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Generalized Weighted Composition Operators from the Bloch-Type Spaces to the Weighted Zygmund Spaces

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Abstract. The boundedness and compactness of generalized weighted composition operators from Bloch-type spaces and little Bloch-type spaces into weighted Zygmund spaces on the unit disc are characterized, in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$, and $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} | f(z) |$. For $0 < \alpha < \infty$, a function $f \in H(\mathbb{D})$ is said to be in the Bloch-type spaces $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$, if

$$b_\alpha(f) = \sup_{z\in\mathbb{D}}(1 - | z |^2)^\alpha | f'(z) |< \infty.$$

The space $\mathcal{B}^\alpha$ becomes a Banach space under the norm $\|f\|_{\mathcal{B}^\alpha} =| f(0) | +b_\alpha(f)$. The little Bloch-type space $\mathcal{B}^\alpha_0$, is a subspace of $\mathcal{B}^\alpha$, consisting of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1}(1 - | z |^2)^\alpha | f'(z) |= 0.$$

When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space, while $\mathcal{B}^1_0 = \mathcal{B}_0$ is the well-known little Bloch space. For more information on Bloch-type spaces see [12, 13].
Every positive and continuous function on $\mathbb{D}$ is called a weight. Let $\mu(z)$ be a weight. The weighted Zygmund space $\mathcal{Z}_\mu = \mathcal{Z}_\mu(\mathbb{D})$ is the space of all analytic functions f on $\mathbb{D}$ such that

$$b_{\mathcal{W}^2_\mu}(f) = \sup_{z\in\mathbb{D}} \mu(z) | f''(z) |< \infty.$$

The space $\mathcal{Z}_\mu$ becomes a Banach space with the following norm

$$\|f\|_{\mathcal{Z}_\mu} =| f(0) | + | f'(0) | +b_{\mathcal{W}^2_\mu}(f).$$

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When $\mu(z) = (1 - |z|^2)$, $\mathcal{Z}_\mu = \mathcal{Z}$ is the well-known Zygmund space. More information on the Zygmund-type space on the unit disc or the unit ball, can be found in [9, 14, 17].
Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator uC_φ, which induced by φ and u, is defined as follows

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

If $u(z) \equiv 1$, then the weighted composition operator is reduced to the composition operator, usually denoted by C_φ, while for $\varphi(z) = z$, it is reduced to the multiplication operator, usually denoted by M_u.
Let D be the differentiation operator and n be a nonnegative integer. Write

$$Df = f', \quad D^n f = f^{(n)}, \quad f \in H(\mathbb{D}).$$

The generalized weighted composition operator, denoted by $D^n_{\varphi,u}$, is defined as follows (see [15, 16, 18, 20])

$$(D^n_{\varphi,u} f)(z) = u(z)f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

When $n = 0$, then $D^n_{\varphi,u}$ becomes the weighted composition operator. If $n = 0$ and $u(z) = 1$, then $D^n_{\varphi,u} = C_\varphi$. If $n = 1$ and $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$. If $n = 1$ and $u(z) = 1$, then $D^n_{\varphi,u} = C_\varphi D$. The operators DC_φ and $C_\varphi D$ were studied in [2, 5, 6, 8, 11]
Stević in [10] has found some characterizations for boundedness and compactness of generalized weighted composition operators $D^n_{\varphi,u}$ from H^∞ and Bloch space to nth weighted-type spaces on the unit disc. In addition, Zhu in [19, 21] has found some characterizations for boundedness and compactness of operator $D^n_{\varphi,u}$ from $\mathcal{B}$ to H^∞_α and $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$. Li and Stević in [7] provide some results for boundedness and compactness of $D^n_{\varphi,u}$ from $\mathcal{B}^\alpha$ to H^∞_μ. In this paper, inspired by previous works, we attempt to study boundedness and compactness of generalized weighted composition operators from $\mathcal{B}^\alpha$ to $\mathcal{Z}_\mu$.
Throughout this paper, C is used to denote a positive constant which may differ from one occurrence to the other. We say that $A \preceq B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \preceq B \preceq A$.

2. Boundedness of $D^n_{\varphi,u} : \mathcal{B}^\alpha(\mathcal{B}^\alpha_0) \to \mathcal{Z}_\mu$

In this section, we give some characterizations for boundedness of generalized weighted composition operators from the Bloch-type spaces into the weighted Zygmund spaces.

For $a \in \mathbb{D}$ and $0 < \alpha < \infty$, set

$$f_a(z) = \frac{1 - |a|^2}{(1 - \bar{a}z)^\alpha}, \quad h_a(z) = \frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{\alpha+1}}, \quad g_a(z) = \frac{(1 - |a|^2)^3}{(1 - \bar{a}z)^{\alpha+2}}, \quad z \in \mathbb{D}. \tag{1}$$

We have

$$f^{(n)}_a(z) = \frac{(1 - |a|^2)\bar{a}^n}{(1 - \bar{a}z)^{\alpha+n}} \prod_{j=0}^{n-1}(\alpha + j),$$

$$h^{(n)}_a(z) = \frac{(1 - |a|^2)^2\bar{a}^n}{(1 - \bar{a}z)^{\alpha+n+1}} \prod_{j=1}^{n}(\alpha + j),$$

$$g^{(n)}_a(z) = \frac{(1 - |a|^2)^3\bar{a}^n}{(1 - \bar{a}z)^{\alpha+n+2}} \prod_{j=2}^{n+1}(\alpha + j).$$

By using f_a, h_a and g_a, for any $n \in \mathbb{N}$ we define $m_{n,a}$, $l_{n,a}$ and $k_{n,a}$ as follows

$$m_{n,a}(z) = f_a(z) - \frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)}h_a(z) + \frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)}g_a(z),$$
$$l_{n,a}(z) = f_a(z) - \frac{2\alpha}{\alpha+n+1}h_a(z) + \frac{\alpha(\alpha+1)}{(\alpha+n+1)(\alpha+n+2)}g_a(z),$$
$$k_{n,a}(z) = f_a(z) - \frac{2\alpha}{\alpha+n}h_a(z) + \frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+1)}g_a(z).$$

Lemma 2.1. *For any $a \in \mathbb{D}$ and $n \in \mathbb{N}$, $m_{n,a}^{(n)}(a) = m_{n,a}^{(n+2)}(a) = 0$ and*

$$m_{n,a}^{(n+1)}(a) = -\frac{\bar{a}^{n+1}}{(\alpha+n+2)(1-|a|^2)^{\alpha+n}}\prod_{j=0}^{n-1}(\alpha+j).$$

