]

\[+ \frac{\alpha^{n+2}}{(1 - |a|^2)^{\alpha + n + 1}} \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} \prod_{j=2}^{n+3} (\alpha + j)
\]

\[= \frac{\alpha^{n+2}}{(1 - |a|^2)^{\alpha + n + 1}} \prod_{j=0}^{n+1} (\alpha + j)(1 - \frac{2\alpha + 2n + 3}{\alpha + n} + \frac{\alpha + n + 3}{\alpha + n})
\]

\[= 0.
\]
The proofs of the next lemmas are similar to the proof of Lemma 2.1 and are omitted.

**Lemma 2.2.** For any \( a \in \mathbb{D} \) and \( n \in \mathbb{N} \), \( l_{n,a}^{(n)}(a) = l_{n,a}^{(n+1)}(a) = 0 \) and

\[
l_{n,a}^{(n)}(a) = \frac{2a^n}{(\alpha + n + 1)(\alpha + n + 2)(1 - |a|^2)^{n+1}} \prod_{j=0}^{n-1} (\alpha + j).
\]

**Lemma 2.3.** For any \( a \in \mathbb{D} \) and \( n \in \mathbb{N} \), \( k_{n,a}^{(n)}(a) = k_{n,a}^{(n+1)}(a) = 0 \) and

\[
k_{n,a}^{(n+1)}(a) = \frac{2a^{n+2}}{(1 - |a|^2)^{n+1}} \prod_{j=0}^{n-1} (\alpha + j).
\]

**Theorem 2.4.** Let \( n \) be a positive integer, \( 0 < \alpha < \infty \), \( u \in H(\mathbb{D}) \), \( \mu \) be a weight and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then the following statements are equivalent.

(a) The operator \( D_{\varphi,\mu}^n : \mathcal{B}_{\varphi} \rightarrow \mathcal{Z}_\mu \) is bounded.
(b) The operator \( D_{\varphi,\mu}^n : \mathcal{B}_0 \rightarrow \mathcal{Z}_\mu \) is bounded.
(c) \( \sup_{j \geq n} \beta_{j-1}||D_{\varphi,\mu}^n||_{\mathcal{B}_0} < \infty \), where \( p_j(z) = z^j \).
(d) \( u \in \mathcal{Z}_\mu \), \( \sup_{z \in \mathbb{D}} \mu(z) | u(z) | | q'v(z) |^2 < \infty \), \( \sup_{z \in \mathbb{D}} \mu(z) | 2q'v(z)u'(z) + q''v(z)u(z) | < \infty \) and

\[
\sup_{z \in \mathbb{D}} |D_{\varphi,\mu}^n f_n|_{\mathcal{Z}_\mu} < \infty, \quad \sup_{z \in \mathbb{D}} |D_{\varphi,\mu}^n h_n|_{\mathcal{Z}_\mu} < \infty, \quad \sup_{z \in \mathbb{D}} |D_{\varphi,\mu}^n g_n|_{\mathcal{Z}_\mu} < \infty,
\]

where \( f_n, h_n \) and \( g_n \) are defined in (1).

(e)

\[
\sup_{z \in \mathbb{D}} \mu(z) \left| \frac{2q'v(z)u'(z) + q''v(z)u(z)}{(1 - |q(z)|^2)^{n+1}} \right| < \infty, \quad \sup_{z \in \mathbb{D}} \mu(z) \left| \frac{u'(z)}{(1 - |q(z)|^2)^{n+1}} \right| < \infty, \quad \sup_{z \in \mathbb{D}} \mu(z) \left| \frac{q''v(z)}{(1 - |q(z)|^2)^{n+1}} \right| < \infty.
\]

**Proof.** (a) \( \Rightarrow \) (b) This implication is obvious.

(b) \( \Rightarrow \) (c) The sequence \( \{ p_{j-1} \}_{j=1}^{\infty} \) is bounded in \( \mathcal{B}_0 \) and \( \lim_{j \rightarrow \infty} \beta_{j-1}||p_j||_{\mathcal{B}_0} = (\frac{2}{\pi})^n \) (see Lemma 2.1 in [3]). Hence,

\[
\sup_{j \geq 1} \beta_{j-1}||D_{\varphi,\mu}^n p_j||_{\mathcal{B}_0} < \infty.
\]

Since for \( j < n \), \( D_{\varphi,\mu}^n p_j = 0 \), we obtain \( \sup_{j \geq n} \beta_{j-1}||D_{\varphi,\mu}^n p_j||_{\mathcal{Z}_\mu} < \infty \).

