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Higher-Order Symmetric Duality in Nondifferentiable Multiobjective Optimization over Cones

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Abstract. In this paper, a new pair of higher-order nondifferentiable multiobjective symmetric dual programs over arbitrary cones is formulated, where each of the objective functions contains a support function of a compact convex set. We identify a function lying exclusively in the class of higher-order K-η-convex and not in the class of K-η-bonvex function already existing in literature. Weak, strong and converse duality theorems are then established under higher-order K-η-convexity assumptions. Self duality is obtained by assuming the functions involved to be skew-symmetric. Several known results are also discussed as special cases.

1. Introduction

Duality is used in many theoretical and computational developments in mathematical programming and in solving different real life problems and mathematical models that require the relative comparison of two magnitudes. In mathematical programming, a pair of primal and dual problem is called symmetric if the dual of dual is primal problem. Unlike linear programming, the majority of dual formulations in nonlinear programming do not possess the symmetry property.

The concept of symmetric duality in quadratic programming was introduced by Dorn [6]. His results were extended to nonlinear convex programming problems in Dantzig et al.[5] and then in Bazaraa and Goode [3] over arbitrary cones by assuming the kernel function $f(x, y)$ to be convex in x and concave in y. Subsequently, Mond and Weir [19] presented a distinct pair of symmetric dual nonlinear programs which admits the relaxation of the convexity/concavity assumption to pseudoconvexity/pseudoconcavity.

Mangasarian [15] formulated a class of higher-order dual problems for the nonlinear programming problems. He has also indicated that the study of higher-order duality is significant due to the computational advantage over the first-order duality as it provides tighter bounds for the value of the objective function when approximations are used. Higher-order duality in nonlinear programs has been studied by several researchers [2, 4, 7, 9, 10, 20].

Mond and Zhang [20] obtained duality results for various higher-order dual problems under invexity assumptions. Chen [4] considered a pair of nondifferentiable programs and established duality theorems under higher-order generalized F-convexity. Wolfe type higher-order nondifferentiable symmetric dual

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programs and their duality relations were discussed by Gulati and Gupta [9]. Later on, Ahmad et al. [2] formulated a general Mond-Weir type higher-order dual for nondifferentiable multiobjective programming problem and established higher-order duality theorems. Scott and Jefferson [22] proved duality results for square root convex programs. Optimality conditions for nonconvex quadratic-exponential minimization problems were discussed by Gao and Ruan [8]. Mishra et al. [16] obtained optimality conditions and relations between primal and dual models for a nonsmooth multiobjective optimization involving generalized type I functions. Usual duality relations has been proved in Saini and Gulati [21] for a pair of Wolfe type multiobjective second-order symmetric dual programs over arbitrary cones for nondifferentiable functions.

Thakur and Priya [25] discussed second-order duality results for nondifferentiable multiobjective programming problems with (ϕ, ρ)-univexity. A new pair of second-order multiobjective symmetric dual programs over arbitrary cones were introduced by Gupta and Kailey [11] and appropriate duality theorems were derived under K-η-bonvexity assumptions. Efficient solutions and optimality conditions for vector equilibrium problems were studied by Luu and Hang [14]. Recently, Gao [7] formulated a pair of Mond-Weir type higher-order symmetric dual programs over arbitrary cones and established duality results under higher-order (strongly) cone pseudoinvexity assumptions. Motivated by [7, 11, 25], we formulate a new pair of symmetric higher-order nondifferentiable multiobjective dual programs over arbitrary cones and establish weak, strong and converse duality theorems under higher-order K-η-convexity assumptions. An example of a non trivial function has been given to show the existence of higher-order K-η-convex functions. Self duality has been discussed by assuming the functions involved to be skew-symmetric. Some special cases are also obtained to show that this paper extends known results of the literature.

2. Notations and preliminaries

Consider the following multiobjective programming problem:

$$\textbf{(P)} \qquad K\text{-Minimize} \quad \phi(x)$$
$$\text{subject to} \quad x \in X^0 = \{x \in S : -g(x) \in Q\},$$

where $S \subseteq R^n$, $\phi : S \to R^k$, $g : S \to R^m$, K is a closed convex pointed cone in R^k with int $K \neq \phi$ and Q is closed convex cone with a nonempty interior in R^m.

Definition 2.1. *[13, 23] A point $\bar{x} \in X^0$ is a weakly efficient solution of* **(P)** *if there exists no other $x \in X^0$ such that*

$$\phi(\bar{x}) - \phi(x) \in int\ K.$$

Definition 2.2. *[13] A point $\bar{x} \in X^0$ is an efficient solution of* **(P)** *if there exists no other $x \in X^0$ such that*

$$\phi(\bar{x}) - \phi(x) \in K \setminus \{0\}.$$

Definition 2.3. *[13, 23] The positive polar cone of C^* of C is defined as*

$$C^* = \{z : \xi^T z \geqq 0,\ for\ all\ \xi \in C\}.$$

Definition 2.4. *[26] Let D be a compact convex set in R^n. The support function of D is defined by*

$$S(x|D) = \max\{x^T y : y \in D\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$S(y|D) \geqq S(x|D) + z^T(y - x) \text{ for all } y \in D.$$

The subdifferential of $S(x|D)$ is given by

$$\partial S(x|D) = \{z \in D : z^T x = S(x|D)\}.$$

For any convex set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \text{ for all } z \in S\}.$$

It can be easily seen that for a compact convex set D, y is in $N_D(x)$ if and only if $S(y|D) = x^T y$, or equivalently, x is in $\partial S(y|D)$.