Proof.

$$\begin{aligned} m_{n,a}^{(n)}(a) =& \frac{\bar{a}^n}{(1-|a|^2)^{\alpha+n-1}}\prod_{j=0}^{n-1}(\alpha+j) - \frac{\bar{a}^n}{(1-|a|^2)^{\alpha+n-1}}\frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)}\prod_{j=1}^{n}(\alpha+j)\\ &+\frac{\bar{a}^n}{(1-|a|^2)^{\alpha+n-1}}\frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)}\prod_{j=2}^{n+1}(\alpha+j)\\ =&\frac{\bar{a}^n}{(1-|a|^2)^{\alpha+n-1}}\prod_{j=0}^{n-1}(\alpha+j)\Big(1-\frac{2\alpha+2n+3}{\alpha+n+2}+\frac{\alpha+n+1}{\alpha+n+2}\Big)\\ =&0, \end{aligned}$$

$$\begin{aligned} m_{n,a}^{(n+1)}(a) =& \frac{\bar{a}^{n+1}}{(1-|a|^2)^{\alpha+n}}\prod_{j=0}^{n}(\alpha+j) - \frac{\bar{a}^{n+1}}{(1-|a|^2)^{\alpha+n}}\frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)}\prod_{j=1}^{n+1}(\alpha+j)\\ &+\frac{\bar{a}^{n+1}}{(1-|a|^2)^{\alpha+n}}\frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)}\prod_{j=2}^{n+2}(\alpha+j)\\ =&\frac{\bar{a}^{n+1}}{(1-|a|^2)^{\alpha+n}}\prod_{j=0}^{n-1}(\alpha+j)\Big(\alpha+n-\frac{(2\alpha+2n+3)(\alpha+n+1)}{\alpha+n+2}+\alpha+n+1\Big)\\ =&-\frac{\bar{a}^{n+1}}{(\alpha+n+2)(1-|a|^2)^{\alpha+n}}\prod_{j=0}^{n-1}(\alpha+j) \end{aligned}$$

and

$$\begin{aligned} m_{n,a}^{(n+2)}(a) =& \frac{\bar{a}^{n+2}}{(1-|a|^2)^{\alpha+n+1}}\prod_{j=0}^{n+1}(\alpha+j) - \frac{\bar{a}^{n+2}}{(1-|a|^2)^{\alpha+n+1}}\frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)}\prod_{j=1}^{n+2}(\alpha+j)\\ &+\frac{\bar{a}^{n+2}}{(1-|a|^2)^{\alpha+n+1}}\frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)}\prod_{j=2}^{n+3}(\alpha+j)\\ =&\frac{\bar{a}^{n+2}}{(1-|a|^2)^{\alpha+n+1}}\prod_{j=0}^{n+1}(\alpha+j)\Big(1-\frac{2\alpha+2n+3}{\alpha+n}+\frac{\alpha+n+3}{\alpha+n}\Big)\\ =&0. \end{aligned}$$

□

The proofs of the next lemmas are similar to the proof of Lemma 2.1 and are omitted.

Lemma 2.2. *For any $a \in \mathbb{D}$ and $n \in \mathbb{N}$, $l_{n,a}^{(n+1)}(a) = l_{n,a}^{(n+2)}(a) = 0$ and*

$$l_{n,a}^{(n)}(a) = \frac{2\bar{a}^n}{(\alpha+n+1)(\alpha+n+2)(1-|a|^2)^{\alpha+n-1}} \prod_{j=0}^{n-1}(\alpha+j).$$

Lemma 2.3. *For any $a \in \mathbb{D}$ and $n \in \mathbb{N}$, $k_{n,a}^{(n)}(a) = k_{n,a}^{(n+1)}(a) = 0$ and*

$$k_{n,a}^{(n+2)}(a) = \frac{2\bar{a}^{n+2}}{(1-|a|^2)^{\alpha+n+1}} \prod_{j=0}^{n-1}(\alpha+j).$$

Theorem 2.4. *Let n be a positive integer, $0 < \alpha < \infty$, $u \in H(\mathbb{D})$, μ be a weight and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) *The operator $D_{\varphi,u}^n : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded.*
(b) *The operator $D_{\varphi,u}^n : \mathcal{B}_0^\alpha \to \mathcal{Z}_\mu$ is bounded.*
(c) $\sup_{j\geq n} j^{\alpha-1}\|D_{\varphi,u}^n p_j\|_{\mathcal{Z}_\mu} < \infty$, *where $p_j(z) = z^j$.*
(d) $u \in \mathcal{Z}_\mu$, $\sup_{z\in\mathbb{D}} \mu(z)|u(z)||\varphi'(z)|^2 < \infty$, $\sup_{z\in\mathbb{D}} \mu(z)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| < \infty$ *and*

$$\sup_{a\in\mathbb{D}} \|D_{\varphi,u}^n f_a\|_{\mathcal{Z}_\mu} < \infty, \quad \sup_{a\in\mathbb{D}} \|D_{\varphi,u}^n h_a\|_{\mathcal{Z}_\mu} < \infty, \quad \sup_{a\in\mathbb{D}} \|D_{\varphi,u}^n g_a\|_{\mathcal{Z}_\mu} < \infty,$$

where f_a, h_a and g_a are defined in (1).
(e)

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|2\varphi'(z)u'(z)+\varphi''(z)u(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}} < \infty, \quad \sup_{z\in\mathbb{D}} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}} < \infty, \quad \sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)|^2|u(z)|}{(1-|\varphi(z)|^2)^{\alpha+n+1}} < \infty.$$

Proof. $(a) \Rightarrow (b)$ This implication is obvious.
$(b) \Rightarrow (c)$ The sequence $\{j^{\alpha-1}p_j\}_1^\infty$ is bounded in $\mathcal{B}_0^\alpha$ and $\lim_{j\to\infty} j^{\alpha-1}\|p_j\|_{\mathcal{B}^\alpha} = \left(\frac{2\alpha}{e}\right)^\alpha$ (see Lemma 2.1 in [3]). Hence,

$$\sup_{j\geq 1} j^{\alpha-1}\|D_{\varphi,u}^n p_j\|_{\mathcal{Z}_\mu} < \infty.$$