(c) \( \Rightarrow \) (d) Suppose (c) holds. Applying the operator \( D_{\varphi,\mu}^n \) for \( p_j \) with \( j = n, n + 1 \) and \( n + 2 \), we obtain

\[
(D_{\varphi,\mu}^n p_n)(z) = n!u(z), \quad (D_{\varphi,\mu}^n p_{n+1})(z) = (n+1)!q(z)u(z), \quad (D_{\varphi,\mu}^n p_{n+2})(z) = \frac{(n + 2)!}{2} q^2(z)u(z).
\]

Thus from (2), we have

\[
\sup_{z \in \mathbb{D}} \mu(z) |u''(z)| \leq \frac{1}{n!} ||D_{\varphi,\mu}^n p_n||_{\mathcal{Z}_\mu} < \infty.
\]

So, \( u \in \mathcal{Z}_\mu \). By using (2), we get

\[
\sup_{z \in \mathbb{D}} \mu(z) \left| q''v(z)u(z) + 2q'v(z)u'(z) + q(z)u''(z) \right| \leq \frac{1}{(n+1)!} ||D_{\varphi,\mu}^n p_{n+1}||_{\mathcal{Z}_\mu} < \infty.
\]

From the boundedness of the function \( \varphi \) and (3),

\[
\sup_{z \in \mathbb{D}} \mu(z) \left| q''v(z)u(z) + 2q'v(z)u'(z) \right| < \infty.
\]
By using (2),
\[ \sup_{z \in D} \mu(z) \left| 2q'(z)^2 u(z) + 2\left(q''(z)u(z) + 2q'(z)u'(z)\right)q(z) + q^2(z)u''(z) \right| \leq \frac{2}{(n + 2)!} \| D^n_{q,p} \|_{z_p} < \infty. \]

Finally, from boundedness of the function \( q \), (3) and (4)
\[ \sup_{z \in D} \mu(z) |q'(z)|^2 |u(z)| < \infty. \]

We set \( Q := \sup_{\beta \leq 1} j^{-1} \| D^n_{q,p} \|_{z_p} \). For any \( a \in D \), it is easy to check that \( f_a, h_a \) and \( g_a \) are in \( \mathcal{B}^a \). By simple calculation, we obtain
\[
\begin{align*}
 f_a(z) &= (1 - |a|) \sum_{j=0}^{\infty} \frac{\Gamma(j + \alpha)}{\beta \Gamma(\alpha)} |\bar{a}|^j z^j, \\
 h_a(z) &= (1 - |a|)^2 \sum_{j=0}^{\infty} \frac{\Gamma(j + 1 + \alpha)}{\beta \Gamma(1 + \alpha)} |\bar{a}|^j z^j, \\
 g_a(z) &= (1 - |a|)^3 \sum_{j=0}^{\infty} \frac{\Gamma(j + 2 + \alpha)}{\beta \Gamma(2 + \alpha)} |\bar{a}|^j z^j.
\end{align*}
\]

From Stirling’s formula, we have \( \frac{\Gamma(j + \alpha)}{\beta \Gamma(\alpha)} = j^{\alpha - 1} \) as \( j \to \infty \). Using linearity, we get
\[
\begin{align*}
 \| D^n_{q,p} f_a \|_{z_p} &\leq C(1 - |a|) \sum_{j=0}^{\infty} \frac{\| D^n_{q,p} \|_{z_p}}{\beta} j^{-1} \| D^n_{q,p} \|_{z_p} \leq \frac{CQ(1 - |a|^2)}{1 - |a|} \leq 2CQ, \\
 \| D^n_{q,p} h_a \|_{z_p} &\leq C(1 - |a|)^2 \sum_{j=0}^{\infty} \frac{\| D^n_{q,p} \|_{z_p}}{\beta} j^{-1} \| D^n_{q,p} \|_{z_p} \leq \frac{CQ(1 - |a|^2)^2}{1 - |a|^2} \leq 4CQ, \\
 \| D^n_{q,p} g_a \|_{z_p} &\leq C(1 - |a|)^3 \sum_{j=0}^{\infty} \frac{\| D^n_{q,p} \|_{z_p}}{\beta} j^{-1} \| D^n_{q,p} \|_{z_p} \leq \frac{CQ(1 - |a|^2)^3}{1 - |a|^3} \leq 16CQ.
\end{align*}
\]

Since \( a \) is arbitrary, so
\[ \sup_{a \in D} \| D^n_{q,p} f_a \|_{z_p} < \infty, \quad \sup_{a \in D} \| D^n_{q,p} h_a \|_{z_p} < \infty \quad \text{and} \quad \sup_{a \in D} \| D^n_{q,p} g_a \|_{z_p} < \infty. \]

\((d) \Rightarrow (e)\) Assume that \((d)\) holds. Set
\[ C_1 = \sup_{a \in D} \| D^n_{q,p} f_a \|_{z_p}, \quad C_2 = \sup_{a \in D} \| D^n_{q,p} h_a \|_{z_p} \quad \text{and} \quad C_3 = \sup_{a \in D} \| D^n_{q,p} g_a \|_{z_p}. \]