Let C_1 and C_2 be closed convex cones with nonempty interiors in R^n and R^m, respectively and $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

Definition 2.5. *A differentiable function $f : S_1 \times S_2 \to R^k$ is said to be higher-order K-η_1-convex in the first variable at $u \in S_1$ for fixed $v \in S_2$ with respect to $g : S_1 \times S_2 \times R^n \to R^k$, if there exists a function $\eta_1 : S_1 \times S_1 \to R^n$ such that for $x \in S_1$, $q_i \in R^n$, $i = 1, 2, \ldots, k$,*

$$\Big(f_1(x,v) - f_1(u,v) - g_1(u,v,q_1) + q_1^T\nabla_{q_1}g_1(u,v,q_1) - \eta_1^T(x,u)[\nabla_x f_1(u,v) + \nabla_{q_1}g_1(u,v,q_1)], \ldots,$$
$$f_k(x,v) - f_k(u,v) - g_k(u,v,q_k) + q_k^T\nabla_{q_k}g_k(u,v,q_k) - \eta_1^T(x,u)[\nabla_x f_k(u,v) + \nabla_{q_k}g_k(u,v,q_k)]\Big) \in K,$$

and $f(x,y)$ is said to be higher-order K-η_2-convex in the second variable at $v \in S_2$ for fixed $u \in S_1$ with respect to $h : S_1 \times S_2 \times R^m \to R^k$, if there exists a function $\eta_2 : S_2 \times S_2 \to R^m$ such that for $y \in S_2$, $p_i \in R^m$, $i = 1, 2, \ldots, k$,

$$\Big(f_1(u,y) - f_1(u,v) - h_1(u,v,p_1) + p_1^T\nabla_{p_1}h_1(u,v,p_1) - \eta_2^T(y,v)[\nabla_y f_1(u,v) + \nabla_{p_1}h_1(u,v,p_1)], \ldots,$$
$$f_k(u,y) - f_k(u,v) - h_k(u,v,p_k) + p_k^T\nabla_{p_k}h_k(u,v,p_k) - \eta_2^T(y,v)[\nabla_y f_k(u,v) + \nabla_{p_k}h_k(u,v,p_k)]\Big) \in K.$$

Remark 2.1. (i) If we take $g_i(u,v,q_i) = \frac{1}{2}q_i^T\nabla_{xx}f_i(u,v)q_i$ and $h_i(u,v,p_i) = \frac{1}{2}p_i^T\nabla_{yy}f_i(u,v)p_i$, $i = 1, 2, \ldots, k$ then higher order K-η_1-convexity and K-η_2-convexity reduces to K-η_1-bonvexity and K-η_2-bonvexity [11] respectively.
(ii) The above definition can be reduced to η-convexity/invexity [18], η-bonvexity [12, 24] and K-convexity [23] as given in *Remark* 1 of [11].

Example 2.1. Let $X = (1.95, 2.4) \subset R$, $n = m = 1$, $k = 2$ and $K = \{(x,y) : x \geq 0,\ y \geq 0\}$. Consider the function $f : X \to R^2$ be defined by $f(x) = (f_1, f_2)$, where
$f_1(x) = 8\cos^2 x$, $\quad f_2(x) = \cos 3x$,

and $\eta : X \times X \to R$ be defined by $\eta(x,u) = -1 - u$. Suppose $g : X \times R \to R^2$ is defined by $g(u,q) = (g_1(u,q_1), g_2(u,q_2))$, where
$g_1(u,q_1) = q_1(u^2+1)$, $\quad g_2(u,q_2) = q_2(u^2-1)$.

To show that f is higher order K-η-convex, we have to prove that

$$\Big(f_1(x) - f_1(u) - g_1(u,q_1) + q_1^T\nabla_{q_1}g_1(u,q_1) - \eta^T(x,u)[\nabla_x f_1(u) + \nabla_{q_1}g_1(u,q_1)],$$
$$f_2(x) - f_2(u) - g_2(u,q_2) + q_2^T\nabla_{q_2}g_2(u,q_2) - \eta^T(x,u)[\nabla_x f_2(u) + \nabla_{q_2}g_2(u,q_2)]\Big) \in K,$$

or

$$(8\cos^2 x - 8\cos^2 u + (1+u)(-8\sin 2u + u^2 + 1),\ \cos 3x - \cos 3u + (1+u)(-3\sin 3u + u^2 - 1)) \in K$$