Since for $j < n$, $D_{\varphi,u}^n p_j = 0$, we obtain $\sup_{j\geq n} j^{\alpha-1}\|D_{\varphi,u}^n p_j\|_{\mathcal{Z}_\mu} < \infty$.
$(c) \Rightarrow (d)$ Suppose (c) holds. Applying the operator $D_{\varphi,u}^n$ for p_j with $j = n$, $n+1$ and $n+2$, we obtain

$$(D_{\varphi,u}^n p_n)(z) = n!u(z), \quad (D_{\varphi,u}^n p_{n+1})(z) = (n+1)!\varphi(z)u(z), \quad (D_{\varphi,u}^n p_{n+2})(z) = \frac{(n+2)!}{2}\varphi^2(z)u(z). \tag{2}$$

Thus from (2), we have

$$\sup_{z\in\mathbb{D}} \mu(z)|u''(z)| \leq \frac{1}{n!}\| D_{\varphi,u}^n p_n \|_{\mathcal{Z}_\mu} < \infty. \tag{3}$$

So, $u \in \mathcal{Z}_\mu$. By using (2), we get

$$\sup_{z\in\mathbb{D}} \mu(z)\left|\varphi''(z)u(z)+2\varphi'(z)u'(z)+\varphi(z)u''(z)\right| \leq \frac{1}{(n+1)!}\| D_{\varphi,u}^n p_{n+1} \|_{\mathcal{Z}_\mu} < \infty.$$

From the boundedness of the function φ and (3),

$$\sup_{z\in\mathbb{D}} \mu(z)\left|\varphi''(z)u(z)+2\varphi'(z)u'(z)\right| < \infty. \tag{4}$$

By using (2),

$$\sup_{z\in\mathbb{D}} \mu(z) \left|2\varphi'(z)^2 u(z) + 2\Big(\varphi''(z)u(z) + 2\varphi'(z)u'(z)\Big)\varphi(z) + \varphi^2(z)u''(z)\right| \le \frac{2}{(n+2)!} \parallel D^n_{\varphi,u} p_{n+2} \parallel_{\mathcal{Z}_\mu} < \infty.$$

Finally, from boundedness of the function φ, (3) and (4)

$$\sup_{z\in D} \mu(z) | \varphi'(z) |^2 | u(z) | < \infty.$$

We set $Q := \sup_{j\ge n} j^{\alpha-1} \parallel D^n_{\varphi,u} p_j \parallel_{\mathcal{Z}_\mu}$. For any $a \in \mathbb{D}$, it is easy to check that f_a, h_a and g_a are in $\mathcal{B}^\alpha$. By simple calculation, we obtain

$$f_a(z) = (1 - | a |^2) \sum_{j=0}^{\infty} \frac{\Gamma(j+\alpha)}{j!\Gamma(\alpha)} \bar{a}^j z^j, \quad h_a(z) = (1 - | a |^2)^2 \sum_{j=0}^{\infty} \frac{\Gamma(j+1+\alpha)}{j!\Gamma(1+\alpha)} \bar{a}^j z^j,$$
$$g_a(z) = (1 - | a |^2)^3 \sum_{j=0}^{\infty} \frac{\Gamma(j+2+\alpha)}{j!\Gamma(2+\alpha)} \bar{a}^j z^j.$$

From Stirling's formula, we have $\frac{\Gamma(j+\alpha)}{j!\Gamma(\alpha)} \approx j^{\alpha-1}$ as $j \to \infty$. Using linearity, we get

$$\parallel D^n_{\varphi,u} f_a \parallel_{\mathcal{Z}_\mu} \le C(1 - | a |^2) \sum_{j=0}^{\infty} | \bar{a} |^j \, j^{\alpha-1} \parallel D^n_{\varphi,u} p_j \parallel_{\mathcal{Z}_\mu} \le \frac{CQ(1-|a|^2)}{1-|\bar{a}|} \le 2CQ, \tag{5}$$

$$\parallel D^n_{\varphi,u} h_a \parallel_{\mathcal{Z}_\mu} \le C(1 - | a |^2)^2 \sum_{j=0}^{\infty} | \bar{a} |^j \, j j^{\alpha-1} \parallel D^n_{\varphi,u} p_j \parallel_{\mathcal{Z}_\mu} \le \frac{CQ|\bar{a}|(1-|a|^2)^2}{(1-|\bar{a}|)^2} \le 4CQ, \tag{6}$$

$$\parallel D^n_{\varphi,u} g_a \parallel_{\mathcal{Z}_\mu} \le C(1 - | a |^2)^3 \sum_{j=0}^{\infty} | \bar{a} |^j \, j^2 j^{\alpha-1} \parallel D^n_{\varphi,u} p_j \parallel_{\mathcal{Z}_\mu} \le \frac{CQ|\bar{a}|(1+|\bar{a}|)(1-|a|^2)^3}{(1-|\bar{a}|)^3} \le 16CQ. \tag{7}$$

Since a is arbitrary, so

$$\sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} f_a \parallel_{\mathcal{Z}_\mu} < \infty, \quad \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} h_a \parallel_{\mathcal{Z}_\mu} < \infty \quad \text{and} \quad \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} g_a \parallel_{\mathcal{Z}_\mu} < \infty.$$

$(d) \Rightarrow (e)$ Assume that (d) holds. Set

$$C_1 = \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} f_a \parallel_{\mathcal{Z}_\mu}, \quad C_2 = \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} h_a \parallel_{\mathcal{Z}_\mu} \quad \text{and} \quad C_3 = \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} g_a \parallel_{\mathcal{Z}_\mu}.$$

It is obvious that for any $a \in \mathbb{D}$ and $n \in \mathbb{N}$ the functions $m_{n,a}$, $l_{n,a}$ and $k_{n,a}$ are in $\mathcal{B}^\alpha$. Moreover

$$\begin{aligned}\sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} m_{n,a} \parallel_{\mathcal{Z}_\mu} \le{} & \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} f_a \parallel_{\mathcal{Z}_\mu} + \frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)} \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} h_a \parallel_{\mathcal{Z}_\mu} \\ & + \frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)} \sup_{a\in\mathbb{D}} \parallel D^n_{\varphi,u} g_a \parallel_{\mathcal{Z}_\mu} \\ \le{} & C_1 + \frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)} C_2 + \frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)} C_3. \end{aligned} \tag{8}$$