It is obvious that for any \( a \in D \) and \( n \in \mathbb{N} \) the functions \( m_{n,a}, h_{n,a} \) and \( k_{n,a} \) are in \( \mathcal{B}^a \). Moreover
\[
\sup_{a \in D} \| D^n_{q,p} m_{n,a} \|_{z_p} \leq \sup_{a \in D} \| D^n_{q,p} f_a \|_{z_p} + \frac{a(\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} \sup_{a \in D} \| D^n_{q,p} h_a \|_{z_p} + \frac{a(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} \sup_{a \in D} \| D^n_{q,p} g_a \|_{z_p} \]
\[ \leq C_1 + \frac{a(\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} C_2 + \frac{a(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} C_3. \] 

Hence, for any \( \lambda \in D \) it follows from Lemma 2.1 and (8) that
\[ m^{(n)}_{n,\mu,\lambda}(\phi(\lambda)) = m^{(n+2)}_{n,\mu,\lambda}(\phi(\lambda)) = 0. \]
\[
\begin{align*}
\mu(\lambda) | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | & \leq \frac{\mu(\lambda)^{(n+1)}}{(1 - |\varphi(\lambda)|^2)^{n+1}} \prod_{j=0}^{n+1} (\alpha + j) \mu(\lambda) \left( |2u'(\lambda)\varphi'(\lambda) + u(\lambda)\varphi''(\lambda)| \right) \|D^n_{\varphi,u,n}\|_{\mathcal{Z}_\varphi} \leq C_1 + \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} C_3.
\end{align*}
\]

So,
\[
\mu(\lambda) | \varphi(\lambda)|^{n+1} | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | \leq \frac{\alpha + n + 2}{\alpha(\alpha + 1) \cdots (\alpha + n)} (C_1 + \frac{\alpha(2\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} C_2 + \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} C_3).
\]

Therefore,
\[
\sup_{\lambda \in \mathbb{D}} \mu(\lambda) | \varphi(\lambda)|^{n+1} | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | < \infty.
\]

For any fixed \( r \in (0, 1) \) from (10), we obtain
\[
\sup_{|\varphi(\lambda)| > r} \mu(\lambda) | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | \leq \frac{1}{r^{n+1}} \sup_{|\varphi(\lambda)| > r} \mu(\lambda) | \varphi(\lambda)|^{n+1} | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | < \infty.
\]

On the other hand from (d),
\[
\sup_{|\varphi(\lambda)| \leq r} \mu(\lambda) | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | \leq \sup_{|\varphi(\lambda)| \leq r} \mu(\lambda) | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | < \infty.
\]

For any \( \sigma \in \mathbb{D} \),
\[
\|D^n_{\varphi,u,n}\|_{\mathcal{Z}_\varphi} \leq C_1 + \frac{2\alpha}{\alpha + n + 1} C_2 + \frac{\alpha(\alpha + 1)}{(\alpha + n + 1)(\alpha + n + 2)} C_3.
\]

So, \( \sup_{\sigma \in \mathbb{D}} \|D^n_{\varphi,u,n}\|_{\mathcal{Z}_\varphi} < \infty \). From Lemma 2.2,
\[
\left( T^{n+1}_{u(\lambda),\varphi(\lambda)} \right)(\varphi(\lambda)) = 0.
\]

Hence, by a similar calculation as in (9), we obtain
\[
\begin{align*}
\mu(\lambda) | \varphi(\lambda)|^{n+1} | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | \leq \frac{\alpha(\alpha + 1)}{(\alpha + n + 1)(\alpha + n + 2)} C_3.
\end{align*}
\]

Therefore,
\[
\sup_{\lambda \in \mathbb{D}} \mu(\lambda) | \varphi(\lambda)|^{n+1} | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda) u(\lambda) | < \infty.
\]
Moreover, and with the similar calculation as in (9), we have

\[ \sup_{\lambda \in \mathcal{D}} \frac{\mu(\lambda) | u''(\lambda) |}{(1 - | \varphi(\lambda) | ^2)^{a+n-1}} < \infty. \]

For any \( a \in \mathbb{D} \),

\[ \|D^n_{\varphi,a}k_{n,a}\|_{Z_\varphi} \leq C_1 + \frac{2a}{\alpha + n} C_2 + \frac{a(\alpha + 1)}{(\alpha + n)(\alpha + n + 1)} C_3. \]

(16)

Hence, \( \sup_{\varphi \in \mathbb{D}} \| D^n_{\varphi,a} k_{n,a} \|_{Z_\varphi} < \infty \). For any \( \lambda \in \mathbb{D} \), from Lemma 2.3

\[ k^{(n)}_{n,\varphi(\lambda)}(\varphi(\lambda)) = k^{(n+1)}_{n,\varphi(\lambda)}(\varphi(\lambda)) = 0 \]

and with the similar calculation as in (9), we have

\[ \frac{2\mu(\lambda) | \varphi(\lambda) | ^{n+2} | \varphi'(\lambda) | ^2 | u(\lambda) |}{(1 - | \varphi(\lambda) | ^2)^{a+n+1}} \leq C_1 + \frac{2a}{\alpha + n} C_2 + \frac{a(\alpha + 1)}{(\alpha + n)(\alpha + n + 1)} C_3. \]