Let $L = (8\cos^2 x - 8\cos^2 u + (1+u)(-8\sin 2u + u^2 + 1),\ \cos 3x - \cos 3u + (1+u)(-3\sin 3u + u^2 - 1))$
$= (L_1, L_2)$,

where
$Ł_1 = 8\cos^2 x - 8\cos^2 u + (1+u)(-8\sin 2u + u^2 + 1)$
$\geq 0\ \forall x,\ u \in X$ as can be seen from Figure 1

and
$L_2 = \cos 3x - \cos 3u + (1+u)(-3\sin 3u + u^2 - 1)$

$\geq 0 \; \forall x,\; u \in X$ as can be seen from Figure 2

Therefore, f is higher-order K-η-convex with respect to g.

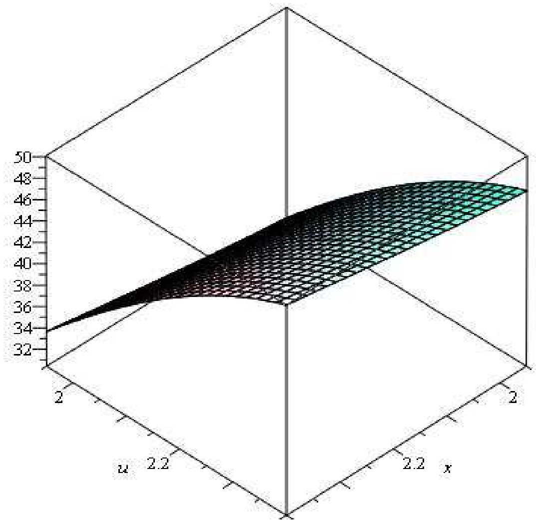

Figure 1: graph of L_1

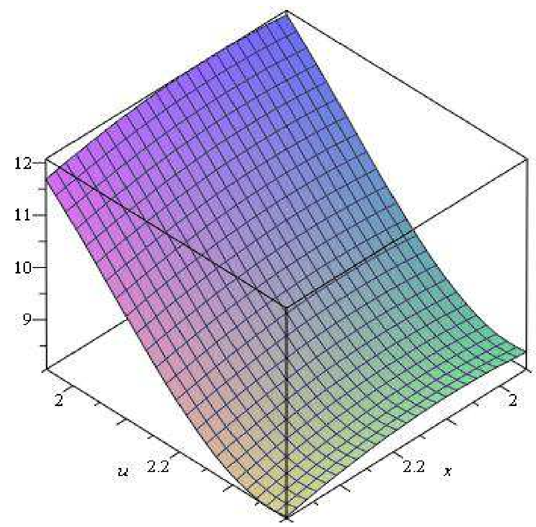

Figure 2: graph of L_2

Next, we need to show that f is not K-η-bonvex. To prove it, we will show that

$M = \Big(f_1(x) - f_1(u) + \dfrac{1}{2}q_1^T(\nabla_{xx}f_1(u)q_1) - \eta^T(x,u)[\nabla_x f_1(u) + \nabla_{xx}f_1(u)q_1],$

$f_2(x) - f_2(u) + \dfrac{1}{2}q_2^T(\nabla_{xx}f_2(u)q_2) - \eta^T(x,u)[\nabla_x f_2(u) + \nabla_{xx}f_2(u)q_2]\Big) \notin K,$

i.e., either

$f_1(x) - f_1(u) + \dfrac{1}{2}q_1^T(\nabla_{xx}f_1(u)q_1) - \eta^T(x,u)[\nabla_x f_1(u) + \nabla_{xx}f_1(u)q_1] \ngeq 0$

or

$f_2(x) - f_2(u) + \dfrac{1}{2}q_2^T(\nabla_{xx}f_2(u)q_2) - \eta^T(x,u)[\nabla_x f_2(u) + \nabla_{xx}f_2(u)q_2] \ngeq 0.$

Since

$$\begin{aligned}
&f_2(x) - f_2(u) + \frac{1}{2}q_2^T(\nabla_{xx}f_2(u)q_2) - \eta^T(x,u)[\nabla_x f_2(u) + \nabla_{xx}f_2(u)q_2]\\
&\quad= \cos 3x - \cos 3u - \frac{9}{2}q_2^2\cos 3u + (1+u)(-3\sin 3u - 9q_2\cos 3u)\\
&\quad\leq 0 \text{ (for } x = 2,\ u = 2.1 \text{ and } q_2 \in (-10^{18}, 10^{18}))
\end{aligned}$$

Therefore $M \notin K$. Hence f is not K-η-bonvex function.