Hence, for any $\lambda \in \mathbb{D}$ it follows from Lemma 2.1 and (8) that

$$m^{(n)}_{n,\varphi(\lambda)}(\varphi(\lambda)) = m^{(n+2)}_{n,\varphi(\lambda)}(\varphi(\lambda)) = 0,$$

$$\frac{\mu(\lambda)\mid 2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\mid\mid\varphi(\lambda)\mid^{n+1}}{(\alpha+n+2)(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n}}\prod_{j=0}^{n-1}(\alpha+j)=\mu(\lambda)\left|\left(2u^{'}(\lambda)\varphi^{'}(\lambda)+u(\lambda)\varphi^{''}(\lambda)\right)m_{n,\varphi(\lambda)}^{(n+1)}(\varphi(\lambda))\right|=$$
$$\mu(\lambda)\left|u^{''}(\lambda)\,m_{n,\varphi(\lambda)}^{(n)}(\varphi(\lambda))+\left(2u^{'}(\lambda)\varphi^{'}(\lambda)+u(\lambda)\varphi^{''}(\lambda)\right)m_{n,\varphi(\lambda)}^{(n+1)}(\varphi(\lambda))+u(\lambda)\varphi'^2(\lambda)\,m_{n,\varphi(\lambda)}^{(n+2)}(\varphi(\lambda)\right|\leq$$
$$\sup_{z\in\mathbb{D}}\mu(z)\left|u^{''}(z)\,m_{n,\varphi(\lambda)}^{(n)}(\varphi(z))+\left(2u^{'}(z)\varphi^{'}(z)+u(z)\varphi^{''}(z)\right)m_{n,\varphi(\lambda)}^{(n+1)}(\varphi(z))+u(z)\varphi'^2(z)\,m_{n,\varphi(\lambda)}^{(n+2)}(\varphi(z))\right|=$$
$$\sup_{z\in\mathbb{D}}\mu(z)\left|\left(u(z)\,m_{n,\varphi(\lambda)}^{(n)}(\varphi(z))\right)^{''}\right|\leq\|D_{\varphi,u}^n m_{n,\varphi(\lambda)}\|_{\mathcal{Z}_\mu}\leq\sup_{a\in\mathbb{D}}\|D_{\varphi,u}^n m_{n,a}\|_{\mathcal{Z}_\mu}\leq C_1+\frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)}C_2+$$
$$\frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)}C_3<\infty. \tag{9}$$

So,

$$\frac{\mu(\lambda)\mid\varphi(\lambda)\mid^{n+1}\mid 2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\mid}{(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n}}\leq$$
$$\frac{\alpha+n+2}{\alpha(\alpha+1)\cdots(\alpha+n-1)}(C_1+\frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)}C_2+\frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)}C_3).$$

Therefore,

$$\sup_{\lambda\in\mathbb{D}}\frac{\mu(\lambda)\mid\varphi(\lambda)\mid^{n+1}\mid 2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\mid}{(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n}}<\infty. \tag{10}$$

For any fixed $r\in(0,1)$ from (10), we obtain

$$\sup_{|\varphi(\lambda)|>r}\frac{\mu(\lambda)\mid 2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\mid}{(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n}}\leq\frac{1}{r^{n+1}}\sup_{|\varphi(\lambda|>r}\frac{\mu(\lambda)\mid\varphi(\lambda)\mid^{n+1}\mid 2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\mid}{(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n}}<\infty. \tag{11}$$

On the other hand from (d),

$$\sup_{|\varphi(\lambda)|\leq r}\frac{\mu(\lambda)\mid 2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\mid}{(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n}}\leq\frac{\sup_{|\varphi(\lambda)|\leq r}\mu(\lambda)\mid 2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\mid}{(1-r^2)^{\alpha+n}}<\infty. \tag{12}$$

For any $a\in\mathbb{D}$,

$$\|D_{\varphi,u}^n l_{n,a}\|_{\mathcal{Z}_\mu}\leq C_1+\frac{2\alpha}{\alpha+n+1}C_2+\frac{\alpha(\alpha+1)}{(\alpha+n+1)(\alpha+n+2)}C_3. \tag{13}$$

So, $\sup_{a\in\mathbb{D}}\|D_{\varphi,u}^n l_{n,a}\|_{\mathcal{Z}_\mu}<\infty$. From Lemma 2.2,

$$l_{n,\varphi(\lambda)}^{(n+1)}(\varphi(\lambda))=l_{n,\varphi(\lambda)}^{(n+2)}(\varphi(\lambda))=0.$$

Hence, by a similar calculation as in (9), we obtain

$$\frac{\mu(\lambda)\mid\varphi(\lambda)\mid^n\mid u''(\lambda)\mid}{(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n-1}}\frac{2}{(\alpha+n+1)(\alpha+n+2)}\prod_{j=0}^{n-1}(\alpha+j)=\mu(\lambda)\left|u^{''}(\lambda)\,l_{n,\varphi(\lambda)}^{(n)}(\varphi(\lambda))\right|\leq\|D_{\varphi,u}^n l_{n,\varphi(\lambda)}\|_{\mathcal{Z}_\mu}$$
$$\leq C_1+\frac{2\alpha}{\alpha+n+1}C_2+\frac{\alpha(\alpha+1)}{(\alpha+n+1)(\alpha+n+2)}C_3. \tag{14}$$

Therefore,

$$\sup_{\lambda\in\mathbb{D}}\frac{\mu(\lambda)\mid\varphi(\lambda)\mid^n\mid u''(\lambda)\mid}{(1-\mid\varphi(\lambda)\mid^2)^{\alpha+n-1}}<\infty. \tag{15}$$

From (15) and $u \in \mathcal{Z}_\mu$ with similar calculation as in (11) and (12), we get

$$\sup_{\lambda\in\mathbb{D}} \frac{\mu(\lambda) \mid u''(\lambda) \mid}{(1-\mid \varphi(\lambda) \mid^2)^{\alpha+n-1}} < \infty.$$

For any $a \in \mathbb{D}$,

$$\|D^n_{\varphi,u}k_{n,a}\|_{\mathcal{Z}_\mu} \le C_1 + \frac{2\alpha}{\alpha+n}C_2 + \frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+1)}C_3. \tag{16}$$

Hence, $\sup_{a\in\mathbb{D}} \| D^n_{\varphi,u}k_{n,a} \|_{\mathcal{Z}_\mu} < \infty$. For any $\lambda \in \mathbb{D}$, from Lemma 2.3

$$k^{(n)}_{n,\varphi(\lambda)}(\varphi(\lambda)) = k^{(n+1)}_{n,\varphi(\lambda)}(\varphi(\lambda)) = 0$$

and with the similar calculation as in (9), we have

$$\begin{aligned}\frac{2\mu(\lambda) \mid \varphi(\lambda) \mid^{n+2}\mid \varphi'(\lambda) \mid^2\mid u(\lambda) \mid}{(1-\mid \varphi(\lambda) \mid^2)^{\alpha+n+1}} \prod_{j=0}^{n-1}(\alpha+j) &= \mu(\lambda)\left|u(\lambda)\varphi'^2(\lambda)\ k^{(n+2)}_{n,\varphi(\lambda)}(\varphi(\lambda))\right| \le \| D^n_{\varphi,u}k_{n,\varphi(\lambda)} \|_{\mathcal{Z}_\mu} \\ &\le C_1 + \frac{2\alpha}{\alpha+n}C_2 + \frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+1)}C_3. \end{aligned} \tag{17}$$