(17)

Thus,

\[ \sup_{\lambda \in \mathcal{D}} \frac{\mu(\lambda) | \varphi'(\lambda) | ^2 | u(\lambda) |}{(1 - | \varphi(\lambda) | ^2)^{a+n+1}} < \infty. \]

(18)

From (d) and (18) with similar calculation as in (11) and (12), we obtain

\[ \sup_{\lambda \in \mathcal{D}} \frac{\mu(\lambda) | \varphi'(\lambda) | ^2 | u(\lambda) |}{(1 - | \varphi(\lambda) | ^2)^{a+n+1}} < \infty. \]

(e) \( \Rightarrow \) (a) Assume (e) holds. For any \( f \in \mathcal{B} \),

\[
\mu(z) \left| (D^n_{\varphi,a} f)'(z) \right| = \mu(z) \left| f^{(n+2)}(\varphi(z)) \varphi'(2z)u(z) + f^{(n+1)}(\varphi(z))(2\varphi'(z)u'(z) + \varphi''(z)u(z)) + f^{(n)}(\varphi(z))u''(z) \right|
\]

\[ \leq \mu(z) \left| f^{(n+2)}(\varphi(z)) \right| \left| \varphi'(2z)u(z) \right| + \mu(z) \left| f^{(n+1)}(\varphi(z)) \right| \left| 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right| + \mu(z) \left| f^{(n)}(\varphi(z)) \right| \left| u''(z) \right|
\]

\[ \leq \frac{\mu(z) | \varphi'(\lambda) | ^2 | u(\lambda) |}{(1 - | \varphi(\lambda) | ^2)^{a+n+1}} C \| f \|_{\mathcal{B}_a} + \frac{\mu(z) | 2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda) |}{(1 - | \varphi(\lambda) | ^2)^{a+n}} C \| f \|_{\mathcal{B}_a} + \frac{\mu(z) | u''(\lambda) |}{(1 - | \varphi(\lambda) | ^2)^{a+n-1}} C \| f \|_{\mathcal{B}_a}.
\]

(19)

In the last inequality we use the fact that Proposition 8 in [13]) for \( f \in \mathcal{B} \)

\[ \sup_{-1 \leq z \leq 1} | z | | f'(z) | = | f(0) | + \cdots + | f^{(n)}(0) | + \sup_{-1 \leq z \leq 1} | z | | f^{(n+1)}(z) |.
\]

(20)

Moreover,

\[
(D^n_{\varphi,a}f)(0) = \left| \frac{f^{(n)}(\varphi(0))u(0)}{(1 - | \varphi(0) | ^2)^{a+n-1}} C \| f \|_{\mathcal{B}_a}\right|
\]

\[
(D^n_{\varphi,a}f)'(0) = \left| \frac{f^{(n+1)}(\varphi(0))\varphi'(0)u(0) + f^{(n)}(\varphi(0))u'(0)}{(1 - | \varphi(0) | ^2)^{a+n}} C \| f \|_{\mathcal{B}_a} + \left| \frac{u'(0)}{(1 - | \varphi(0) | ^2)^{a+n-1}} C \| f \|_{\mathcal{B}_a}\right|
\]

(21)

From (e), (19) and (21), we conclude that the operator \( D^n_{\varphi,a} : \mathcal{B} \rightarrow \mathcal{Z}_\mu \) is bounded. The proof is complete.
In this section, we obtain several characterizations for compactness of generalized weighted composition operators from the Bloch-type spaces into the weighted Zygmund spaces. To study compactness, we need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [1]).

**Lemma 3.1.** Let \( n \) be a positive integer, \( 0 < \alpha < \infty \), \( u \) be a weight, \( u \in H(\mathbb{D}) \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then \( D^n_{\varphi,u} : \mathcal{B}^\alpha \to Z_\mu \) is compact if and only if \( D^n_{\varphi,u} : \mathcal{B}^\alpha \to Z_\mu \) is bounded and for any bounded sequence \( (f_k)_{k \in \mathbb{N}} \) in \( \mathcal{B}^\alpha \), which converges to zero uniformly on compact subsets of \( \mathbb{D} \),

\[
\lim_{k \to \infty} \|D^n_{\varphi,u}f_k\|_{Z_\mu} = 0.
\]

**Theorem 3.2.** Let \( n \) be a positive integer, \( 0 < \alpha < \infty \), \( u \in H(\mathbb{D}) \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( D^n_{\varphi,u} : \mathcal{B}^\alpha \to Z_\mu \) is bounded. Then the following statements are equivalent.

(a) The operator \( D^n_{\varphi,u} : \mathcal{B}^\alpha \to Z_\mu \) is compact.