3. Problem Formulation

Consider the following pair of higher-order nondifferentiable multiobjective symmetric dual programs:

Primal Problem (PP)

K-minimize

$$\begin{aligned}
S(x,y,\lambda,p) = \quad & \Big(f(x,y) + S(x|E)e_k - y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) + \nabla_{p_i}h_i(x,y,p_i))e_k\\
& + \sum_{i=1}^{k}\lambda_i h_i(x,y,p_i)e_k - \sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{p_i}h_i(x,y,p_i))e_k\Big)\\
& \text{subject to } -\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y) - z + \nabla_{p_i}h_i(x,y,p_i)) \in C_2^*, && (1)\\
& z \in D && (2)\\
& \lambda^T e_k = 1 && (3)\\
& \lambda \in \operatorname{int} K^*,\ x \in C_1 && (4)
\end{aligned}$$

Dual Problem (DP)

K-maximize

$$\begin{aligned}
T(u,v,\lambda,q) = \quad & f(u,v) - S(v|D)e_k - u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + \nabla_{q_i}g_i(u,v,q_i))e_k\\
& + \sum_{i=1}^{k}\lambda_i g_i(u,v,q_i)e_k - \sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{q_i}g_i(u,v,q_i))e_k\\
& \text{subject to } \sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v) + w + \nabla_{q_i}g_i(u,v,q_i)) \in C_1^*, && (5)\\
& w \in E && (6)\\
& \lambda^T e_k = 1 && (7)\\
& \lambda \in \operatorname{int} K^*,\ v \in C_2 && (8)
\end{aligned}$$

where

(i) $f_i : S_1 \times S_2 \to R$, $h_i : S_1 \times S_2 \times R^m \to R$ and $g_i : S_1 \times S_2 \times R^n \to R$, $i = 1, 2, \ldots, k$ are differentiable functions, where $h(x, y, p)$ denotes $(h_1(x, y, p_1), h_2(x, y, p_2), \ldots, h_k(x, y, p_k))$ and $g(u, v, q)$ denotes $(g_1(u, v, q_1), g(u, v, q_2), \ldots, g(u, v, q_k))$, $e_k = (1, \ldots, 1)^T \in R^k$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$,

(ii) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively,

(iii) q_i and p_i are vectors in R^n and R^m, respectively for $i = 1, 2, \ldots, k$.

(iv) E and D are compact convex sets in R^n and R^m, respectively, and

(v) $S(x|E)$ and $S(x|D)$ are the support functions of E and D, respectively.

Theorem 3.1. *(Weak Duality) Let (x, y, λ, z, p) be feasible for (PP) and (u, v, λ, w, q) be feasible for (DP). Let*

(i) *$f(\cdot, v) + (\cdot)^T we_k$ be higher-order $K - \eta_1$-convex at u with respect to $g(u, v, q)$ for fixed v,*

(ii) *$-f(x, \cdot) + (\cdot)^T ze_k$ be higher-order $K - \eta_2$-convex at y with respect to $-h(x, y, p)$ for fixed x,*

(iii) *$R_+^k \subseteq K$,*

(iv) *$\eta_1(x, u) + u \in C_1$, for all $x \in C_1$,*

(v) *$\eta_2(v, y) + y \in C_2$, for all $v \in C_2$.*

Then

$$S(x, y, \lambda, p) - T(u, v, \lambda, q) \notin -K \setminus \{0\}.$$

Proof. Suppose, to the contrary, that

$$S(x, y, \lambda, p) - T(u, v, \lambda, q) \in -K \setminus \{0\}.$$

that is

$$\begin{aligned}
&\Bigg\{\Bigg[f(x, y) + S(x|E)e_k - y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i))e_k \\
&+ \sum_{i=1}^{k} \lambda_i h_i(x, y, p_i)e_k - \sum_{i=1}^{k} \lambda_i(p_i^T \nabla_{p_i} h_i(x, y, p_i))e_k\Bigg] \\
&-\Bigg[f(u, v) - S(v|D)e_k - u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + \nabla_{q_i} g_i(u, v, q_i))e_k \\
&+ \sum_{i=1}^{k} \lambda_i g_i(u, v, q_i)e_k - \sum_{i=1}^{k} \lambda_i(q_i^T \nabla_{q_i} g_i(u, v, q_i))e_k\Bigg]\Bigg\} \in -K \setminus \{0\}
\end{aligned}$$

Since $\lambda \in \operatorname{int} K^* \subseteq \operatorname{int} R_+^k$ (by hypothesis (iii)), hence $\lambda > 0$. Therefore, we get

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i f_i(x, y) + S(x|E) - y^T \sum_{i=1}^{k} \lambda_i(\nabla_y f_i(x, y) + \nabla_{p_i} h_i(x, y, p_i)) \\
&+ \sum_{i=1}^{k} \lambda_i h_i(x, y, p_i) - \sum_{i=1}^{k} \lambda_i(p_i^T \nabla_{p_i} h_i(x, y, p_i)) \\
&-(\sum_{i=1}^{k} \lambda_i f_i(u, v) - S(v|D) - u^T \sum_{i=1}^{k} \lambda_i(\nabla_x f_i(u, v) + \nabla_{q_i} g_i(u, v, q_i)) \\
&+ \sum_{i=1}^{k} \lambda_i g_i(u, v, q_i) - \sum_{i=1}^{k} \lambda_i(q_i^T \nabla_{q_i} g_i(u, v, q_i))) < 0.
\end{aligned} \tag{9}$$