Thus,

$$\sup_{\lambda\in\mathbb{D}} \frac{\mu(\lambda) \mid \varphi(\lambda) \mid^{n+2}\mid \varphi'(\lambda) \mid^2\mid u(\lambda) \mid}{(1-\mid \varphi(\lambda) \mid^2)^{\alpha+n+1}} < \infty. \tag{18}$$

From (d) and (18) with similar calculation as in (11) and (12), we obtain

$$\sup_{\lambda\in\mathbb{D}} \frac{\mu(\lambda) \mid \varphi'(\lambda) \mid^2\mid u(\lambda) \mid}{(1-\mid \varphi(\lambda) \mid^2)^{\alpha+n+1}} < \infty.$$

(e) ⇒ (a) Assume (e) holds. For any $f \in \mathcal{B}^\alpha$,

$$\begin{aligned}\mu(z)\left|(D^n_{\varphi,u}f)''(z)\right| &= \mu(z)\left|f^{(n+2)}(\varphi(z))\varphi'^2(z)u(z) + f^{(n+1)}(\varphi(z))\big(2\varphi'(z)u'(z)+\varphi''(z)u(z)\big) + f^{(n)}(\varphi(z))u''(z)\right| \\ &\le \mu(z)\left|f^{(n+2)}(\varphi(z))\right|\left|\varphi'^2(z)u(z)\right| + \mu(z)\left|f^{(n+1)}(\varphi(z))\right|\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| + \mu(z)\left|f^{(n)}(\varphi(z))\right|\left|u''(z)\right| \\ &\le \frac{\mu(z)\mid \varphi'(z) \mid^2 \mid u(z) \mid}{(1-\mid \varphi(z) \mid^2)^{\alpha+n+1}}C \| f \|_{\mathcal{B}^\alpha} + \frac{\mu(z) \mid 2\varphi'(z)u'(z)+\varphi''(z)u(z) \mid}{(1-\mid \varphi(z) \mid^2)^{\alpha+n}}C \| f \|_{\mathcal{B}^\alpha} + \frac{\mu(z) \mid u''(z) \mid}{(1-\mid \varphi(z) \mid^2)^{\alpha+n-1}}C \| f \|_{\mathcal{B}^\alpha}. \end{aligned} \tag{19}$$

In the last inequality we use the fact that (Proposition 8 in [13]) for $f \in \mathcal{B}^\alpha$

$$\sup(1-\mid z \mid^2)^\alpha \mid f'(z) \mid \approx \mid f(0) \mid + \cdots + \mid f^{(n)}(0) \mid + \sup(1-\mid z \mid^2)^{\alpha+n} \mid f^{(n+1)}(z) \mid. \tag{20}$$

Moreover,

$$\begin{aligned}\left|(D^n_{\varphi,u}f)(0)\right| &= \left|f^{(n)}(\varphi(0))u(0)\right| \le \frac{\mid u(0) \mid}{(1-\mid \varphi(0) \mid^2)^{\alpha+n-1}}C \| f \|_{\mathcal{B}^\alpha}, \\ \left|(D^n_{\varphi,u}f)'(0)\right| &= \left|f^{(n+1)}(\varphi(0))\varphi'(0)u(0) + f^{(n)}(\varphi(0))u'(0)\right| \\ &\le \left|f^{(n+1)}(\varphi(0))\varphi'(0)u(0)\right| + \left|f^{(n)}(\varphi(0))u'(0)\right| \\ &\le \frac{\mid \varphi'(0) \mid\mid u(0) \mid}{(1-\mid \varphi(0) \mid^2)^{\alpha+n}}C \| f \|_{\mathcal{B}^\alpha} + \frac{\mid u'(0) \mid}{(1-\mid \varphi(0) \mid^2)^{\alpha+n-1}}C \| f \|_{\mathcal{B}^\alpha}. \end{aligned} \tag{21}$$

From (e), (19) and (21), we conclude that the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded. The proof is complete. □

3. Compactness of $D^n_{\varphi,u}:\mathcal{B}^\alpha(\mathcal{B}^\alpha_0)\to\mathcal{Z}_\mu$

In this section, we obtain several characterizations for compactness of generalized weighted composition operators from the Bloch-type spaces into the weighted Zygmund spaces. To study compactness, we need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [1]).

Lemma 3.1. *Let n be a positive integer, $0<\alpha<\infty$, μ be a weight, $u\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi,u}:\mathcal{B}^\alpha\to\mathcal{Z}_\mu$ is compact if and only if $D^n_{\varphi,u}:\mathcal{B}^\alpha\to\mathcal{Z}_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in N}$ in $\mathcal{B}^\alpha$, which converges to zero uniformly on compact subsets of $\mathbb{D}$,*

$$\lim_{k\to\infty}\|D^n_{\varphi,u}f_k\|_{\mathcal{Z}_\mu}=0.$$

Theorem 3.2. *Let n be a positive integer, $0<\alpha<\infty$, $u\in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ and $D^n_{\varphi,u}:\mathcal{B}^\alpha\to\mathcal{Z}_\mu$ is bounded. Then the following statements are equivalent.*