(b) The operator \( D^n_{\varphi,u} : \mathcal{B}^\alpha_0 \to Z_\mu \) is compact.

(c) \( \lim_{j \to \infty} j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu} = 0 \).

(d) \( \lim_{|\varphi(z)| \to 1} \|D^n_{\varphi,u}f(\varphi(z))\|_{Z_\mu} = 0 \), \( \lim_{|\varphi(z)| \to 1} \|D^{n+1}_{\varphi,u}h(\varphi(z))\|_{Z_\mu} = 0 \), \( \lim_{|\varphi(z)| \to 1} \|D^n_{\varphi,u}g(\varphi(z))\|_{Z_\mu} = 0 \).

(e) \( \lim_{|\varphi(z)| \to 1} \frac{\mu(z)}{|\varphi(z)|} \left| 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right| = 0 \), \( \lim_{|\varphi(z)| \to 1} \frac{\mu(z)}{|\varphi(z)|^2} \left| u'(z) + \varphi'(z)u(z) \right| = 0 \), \( \lim_{|\varphi(z)| \to 1} \frac{\mu(z)}{|\varphi(z)|^3} \left| u(z) \right| = 0 \).

**Proof.** (a) \( \Rightarrow \) (b) This implication is clear.

(b) \( \Rightarrow \) (c) The sequence \( \{j^{\alpha-1}p_j\}_{j=1}^\infty \) is bounded in \( \mathcal{B}^\alpha_0 \) and converges to 0 uniformly on compact subsets of \( \mathbb{D} \).

By Lemma 3.1 it follows that \( \lim_{j \to \infty} j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu} = 0 \).

(c) \( \Rightarrow \) (d) Suppose (c) holds. Since for \( j < n \), \( D^n_{\varphi,u}p_j = 0 \), hence for given \( \epsilon > 0 \) there exists a positive integer \( N \geq n \), such that

\[
j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu} < \epsilon,
\]

for all \( j \geq N \). Also, from Theorem 2.4 (c), \( Q = \sup_{|z| < 1} j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu} < \infty \). Let \( \{z_k\}_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( \lim_{k \to \infty} |\varphi(z_k)| = 1 \). Similar to the proof of (5), there exists a constant \( C \) such that

\[
\|D^n_{\varphi,u}f(\varphi(z_k))\|_{Z_\mu} \leq C(1 - |\varphi(z_k)|^2) \sum_{j=0}^{\infty} |\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu} = C(1 - |\varphi(z_k)|^2) \sum_{j=N}^{\infty} |\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu}
\]

\[
+ C(1 - |\varphi(z_k)|^2) \sum_{j=N}^{\infty} |\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu} = C(1 - |\varphi(z_k)|^2) \sum_{j=N}^{\infty} \sum_{j=0}^{\infty} |\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{Z_\mu} \leq 2CQ(1 - |\varphi(z_k)|^N) + 2Ce.
\]

Since \( \lim_{k \to \infty} |\varphi(z_k)| = 1 \), so

\[
\lim_{k \to \infty} \|D^n_{\varphi,u}f(\varphi(z_k))\|_{Z_\mu} \leq 2Ce.
\]

Hence, \( \lim_{|\varphi(z)| \to 1} \|D^n_{\varphi,u}f(\varphi(z))\|_{Z_\mu} = 0 \), because \( \epsilon \) is an arbitrary positive number.

Notice that

\[
\sum_{j=0}^{N-1} (j + 1)r^j = \frac{1 - r^{N+1}}{1 - r^2}, \quad 0 \leq r < 1.
\]
Arguing as in the proof of (6), we get

\begin{align*}
\|D^n_{\psi,u}h_{\psi(z_k)}\|_{z_\nu} & \leq C(1 - |\psi(z_k)|^2) \sum_{j=0}^{\infty} |\psi(z_k)|^j \|D^n_{\psi,u}p_j\|_{z_\nu} \\
& \leq C(1 - |\psi(z_k)|^2) \left( \sum_{j=0}^{N-1} (j+1) |\psi(z_k)|^j \|D^n_{\psi,u}p_j\|_{z_\nu} + \sum_{j=N}^{\infty} (j+1) |\psi(z_k)|^j \|D^n_{\psi,u}p_j\|_{z_\nu} \right) \\
& \leq 4CQ \left( 1 - |\psi(z_k)|^N - N|\psi(z_k)|^N(1 - |\psi(z_k)|) \right) + 4Ce.
\end{align*}

Therefore,

\[ \lim_{k \to \infty} \|D^n_{\psi,u}h_{\psi(z_k)}\|_{z_\nu} \leq 4Ce \]

and arbitrariness of \( e \) gives us \( \lim_{\|\psi(\sigma)\|_1 \to 1} \|D^n_{\psi,u}h_{\psi(\sigma)}\|_{z_\nu} = 0 \).