By higher-order $K-\eta_1$-convexity of $f(\cdot,v)+(\cdot)^T we_k$ with respect to $g(u,v,q)$, we have

$$\Big(f_1(x,v)+x^Tw-f_1(u,v)-u^Tw-g_1(u,v,q_1)+q_1^T\nabla_{q_1}g_1(u,v,q_1)$$
$$-\eta_1^T(x,u)[\nabla_x f_1(u,v)+w+\nabla_{q_1}g_1(u,v,q_1)],\ldots,$$
$$f_k(x,v)+x^Tw-f_k(u,v)-u^Tw-g_k(u,v,q_k)+q_k^T\nabla_{q_k}g_k(u,v,q_k)$$
$$-\eta_1^T(x,u)[\nabla_x f_k(u,v)+w+\nabla_{q_k}g_k(u,v,q_k)]\Big)\in K.$$

Using $\lambda\in\operatorname{int}K^*$, we get

$$\sum_{i=1}^k\lambda_i\Big\{f_i(x,v)+x^Tw-f_i(u,v)-u^Tw-g_i(u,v,q_i)+q_i^T\nabla_{q_i}g_i(u,v,q_i)$$
$$-\eta_1^T(x,u)[\nabla_x f_i(u,v)+w+\nabla_{q_i}g_i(u,v,q_i)]\Big\}\geqq 0. \tag{10}$$

Since (u,v,λ,w,q) is feasible for (DP), from the dual constraint (5) and hypothesis (iv), it follows that

$$[\eta_1(x,u)+u]^T\sum_{i=1}^k\lambda_i(\nabla_x f_i(u,v)+w+\nabla_{q_i}g_i(u,v,q_i))\geqq 0.$$

which implies

$$\eta_1^T(x,u)\sum_{i=1}^k\lambda_i(\nabla_x f_i(u,v)+w+\nabla_{q_i}g_i(u,v,q_i))$$
$$\geqq -u^T\sum_{i=1}^k\lambda_i(\nabla_x f_i(u,v)+w+\nabla_{q_i}g_i(u,v,q_i)). \tag{11}$$

Using (10), (11) and $\lambda^Te_k=1$, we obtain

$$\sum_{i=1}^k\lambda_i(f_i(x,v)-f_i(u,v)-g_i(u,v,q_i)+q_i^T\nabla_{q_i}g_i(u,v,q_i))+x^Tw-u^Tw$$
$$\geqq -u^T\sum_{i=1}^k\lambda_i(\nabla_x f_i(u,v)+\nabla_{q_i}g_i(u,v,q_i))-u^Tw. \tag{12}$$

Similarly, by higher-order $K-\eta_2$-convexity of $-f(x,\cdot)+(\cdot)^Tze_k$ with respect to $-h(x,y,p)$, from (1) and hypothesis (v), we get

$$\sum_{i=1}^k\lambda_i(f_i(x,y)-f_i(x,v)+h_i(x,y,p_i)-p_i^T\nabla_{p_i}h_i(x,y,p_i))-y^Tz+v^Tz$$
$$\geqq y^T\sum_{i=1}^k\lambda_i(\nabla_y f_i(x,y)+\nabla_{p_i}h_i(x,y,p_i))-y^Tz. \tag{13}$$

Adding inequalities (12) and (13), we have

$$\Big(\sum_{i=1}^{k}\lambda_i f_i(x,y)+x^Tw-y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{p_i}h_i(x,y,p_i))$$
$$+\sum_{i=1}^{k}\lambda_i h_i(x,y,p_i)-\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{p_i}h_i(x,y,p_i))\Big)$$
$$\geqq\Big(\sum_{i=1}^{k}\lambda_i f_i(u,v)-v^Tz-u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{q_i}g_i(u,v,q_i))$$
$$+\sum_{i=1}^{k}\lambda_i g_i(u,v,q_i)-\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{q_i}g_i(u,v,q_i))\Big),$$

By using $x^Tw\leqq S(x|E)$ and $v^Tz\leqq S(v|D)$ in above inequality, we obtain

$$\Big(\sum_{i=1}^{k}\lambda_i f_i(x,y)+S(x|E)-y^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(x,y)+\nabla_{p_i}h_i(x,y,p_i))$$
$$+\sum_{i=1}^{k}\lambda_i h_i(x,y,p_i)-\sum_{i=1}^{k}\lambda_i(p_i^T\nabla_{p_i}h_i(x,y,p_i))\Big)$$
$$-\Big(\sum_{i=1}^{k}\lambda_i f_i(u,v)-S(v|D)-u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{q_i}g_i(u,v,q_i))$$
$$+\sum_{i=1}^{k}\lambda_i g_i(u,v,q_i)-\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{q_i}g_i(u,v,q_i))\Big)\geqq 0,$$

which contradicts (9). Hence the result.