(a) *The operator $D^n_{\varphi,u}:\mathcal{B}^\alpha\to\mathcal{Z}_\mu$ is compact.*
(b) *The operator $D^n_{\varphi,u}:\mathcal{B}^\alpha_0\to\mathcal{Z}_\mu$ is compact.*
(c) $\lim_{j\to\infty} j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}=0$.
(d) $\lim_{|\varphi(a)|\to\bar{1}}\|D^n_{\varphi,u}f_{\varphi(a)}\|_{\mathcal{Z}_\mu}=0,\quad \lim_{|\varphi(a)|\to\bar{1}}\|D^n_{\varphi,u}h_{\varphi(a)}\|_{\mathcal{Z}_\mu}=0,\quad \lim_{|\varphi(a)|\to\bar{1}}\|D^n_{\varphi,u}g_{\varphi(a)}\|_{\mathcal{Z}_\mu}=0.$
(e)

$$\lim_{|\varphi(z)|\to\bar{1}}\frac{\mu(z)\mid 2\varphi'(z)u'(z)+\varphi''(z)u(z)\mid}{(1-\mid\varphi(z)\mid^2)^{\alpha+n}}=0,\quad \lim_{|\varphi(z)|\to\bar{1}}\frac{\mu(z)\mid u''(z)\mid}{(1-\mid\varphi(z)\mid^2)^{\alpha+n-1}}=0,\quad \lim_{|\varphi(z)|\to\bar{1}}\frac{\mu(z)\mid\varphi'(z)\mid^2\mid u(z)\mid}{(1-\mid\varphi(z)\mid^2)^{\alpha+n+1}}=0.$$

Proof. $(a)\Rightarrow(b)$ This implication is clear.
$(b)\Rightarrow(c)$ The sequence $\{j^{\alpha-1}p_j\}_{j=1}^\infty$ is bounded in $\mathcal{B}^\alpha_0$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 3.1 it follows that $\lim_{j\to\infty}j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}=0$.
$(c)\Rightarrow(d)$ Suppose (c) holds. Since for $j<n$, $D^n_{\varphi,u}p_j=0$, hence for given $\epsilon>0$ there exists a positive integer $N\ge n$, such that

$$j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}<\epsilon,$$

for all $j\ge N$. Also from Theorem 2.4 (c), $Q=\sup_{j\ge n}j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}<\infty$. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty}\mid\varphi(z_k)\mid=1$. Similar to the proof of (5), there exists a constant C such that

$$\begin{aligned}\|D^n_{\varphi,u}f_{\varphi(z_k)}\|_{\mathcal{Z}_\mu}&\le C(1-\mid\varphi(z_k)\mid^2)\sum_{j=0}^{\infty}\mid\varphi(z_k)\mid^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}=C(1-\mid\varphi(z_k)\mid^2)\underbrace{\sum_{j=0}^{n-1}\mid\varphi(z_k)\mid^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}}_{0}\\&+C(1-\mid\varphi(z_k)\mid^2)\Big(\sum_{j=n}^{N-1}\mid\varphi(z_k)\mid^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}+\sum_{j=N}^{\infty}\mid\varphi(z_k)\mid^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}\Big)\\&\le C(1-\mid\varphi(z_k)\mid^2)\Big(Q\sum_{j=0}^{N-1}\mid\varphi(z_k)\mid^j+\varepsilon\sum_{j=0}^{\infty}\mid\varphi(z_k)\mid^j\Big)\\&\le 2CQ(1-\mid\varphi(z_k)\mid^N)+2C\epsilon.\end{aligned}$$

Since $\lim_{k\to\infty}\mid\varphi(z_k)\mid=1$, so

$$\lim_{k\to\infty}\|\,D^n_{\varphi,u}f_{\varphi(z_k)}\,\|_{\mathcal{Z}_\mu}\le 2C\epsilon.$$

Hence, $\lim_{|\varphi(a)|\to\bar{1}}\|D^n_{\varphi,u}f_{\varphi(a)}\|_{\mathcal{Z}_\mu}=0$, because ϵ is an arbitrary positive number.
Notice that

$$\sum_{j=0}^{N-1}(j+1)r^j=\frac{1-r^N-Nr^N(1-r)}{(1-r)^2},\quad 0\le r<1.$$

Arguing as in the proof of (6), we get

$$\begin{aligned}
\|D^n_{\varphi,u}h_{\varphi(z_k)}\|_{\mathcal{Z}_\mu} &\le C(1-|\varphi(z_k)|^2)^2\sum_{j=0}^{\infty}|\varphi(z_k)|^j j^{\alpha}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}\\
&\le C(1-|\varphi(z_k)|^2)^2\Big(\sum_{j=0}^{N-1}(j+1)|\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}+\sum_{j=N}^{\infty}(j+1)|\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}\Big)\\
&\le 4CQ\Big(1-|\varphi(z_k)|^N-N|\varphi(z_k)|^N(1-|\varphi(z_k)|)\Big)+4C\epsilon.
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\|D^n_{\varphi,u}h_{\varphi(z_k)}\|_{\mathcal{Z}_\mu}\le 4C\epsilon$$

and arbitrariness of ϵ gives us $\lim_{|\varphi(a)|\to\bar{1}}\|D^n_{\varphi,u}h_{\varphi(a)}\|_{\mathcal{Z}_\mu}=0$.
Notice that

$$\sum_{j=1}^{N}j^2r^j=\frac{r(1+r-(N+1)^2r^N+(2N^2+2N-1)r^{N+1}-N^2r^{N+2})}{(1-r)^3},\quad 0\le r<1.$$

Similar to the proof of (7), we get

$$\begin{aligned}
&\|D^n_{\varphi,u}g_{\varphi(z_k)}\|_{\mathcal{Z}_\mu}\le C(1-|\varphi(z_k)|^2)^3\sum_{j=0}^{\infty}|\varphi(z_k)|^j j^{\alpha+1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}=\\
&C(1-|\varphi(z_k)|^2)^3\Big(\sum_{j=0}^{N}j^2|\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}+\sum_{j=N+1}^{\infty}j^2|\varphi(z_k)|^j j^{\alpha-1}\|D^n_{\varphi,u}p_j\|_{\mathcal{Z}_\mu}\Big)\\
&\le 8CQ|\varphi(z_k)|\left(1+|\varphi(z_k)|-(N+1)^2|\varphi(z_k)|^N+(2N^2+2N-1)|\varphi(z_k)|^{N+1}-N^2|\varphi(z_k)|^{N+2}\right)+16C\epsilon.
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\|D^n_{\varphi,u}g_{\varphi(z_k)}\|_{\mathcal{Z}_\mu}\le 16C\epsilon.$$

Since ϵ is arbitrary, we obtain $\lim_{|\varphi(a)|\to\bar{1}}\|D^n_{\varphi,u}g_{\varphi(a)}\|_{\mathcal{Z}_\mu}=0$.
$(d)\Rightarrow(e)$ To prove (e), it is sufficient to prove that for any sequence $\{z_k\}_{k\in\mathbb{N}}$ in $\mathbb{D}$ with $\lim_{k\to\infty}|\varphi(z_k)|=1$,

$$\lim_{k\to\infty}\frac{\mu(z_k)|2\varphi'(z_k)u'(z_k)+\varphi''(z_k)u(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n}}=0,\ \lim_{k\to\infty}\frac{\mu(z_k)|u''(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n-1}}=0,\ \lim_{k\to\infty}\frac{\mu(z_k)|\varphi'(z_k)|^2|u(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n+1}}=0.$$