Notice that

\[ \sum_{j=1}^{N} j^2 r^j = \frac{r(1 + r - (N + 1)^2r^N + (2N^2 + 2N - 1)r^{N+1} - N^2)}{(1 - r)^3}, \quad 0 \leq r < 1. \]

Similar to the proof of (7), we get

\begin{align*}
\|D^n_{\psi,u}g_{\psi(z_k)}\|_{z_\nu} & \leq C(1 - |\psi(z_k)|^2) \sum_{j=0}^{\infty} |\psi(z_k)|^j \|D^n_{\psi,u}p_j\|_{z_\nu} = \\
& \leq C(1 - |\psi(z_k)|^2) \left( \sum_{j=0}^{N-1} j^2 |\psi(z_k)|^j \|D^n_{\psi,u}p_j\|_{z_\nu} + \sum_{j=N}^{\infty} j^2 |\psi(z_k)|^j \|D^n_{\psi,u}p_j\|_{z_\nu} \right) \\
& \leq 8CQ |\psi(z_k)| \left( 1 + |\psi(z_k)| - (N + 1)^2 |\psi(z_k)|^N + (2N^2 + 2N - 1) \right) |\psi(z_k)|^{N+1} - N^2 |\psi(z_k)|^{N+2} + 16Ce.
\end{align*}

Therefore,

\[ \lim_{k \to \infty} \|D^n_{\psi,u}g_{\psi(z_k)}\|_{z_\nu} \leq 16Ce. \]

Since \( e \) is arbitrary, we obtain \( \lim_{\|\psi(\sigma)\|_1 \to 1} \|D^n_{\psi,u}g_{\psi(\sigma)}\|_{z_\nu} = 0 \).

(d) \( \Rightarrow \) (e) To prove (e), it is sufficient to prove that for any sequence \( \{z_k\} \in \mathbb{D} \) with \( \lim_{k \to \infty} |\psi(z_k)| = 1, \)

\[ \lim_{k \to \infty} \frac{\mu(z_k) |2\psi'(z_k)u'(z_k) + \psi''(z_k)u(z_k)|}{(1 - |\psi(z_k)|^2)^{a+n}} = 0, \quad \lim_{k \to \infty} \frac{\mu(z_k) |u''(z_k)|}{(1 - |\psi(z_k)|^2)^{a+n-1}} = 0, \quad \lim_{k \to \infty} \frac{\mu(z_k) |\psi'(z_k)|^2 |u(z_k)|}{(1 - |\psi(z_k)|^2)^{a+n+1}} = 0. \]

Let \( \{z_k\} \in \mathbb{D} \) be any sequence in \( \mathbb{D} \) such that \( \lim_{k \to \infty} |\psi(z_k)| = 1 \). Similar to the proof of (8), we obtain

\[ \lim_{k \to \infty} \|D^n_{\psi,u}m_{\sigma_p}(z_k)\|_{z_\nu} \leq \lim_{k \to \infty} \|D^n_{\psi,u}f_{\psi(z_k)}\|_{z_\nu} + \frac{\alpha(2\alpha + 2n + 3)}{(\alpha + n)(\alpha + n + 2)} \lim_{k \to \infty} \|D^n_{\psi,u}h_{\psi(z_k)}\|_{z_\nu} \]

\[ + \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} \lim_{k \to \infty} \|D^n_{\psi,u}g_{\psi(z_k)}\|_{z_\nu} = 0. \] (22)

From (22) and (9), we get

\[ \lim_{k \to \infty} \frac{\mu(z_k) |\psi(z_k)|^{p+1} |2\psi'(z_k)u'(z_k) + \psi''(z_k)u(z_k)|}{(1 - |\psi(z_k)|^2)^{a+n}} = 0. \]
Since \( \lim_{k \to \infty} |\varphi(z_k)| = 1 \), therefore
\[
\lim_{k \to \infty} \frac{\mu(z_k) | 2\varphi'(z_k)u'(z_k) + \varphi''(z_k)u(z_k) |}{(1 - |\varphi(z_k)|^2)^{a+n}} = 0.
\]

Similar to the proof of (13), we have
\[
\lim_{k \to \infty} ||D^n_{\varphi,u} f_{\varphi(z_k)}||_{Z_\mu} \leq \lim_{k \to \infty} ||D^n_{\varphi,u} f_{\varphi(z_k)}||_{Z_\mu} + \frac{2a}{\alpha + n} \lim_{k \to \infty} ||D^n_{\varphi,u} h_{\varphi(z_k)}||_{Z_\mu} + \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} \lim_{k \to \infty} ||D^n_{\varphi,u} g_{\varphi(z_k)}||_{Z_\mu} = 0.
\]

By using (14) and (23), we get
\[
\lim_{k \to \infty} \frac{\mu(z_k) | \varphi(z_k) |^2 | u'(z_k) |}{(1 - |\varphi(z_k)|^2)^{a+n-1}} = 0,
\]

since \( \lim_{k \to \infty} |\varphi(z_k)| = 1 \), from the above equation, we obtain
\[
\lim_{k \to \infty} \frac{\mu(z_k) | u''(z_k) |}{(1 - |\varphi(z_k)|^2)^{a+n}} = 0.
\]