If the variable λ in the problems (PP) and (DP) is fixed to be $\bar\lambda$, we shall denote these problems by $(PP)_{\bar\lambda}$ and $(DP)_{\bar\lambda}$, respectively. □

Theorem 3.2. *(Strong Duality) Let $(\bar x,\bar y,\bar\lambda,\bar z,\bar p)$ be a weak efficient solution of (PP). Suppose that*

(i) *the Hessian matrix $\nabla_{p_ip_i}h_i$, $\forall\ i=1,2,\ldots,k$, is positive or negative definite.*
(ii) *the set of vectors $\nabla_y f_1(\bar x,\bar y),\ldots,\nabla_y f_k(\bar x,\bar y)$ is linearly independent,*
(iii) $\sum_{i=1}^{k}\bar\lambda_i\nabla_{yy}f_i\bar p_i\notin span\{\nabla_y f_i+\nabla_{p_i}h_i-\nabla_y h_i,\nabla_y f_i,\ i=1,2,\ldots,k\}\setminus\{0\}$,
(iv) *$\bar p_i\neq 0$, for some $i\in\{1,2,\ldots,k\}$ imply that $\sum_{i=1}^{k}\bar\lambda_i\nabla_{yy}f_i\bar p_i\neq 0$,*
(v) $\sum_{i=1}^{k}\bar\lambda_i h_i(\bar x,\bar y,0)=\sum_{i=1}^{k}\bar\lambda_i g_i(\bar x,\bar y,0)$, $\sum_{i=1}^{k}\bar\lambda_i\nabla_y h_i(\bar x,\bar y,0)=0$, $\sum_{i=1}^{k}\bar\lambda_i\nabla_{p_i}h_i(\bar x,\bar y,0)=0$, $\sum_{i=1}^{k}\bar\lambda_i\nabla_x h_i(\bar x,\bar y,0)=\sum_{i=1}^{k}\bar\lambda_i\nabla_{q_i}g_i(\bar x,\bar y,0)$ *and*
(vi) *K is a closed convex pointed cone with $R_+^k\subseteq K$.*

Then,

(I) *there exists $\bar w\in E$ such that $(\bar x,\bar y,\bar\lambda,\bar w,\bar q=0)$ is feasible for $(DP)_{\bar\lambda}$, and*
(II) $S(\bar x,\bar y,\bar\lambda,\bar p)=T(\bar x,\bar y,\bar\lambda,\bar q)$.

Furthermore, if the hypotheses of Theorem 3.1. are satisfied for all feasible solutions of (PP) and $(DP)_{\bar\lambda}$, then $(\bar x,\bar y,\bar\lambda,\bar w,\bar q=0)$ is an efficient solution for $(DP)_{\bar\lambda}$.

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is a weakly efficient solution of (PP), there exist $\bar{\alpha}\in K^*$, $\bar{\beta}\in C_2$, $\bar{\eta}\in R$, such that the following by Fritz-John optimality conditions ([23], Lemma 1) are satisfied at $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ (for simplicity, we write $\nabla_x f_i$, $\nabla_{xy} f_i$ instead of $\nabla_x f_i(\bar{x},\bar{y})$, $\nabla_{xy} f_i(\bar{x},\bar{y})$ etc.):

$$(x-\bar{x})^T\Big\{\sum_{i=1}^{k}\bar{\alpha}_i(\nabla_x f_i+\bar{\gamma})+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{xy}f_i)^T[\bar{\beta}-(\bar{\alpha}^Te_k)\bar{y}]$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x h_i(\bar{\alpha}^Te_k)+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{xp_i}h_i)^T\Big[\bar{\beta}-(\bar{\alpha}^Te_k)(\bar{y}+\bar{p}_i)\Big]\Big\}\geqq 0,\ \text{for all } x\in C_1, \tag{14}$$

$$\sum_{i=1}^{k}\bar{\alpha}_i\nabla_y f_i+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i)^T[\bar{\beta}-(\bar{\alpha}^Te_k)\bar{y}]+\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y h_i(\bar{\alpha}^Te_k)$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yp_i}h_i)^T\Big[\bar{\beta}-(\bar{\alpha}^Te_k)(\bar{y}+\bar{p}_i)\Big]-\sum_{i=1}^{k}\bar{\lambda}_i[\nabla_y f_i+\nabla_{p_i}h_i](\bar{\alpha}^Te_k)=0, \tag{15}$$

$$(\nabla_y f_i)^T[\bar{\beta}-(\bar{\alpha}^Te_k)\bar{y}]+h_i(\bar{x},\bar{y},\bar{p}_i)(\bar{\alpha}^Te_k)+(\nabla_{p_i}h_i(\bar{x},\bar{y},\bar{p}_i))^T\Big[\bar{\beta}-\bar{\alpha}^Te_k(\bar{y}+\bar{p}_i)\Big]$$
$$+\bar{\eta}=0,\ i=1,2,\dots,k, \tag{16}$$