Let $\{z_k\}_{k\in\mathbb{N}}$ be any sequence in $\mathbb{D}$ such that $\lim_{k\to\infty}|\varphi(z_k)|=1$. Similar to the proof (8), we obtain

$$\begin{aligned}
\lim_{k\to\infty}\|D^n_{\varphi,u}m_{n,\varphi(z_k)}\|_{\mathcal{Z}_\mu}&\le\lim_{k\to\infty}\|D^n_{\varphi,u}f_{\varphi(z_k)}\|_{\mathcal{Z}_\mu}+\frac{\alpha(2\alpha+2n+3)}{(\alpha+n)(\alpha+n+2)}\lim_{k\to\infty}\|D^n_{\varphi,u}h_{\varphi(z_k)}\|_{\mathcal{Z}_\mu}\\
&+\frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+2)}\lim_{k\to\infty}\|D^n_{\varphi,u}g_{\varphi(z_k)}\|_{\mathcal{Z}_\mu}\\
&=0.
\end{aligned}\tag{22}$$

From (22) and (9), we get

$$\lim_{k\to\infty}\frac{\mu(z_k)|\varphi(z_k)|^{n+1}|2\varphi'(z_k)u'(z_k)+\varphi''(z_k)u(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n}}=0.$$

Since $\lim_{k\to\infty} \mid \varphi(z_k) \mid= 1$, therefore

$$\lim_{k\to\infty} \frac{\mu(z_k) \mid 2\varphi'(z_k)u'(z_k) + \varphi''(z_k)u(z_k) \mid}{(1- \mid \varphi(z_k) \mid^2)^{\alpha+n}} = 0.$$

Similar to the proof of (13), we have

$$\begin{aligned}\lim_{k\to\infty} \| D^n_{\varphi,u} l_{n,\varphi(z_k)} \|_{\mathcal{Z}_\mu} &\le \lim_{k\to\infty} \| D^n_{\varphi,u} f_{\varphi(z_k)} \|_{\mathcal{Z}_\mu} + \frac{2\alpha}{\alpha+n+1} \lim_{k\to\infty} \| D^n_{\varphi,u} h_{\varphi(z_k)} \|_{\mathcal{Z}_\mu} \\ &+ \frac{\alpha(\alpha+1)}{(\alpha+n+1)(\alpha+n+2)} \lim_{k\to\infty} \| D^n_{\varphi,u} g_{\varphi(z_k)} \|_{\mathcal{Z}_\mu} \\ &= 0. \end{aligned} \tag{23}$$

By using (14) and (23), we get

$$\lim_{k\to\infty} \frac{\mu(z_k) \mid \varphi(z_k) \mid^n \mid u''(z_k) \mid}{(1- \mid \varphi(z_k) \mid^2)^{\alpha+n-1}} = 0,$$

since $\lim_{k\to\infty} \mid \varphi(z_k) \mid= 1$, from the above equation, we obtain

$$\lim_{k\to\infty} \frac{\mu(z_k) \mid u''(z_k) \mid}{(1- \mid \varphi(z_k) \mid^2)^{\alpha+n-1}} = 0.$$

Finally, similar to the proof (16)

$$\begin{aligned}\lim_{k\to\infty} \| D^n_{\varphi,u} k_{n,\varphi(z_k)} \|_{\mathcal{Z}_\mu} &\le \lim_{k\to\infty} \| D^n_{\varphi,u} f_{\varphi(z_k)} \|_{\mathcal{Z}_\mu} + \frac{2\alpha}{\alpha+n} \lim_{k\to\infty} \| D^n_{\varphi,u} h_{\varphi(z_k)} \|_{\mathcal{Z}_\mu} \\ &+ \frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+1)} \lim_{k\to\infty} \| D^n_{\varphi,u} g_{\varphi(z_k)} \|_{\mathcal{Z}_\mu} \\ &= 0. \end{aligned} \tag{24}$$

So, by using (17) and (24),

$$\lim_{k\to\infty} \frac{\mu(z_k) \mid \varphi(z_k) \mid^{n+2} \mid \varphi'(z_k) \mid^2 \mid u(z_k) \mid}{(1- \mid \varphi(z_k) \mid^2)^{\alpha+n+1}} = 0.$$

Since $\lim_{k\to\infty} \mid \varphi(z_k) \mid= 1$, we get

$$\lim_{k\to\infty} \frac{\mu(z_k) \mid \varphi'(z_k) \mid^2 \mid u(z_k) \mid}{(1- \mid \varphi(z_k) \mid^2)^{\alpha+n+1}} = 0.$$

$(e) \Rightarrow (a)$ Assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}^\alpha$ converging to 0 uniformly on compact subsets of $\mathbb{D}$. For any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z) \mid 2\varphi'(z)u'(z) + \varphi''(z)u(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+n}} < \epsilon, \quad \frac{\mu(z) \mid u''(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+n-1}} < \epsilon, \quad \frac{\mu(z) \mid \varphi'(z) \mid^2 \mid u(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+n+1}} < \epsilon \tag{25}$$

when $\delta <\mid \varphi(z) \mid< 1$. Since $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is bounded, from Theorem 2.4, we have

$$\begin{aligned} C_4 &= \sup_{z\in\mathbb{D}} \mu(z) \mid u''(z) \mid< \infty, \\ C_5 &= \sup_{z\in\mathbb{D}} \mu(z) \mid u(z) \mid\mid \varphi'(z) \mid^2< \infty, \\ C_6 &= \sup_{z\in\mathbb{D}} \mu(z) \mid 2\varphi'(z)u'(z) + \varphi''(z)u(z) \mid< \infty. \end{aligned} \tag{26}$$