Finally, similar to the proof (16)
\[
\lim_{k \to \infty} ||D^n_{\varphi,u} k_{\varphi(z_k)}||_{Z_\mu} \leq \lim_{k \to \infty} ||D^n_{\varphi,u} f_{\varphi(z_k)}||_{Z_\mu} + \frac{2a}{\alpha + n} \lim_{k \to \infty} ||D^n_{\varphi,u} h_{\varphi(z_k)}||_{Z_\mu} + \frac{\alpha(\alpha + 1)}{(\alpha + n)(\alpha + n + 2)} \lim_{k \to \infty} ||D^n_{\varphi,u} g_{\varphi(z_k)}||_{Z_\mu} = 0.
\]

So, by using (17) and (24),
\[
\lim_{k \to \infty} \frac{\mu(z_k) | \varphi(z_k) |^{a+2} | \varphi'(z_k) |^{2} | u(z_k) |}{(1 - |\varphi(z_k)|^2)^{a+n+1}} = 0.
\]

Since \( \lim_{k \to \infty} |\varphi(z_k)| = 1 \), we get
\[
\lim_{k \to \infty} \frac{\mu(z_k) | \varphi'(z_k) |^{2} | u(z_k) |}{(1 - |\varphi(z_k)|^2)^{a+n+1}} = 0.
\]

(e) \( \Rightarrow \) (a) Assume that \( (f_k)_{k \in \mathbb{N}} \) is a bounded sequence in \( B^n \) converging to 0 uniformly on compact subsets of \( D \). For any \( \epsilon > 0 \), there exists \( \delta \in (0, 1) \) such that
\[
\frac{\mu(z) | 2\varphi'(z)u'(z) + \varphi''(z)u(z) |}{(1 - |\varphi(z)|^2)^{a+n}} < \epsilon, \quad \frac{\mu(z) | u''(z) |}{(1 - |\varphi(z)|^2)^{a+n}} < \epsilon, \quad \frac{\mu(z) | \varphi'(z) |^{2} | u(z) |}{(1 - |\varphi(z)|^2)^{a+n+1}} < \epsilon
\]
when \( \delta < |\varphi(z)| < 1 \). Since \( D^n_{\varphi,u} : B^n \to Z_\mu \) is bounded, from Theorem 2.4, we have
\[
C_4 = \sup_{z \in \mathbb{D}} \mu(z) | u''(z) | < \infty,
\]
\[
C_5 = \sup_{z \in \mathbb{D}} \mu(z) | u(z) | | \varphi'(z) |^2 < \infty,
\]
\[
C_6 = \sup_{z \in \mathbb{D}} \mu(z) | 2\varphi'(z)u'(z) + \varphi''(z)u(z) | < \infty.
\]
Let \( V = \{ z \in \mathbb{D} : |\varphi(z)| \leq \delta \} \). From (20), (25) and (26), we obtain

\[
\sup_{z \in \mathbb{D}} \mu(z) | (D^n_{\psi, u} f_k)'(z) | \leq \sup_{z \in \mathbb{V}} \mu(z) | f^{(n+2)}_k(\varphi(z)) | + | \varphi'(z) |^2 | u(z) | \\
+ \sup_{z \in \mathbb{V}} \mu(z) | f^{(n+1)}_k(\varphi(z)) | 2 \varphi'(z) u'(z) + \varphi''(z) u(z) | \\
+ \sup_{z \in \mathbb{V}} \mu(z) | f^{(n)}_k(\varphi(z)) | u''(z) + C \sup_{z \in \mathbb{D}-V} \frac{\mu(z) | \varphi'(z) |^2 | u(z) |}{(1 - |\varphi(z)|^2)^n+1} \| f_k \|_{L^p} \\
+ C \sup_{z \in \mathbb{D}-V} \frac{\mu(z) | u''(z) |}{(1 - |\varphi(z)|^2)^n+2} \| f_k \|_{L^p} \\
\leq C_5 \sup_{z \in \mathbb{V}} | f^{(n+2)}_k(\varphi(z)) | + C_6 \sup_{z \in \mathbb{V}} | f^{(n+1)}_k(\varphi(z)) | + C_4 \sup_{z \in \mathbb{V}} | f^{(n)}_k(\varphi(z)) | + C \sup_{z \in \mathbb{V}} \| f_k \|_{L^p}.
\]