$$[(\bar{\beta}-(\bar{\alpha}^Te_k)(\bar{y}+\bar{p}_i))\bar{\lambda}_i]^T\nabla_{p_ip_i}h_i=0,\ i=1,2,\dots,k, \tag{17}$$

$$\bar{\beta}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i-\bar{z}+\nabla_{p_i}h_i)=0, \tag{18}$$

$$\bar{\eta}^T[\bar{\lambda}^Te_k-1]=0, \tag{19}$$

$$\bar{\beta}\in N_D(\bar{z}), \tag{20}$$

$$\bar{\gamma}\in E,\ \bar{\gamma}^T\bar{x}=S(\bar{x}|E) \tag{21}$$

$$(\bar{\alpha},\bar{\beta},\bar{\eta})\neq 0. \tag{22}$$

Since $R^k_+\subseteq K\Rightarrow K^*\subseteq R^k_+$ which implies int$(K^*)\subseteq$int(R^k_+).
As $\bar{\lambda}\in$ int(K^*), therefore $\bar{\lambda}>0$.
Now hypothesis (i), $\bar{\lambda}_i>0$ for $i=1,2,\dots,k$, and (17) imply that

$$\bar{\beta}=(\bar{\alpha}^Te_k)(\bar{y}+\bar{p}_i),i=1,2,\dots,k, \tag{23}$$

If $\bar{\alpha}=0$ then (23) yields $\bar{\beta}=0$. Further, the Eq. (16) gives $\bar{\eta}=0$. Consequently $(\bar{\alpha},\bar{\beta},\bar{\eta})=0$, contradicting (22). Hence $\bar{\alpha}\neq 0$. Further, $\bar{\alpha}\in K^*\subseteq R^k_+$ implies

$$\bar{\alpha}^Te_k>0. \tag{24}$$

Using (23) and (24) in (15), we get

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{yy}f_i\bar{p}_i=\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i+\nabla_{p_i}h_i-\nabla_y h_i)-\frac{1}{(\bar{\alpha}^Te_k)}\sum_{i=1}^{k}\bar{\alpha}_i\nabla_y f_i, \tag{25}$$

which yields

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{yy}f_i\bar{p}_i\in \text{span}\{\nabla_y f_i+\nabla_{p_i}h_i-\nabla_y h_i,\nabla_y f_i,\ i=1,2,\dots,k\}. \tag{26}$$

Now we claim $\bar{p}_i=0$ for all $i=1,2,\dots,k$. On the contrary, suppose that for some $i\in\{1,2,\dots,k\}$, $\bar{p}_i\neq 0$, then using hypothesis (iv), we have

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{yy}f_i\bar{p}_i\neq 0. \tag{27}$$

This contradicts hypothesis (iii) (by (26) and (27)). Hence

$$\bar{p}_i=0 \text{ for } i=1,2,\dots,k, \tag{28}$$

and thus relation (23) gives

$$\bar{\beta} = (\bar{\alpha}^T e_k)\bar{y}. \tag{29}$$

Using hypothesis (v) and (28) in (25) yields

$$\sum_{i=1}^{k} \nabla_y f_i[\bar{\alpha}_i - (\bar{\alpha}^T e_k)\bar{\lambda}_i] = 0,$$

which on using hypothesis (ii) gives

$$\bar{\alpha}_i = (\bar{\alpha}^T e_k)\bar{\lambda}_i, \; i = 1, 2, \ldots, k, \tag{30}$$

Using (24), (28) – (30) in (14), we have

$$(x - \bar{x})^T \left\{ \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x h_i(\bar{x}, \bar{y}, \bar{p}_i) \right\} \geqq 0, \text{for all } x \in C_1.$$

For $\bar{q}_i = 0$, it follows from the hypothesis (v) and (28) that

$$(x - \bar{x})^T \left\{ \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(\bar{x}, \bar{y}) + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{q_i} g_i(\bar{x}, \bar{y}, \bar{q}_i) \right\} \geqq 0. \tag{31}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$ and so (31) implies

$$x^T \left\{ \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{q_i} g_i \right\} \geqq 0, \text{for all } x \in C_1.$$

Therefore,

$$\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{q_i} g_i \in C_1^*. \tag{32}$$

Also from (24) and (29), we have

$$\bar{y} = \frac{\bar{\beta}}{\bar{\alpha}^T e_k} \in C_2.$$

Thus $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w} = \bar{\gamma}, \bar{q} = 0)$ satisfies the constraints of $(\mathrm{DP})_{\bar{\lambda}}$ and so it is a feasible solution for the dual problem $(\mathrm{DP})_{\bar{\lambda}}$.