Let $V = \{z \in \mathbb{D} : | \varphi(z) | \leq \delta\}$. From (20), (25) and (26), we obtain

$$\begin{aligned}
\sup_{z\in\mathbb{D}} \mu(z) \mid (D^n_{\varphi,u} f_k)''(z) \mid &\leq \sup_{z\in V} \mu(z) \mid f_k^{(n+2)}(\varphi(z)) \mid \mid \varphi'(z) \mid^2 \mid u(z) \mid \\
&+ \sup_{z\in V} \mu(z) \mid f_k^{(n+1)}(\varphi(z)) \mid\mid 2\varphi'(z)u'(z) + \varphi''(z)u(z) \mid \\
&+ \sup_{z\in V} \mu(z) \mid f_k^{(n)}(\varphi(z)) \mid\mid u''(z) \mid + C \sup_{z\in\mathbb{D}-V} \frac{\mu(z) \mid \varphi'(z) \mid^2 \mid u(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+n+1}} \|f_k\|_{\mathcal{B}^\alpha} \\
&+ C \sup_{z\in\mathbb{D}-V} \frac{\mu(z) \mid 2\varphi'(z)u'(z) + \varphi''(z)u(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+n}} \|f_k\|_{\mathcal{B}^\alpha} + C \sup_{z\in\mathbb{D}-V} \frac{\mu(z) \mid u''(z) \mid}{(1- \mid \varphi(z) \mid^2)^{\alpha+n-1}} \|f_k\|_{\mathcal{B}^\alpha} \\
&\leq C_5 \sup_{z\in V} \mid f_k^{(n+2)}(\varphi(z)) \mid + C_6 \sup_{z\in V} \mid f_k^{(n+1)}(\varphi(z)) \mid + C_4 \sup_{z\in V} \mid f_k^{(n)}(\varphi(z)) \mid + C\epsilon \|f_k\|_{\mathcal{B}^\alpha}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|D^n_{\varphi,u} f_k\|_{\mathcal{Z}_\mu} &\leq C_5 \sup_{|w|\leq\delta} \mid f_k^{(n+2)}(w) \mid + C_6 \sup_{|w|\leq\delta} \mid f_k^{(n+1)}(w) \mid + C_4 \sup_{|w|\leq\delta} \mid f_k^{(n)}(w) \mid + C\epsilon \|f_k\|_{\mathcal{B}^\alpha} + \mid u(0) \mid\mid f_k^{(n)}(\varphi(0)) \mid \\
&+ \mid f_k^{(n+1)}(\varphi(0)) \mid\mid \varphi'(0) \mid\mid u(0) \mid + \mid f_k^{(n)}(\varphi(0)) \mid\mid u'(0) \mid . \qquad (27)
\end{aligned}$$

Since $(f_k)_{k\in\mathbb{N}}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, by Cauchy's estimates so do the sequences $(f_k^{(n)})_{k\in\mathbb{N}}$. From (27), letting $k \to \infty$ and using the fact that ϵ is an arbitrary positive number, we get

$$\lim_{k\to\infty} \|D^n_{\varphi,u} f_k\|_{\mathcal{Z}_\mu} = 0.$$

From Lemma 3.1, we deduce that the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{Z}_\mu$ is compact. □

Setting $\alpha = 1$ and $\mu(z) = 1- \mid z \mid^2$, in (1) and Theorems 2.4 and 3.2, we obtain the following corollaries.

Corollary 3.3. *Let n be a positive integer, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) *The operator $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is bounded.*
(b) *The operator $D^n_{\varphi,u} : \mathcal{B}_0 \to \mathcal{Z}$ is bounded.*
(c) $\sup_{j\geq n} \|D^n_{\varphi,u} p_j\|_{\mathcal{Z}} < \infty$, *where* $p_j(z) = z^j$.
(d) $u \in \mathcal{Z}$, $\sup_{z\in\mathbb{D}}(1- \mid z \mid^2) \mid u(z) \mid\mid \varphi'(z) \mid^2 < \infty$, $\sup_{z\in\mathbb{D}}(1- \mid z \mid^2) \mid 2\varphi'(z)u'(z) + \varphi''(z)u(z) \mid < \infty$ *and*

$$\sup_{a\in\mathbb{D}} \|\mathcal{D}^n_{\varphi,u} f_a\|_{\mathcal{Z}} < \infty, \quad \sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} h_a\|_{\mathcal{Z}} < \infty, \quad \sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} g_a\|_{\mathcal{Z}} < \infty$$

where f_a, h_a and g_a are defined in (1).

(e)

$$\sup_{z\in\mathbb{D}} \frac{(1- \mid z \mid^2) \mid 2\varphi'(z)u'(z) + \varphi''(z)u(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n+1}} < \infty, \qquad \sup_{z\in\mathbb{D}} \frac{(1- \mid z \mid^2) \mid u''(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n}} < \infty \qquad \textit{and}$$
$$\sup_{z\in\mathbb{D}} \frac{(1- \mid z \mid^2) \mid \varphi'(z) \mid^2 \mid u(z) \mid}{(1- \mid \varphi(z) \mid^2)^{n+2}} < \infty.$$

Corollary 3.4. *Let n be a positive integer, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$ and $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is bounded. Then the following statements are equivalent.*

(a) *The operator $D^n_{\varphi,u} : \mathcal{B} \to \mathcal{Z}$ is compact.*

(b) *The operator* $D^n_{\varphi,u} : \mathcal{B}_0 \to \mathcal{Z}$ *is compact.*
(c) $\lim_{j\to\infty} \|D^n_{\varphi,u} p_j\|_{\mathcal{Z}} = 0.$
(d) $\lim_{|\varphi(a)|\to \bar{1}} \|D^n_{\varphi,u} f_{\varphi(a)}\|_{\mathcal{Z}} = 0, \quad \lim_{|\varphi(a)|\to \bar{1}} \|D^n_{\varphi,u} h_{\varphi(a)}\|_{\mathcal{Z}} = 0, \quad \lim_{|\varphi(a)|\to \bar{1}} \|D^n_{\varphi,u} g_{\varphi(a)}\|_{\mathcal{Z}} = 0.$
(e)

$$\lim_{|\varphi(z)|\to \bar{1}} \frac{(1-|z|^2)\,|\,2\varphi'(z)u'(z) + \varphi''(z)u(z)\,|}{(1-|\varphi(z)|^2)^{n+1}} = 0, \qquad \lim_{|\varphi(z)|\to \bar{1}} \frac{(1-|z|^2)\,|\,u''(z)\,|}{(1-|\varphi(z)|^2)^{n}} = 0 \qquad \textit{and}$$

$$\lim_{|\varphi(z)|\to \bar{1}} \frac{(1-|z|^2)|\,\varphi'(z)\,|^2\,|\,u(z)\,|}{(1-|\varphi(z)|^2)^{n+2}} = 0.$$

The equivalence of conditions (*a*), (*b*) and (*d*) of corollaries 3.3 and 3.4 was proved in [4]. Also Stević in [10] proved that the conditions (*a*), (*b*) and (*e*) of above two corollaries are equivalent.

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