Hence,

\[
||D^n_{\psi, u} f_k||_{L^p} \leq C_5 \sup_{|z| \leq \delta} | f^{(n+2)}_k(\varphi(z)) | + C_6 \sup_{|z| \leq \delta} | f^{(n+1)}_k(\varphi(z)) | + C_4 \sup_{|z| \leq \delta} | f^{(n)}_k(\varphi(z)) | + C \sup_{|z| \leq \delta} \| f_k \|_{L^p} + ||u(0)|| \| f^{(n)}_k(\varphi(0)) ||
\]

\[
+ | f^{(n+1)}_k(\varphi(0)) || \varphi'(0) || u(0) || + | f^{(n)}_k(\varphi(0)) || u'(0) ||.
\]

\[\text{(27)}\]

Since \((f_k)_{k \in \mathbb{N}}\) converges to 0 uniformly on compact subsets of \( \mathbb{D} \), by Cauchy’s estimates so do the sequences \((f^{(n)}_k)_{k \in \mathbb{N}}\). From (27), letting \( k \to \infty \) and using the fact that \( \epsilon \) is an arbitrary positive number, we get

\[
\lim_{k \to \infty} ||D^n_{\psi, u} f_k||_{L^p} = 0.
\]

From Lemma 3.1, we deduce that the operator \( D^n_{\psi, u} : \mathcal{B}^n \to \mathcal{Z}_\mu \) is compact. \( \square \)

**Setting** \( \alpha = 1 \) and \( \mu(z) = 1 - |z|^2 \), in (1) and Theorems 2.4 and 3.2, we obtain the following corollaries.

**Corollary 3.3.** Let \( n \) be a positive integer, \( u \in H(\mathbb{D}) \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \). Then the following statements are equivalent.

(a) The operator \( D^n_{\psi, u} : \mathcal{B} \to \mathcal{Z} \) is bounded.

(b) The operator \( D^n_{\psi, u} : \mathcal{B}_0 \to \mathcal{Z} \) is bounded.

(c) \( \sup_{f \in \mathcal{B}_0} ||D^n_{\psi, u} f||_{L^p} \leq \alpha < \infty \), where \( p(z) = z^\alpha \).

(d) \( u \in \mathcal{Z}, \sup_{z \in \mathbb{D}} (1 - |z|^2) | u(z) | \ ||\varphi'(z) || < \infty \) and \( \sup_{z \in \mathbb{D}} (1 - |z|^2) | 2 \varphi'(z) u'(z) + \varphi''(z) u(z) | < \infty \) and

\[
\sup_{z \in \mathbb{D}} ||D^n_{\psi, u} f_k||_{L^p} < \infty, \quad \sup_{z \in \mathbb{D}} ||D^n_{\psi, u} h_k||_{L^p} < \infty, \quad \sup_{z \in \mathbb{D}} ||D^n_{\psi, u} g_k||_{L^p} < \infty
\]

where \( f_k, h_k \) and \( g_k \) are defined in (1).

(e) \( \sup_{z \in \mathbb{D}} (1 - |z|^2) | 2 \varphi'(z) u'(z) + \varphi''(z) u(z) | (1 - |\varphi(z)|^2)^n+2 < \infty \) and \( \sup_{z \in \mathbb{D}} (1 - |z|^2) | \varphi''(z) | (1 - |\varphi(z)|^2)^n+2 < \infty \).

**Corollary 3.4.** Let \( n \) be a positive integer, \( u \in H(\mathbb{D}) \) and \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( D^n_{\psi, u} : \mathcal{B} \to \mathcal{Z} \) is bounded. Then the following statements are equivalent.

(a) The operator \( D^n_{\psi, u} : \mathcal{B} \to \mathcal{Z} \) is compact.
(b) The operator \( D_{\phi,u}^n : B_0 \to \mathcal{Z} \) is compact.
(c) \( \lim_{j \to \infty} \| D_{\phi,u,\phi_0,\phi_j}^n \|_{\mathcal{Z}} = 0 \).
(d) \( \lim_{|r| \to 1} \| D_{\phi,u,\phi_0}^n \|_{\mathcal{Z}} = 0 \), \( \lim_{|r| \to 1} \| D_{\phi,u,\phi_0,\phi}^n \|_{\mathcal{Z}} = 0 \), \( \lim_{|r| \to 1} \| D_{\phi,u,\phi_0,\phi}^n \|_{\mathcal{Z}} = 0 \).
(e)

\[
\lim_{|r| \to 1} (1 - |z|^2) \left| \frac{2q'(z)(u'(z) + q''(z)u(z))}{(1 - |q(z)|^2)^{n+1}} \right| = 0,
\lim_{|r| \to 1} (1 - |z|^2) \left| \frac{u''(z)}{(1 - |q(z)|^2)^n} \right| = 0 \quad \text{and}
\lim_{|r| \to 1} (1 - |z|^2) \left| \frac{q'(z)}{(1 - |q(z)|^2)^{n+2}} \right| = 0.
\]

The equivalence of conditions (a), (b) and (d) of corollaries 3.3 and 3.4 was proved in [4]. Also Stević in [10] proved that the conditions (a), (b) and (e) of above two corollaries are equivalent.

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References