Now, letting $x = 0$ and $x = 2\bar{x}$ in (31), we get

$$\bar{x}^T \left\{ \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i + \bar{\gamma}) + \sum_{i=1}^{k} \bar{\lambda}_i \nabla_{q_i} g_i \right\} = 0$$

or

$$\bar{x}^T \left\{ \sum_{i=1}^{k} \bar{\lambda}_i \nabla_x f_i + \nabla_{q_i} g_i \right\} = -\bar{x}^T \bar{\gamma} = -S(\bar{x}|E). \tag{33}$$

From (20) and (29), $(\bar{\alpha}^T e_k)\bar{y} \in N_D(\bar{z})$. Since $\bar{\alpha}^T e_k > 0$, $\bar{y} \in N_D(\bar{z})$. Also, as D is a compact convex set in R^m, $\bar{y}^T \bar{z} = S(\bar{y}|D)$.

Further from (18), (24) and (29) and the above relation, we obtain

$$\bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_y f_i + \nabla_{p_i} h_i) = \bar{y}^T \bar{z} = S(\bar{y}|D). \tag{34}$$

Therefore, using (28), (33), (34) and the hypothesis (v), for $\bar{q}_i = 0$, we get

$$\begin{aligned}
& f(\bar{x},\bar{y}) + S(\bar{x}|E)e_k - \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) + \nabla_{p_i} h_i(\bar{x},\bar{y},\bar{p}_i))e_k \\
& + \sum_{i=1}^{k} \bar{\lambda}_i h_i(\bar{x},\bar{y},\bar{p}_i)e_k - \sum_{i=1}^{k} \bar{\lambda}_i(\bar{p}_i^T \nabla_{p_i} h_i(\bar{x},\bar{y},\bar{p}_i))e_k \\
& = f(\bar{x},\bar{y}) - S(\bar{y}|D)e_k - \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \nabla_{q_i} g_i(\bar{x},\bar{y},\bar{q}_i))e_k \\
& + \sum_{i=1}^{k} \bar{\lambda}_i g_i(\bar{x},\bar{y},\bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{q_i} g_i(\bar{x},\bar{y},\bar{q}_i))e_k
\end{aligned}$$

that is, the two objective values are equal.
Now let $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0)$ be not an efficient solution of $(\mathrm{DP})_{\bar{\lambda}}$, then there exists $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{q}=0)$ feasible for $(\mathrm{DP})_{\bar{\lambda}}$ such that

$$\begin{aligned}
& \Big\{ f(\bar{x},\bar{y}) - S(\bar{y}|D)e_k - \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{x},\bar{y}) + \nabla_{q_i} g_i(\bar{x},\bar{y},\bar{q}_i))e_k \\
& + \sum_{i=1}^{k} \bar{\lambda}_i g_i(\bar{x},\bar{y},\bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{q_i} g_i(\bar{x},\bar{y},\bar{q}_i))e_k \Big\} \\
& - \Big\{ f(\bar{u},\bar{v}) - S(\bar{v}|D)e_k - \bar{u}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{q_i} g_i(\bar{u},\bar{v},\bar{q}_i))e_k \\
& + \sum_{i=1}^{k} \bar{\lambda}_i g_i(\bar{u},\bar{v},\bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{q_i} g_i(\bar{u},\bar{v},\bar{q}_i))e_k \Big\} \in -K \setminus \{0\},
\end{aligned}$$

Using (28), (33), (34) and the hypothesis (v), for $\bar{q}_i = 0$, we obtain

$$\begin{aligned}
& \{ f(\bar{x},\bar{y}) + S(\bar{x}|E)e_k - \bar{y}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y}) + \nabla_{p_i} h_i(\bar{x},\bar{y},\bar{p}_i))e_k \\
& + \sum_{i=1}^{k} \bar{\lambda}_i h_i(\bar{x},\bar{y},\bar{p}_i)e_k - \sum_{i=1}^{k} \bar{\lambda}_i(\bar{p}_i^T \nabla_{p_i} h_i(\bar{x},\bar{y},\bar{p}_i))e_k \} \\
& - \{ f(\bar{u},\bar{v}) - S(\bar{v}|D)e_k - \bar{u}^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_x f_i(\bar{u},\bar{v}) + \nabla_{q_i} g_i(\bar{u},\bar{v},\bar{q}_i))e_k \\
& + \sum_{i=1}^{k} \bar{\lambda}_i g_i(\bar{u},\bar{v},\bar{q}_i)e_k - \sum_{i=1}^{k} \bar{\lambda}_i(\bar{q}_i^T \nabla_{q_i} g_i(\bar{u},\bar{v},\bar{q}_i))e_k \} \in -K \setminus \{0\},
\end{aligned}$$

that is

$$S(\bar{x},\bar{y},\bar{\lambda},\bar{p}) - T(\bar{u},\bar{v},\bar{\lambda},\bar{q}) \in -K \setminus \{0\}.$$

which contradicts Theorem 3.1. Hence $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{q}=0)$ is an efficient solution of $(\mathrm{DP})_{\bar{\lambda}}$.

Theorem 3.3. *(Converse Duality) Let $(\bar{u},\bar{v},\bar{\lambda},\bar{w},\bar{q})$ be a weak efficient solution of (DP). Suppose that*

(i) *the Hessian matrix $\nabla_{q_i q_i} g_i$, $\forall\, i = 1,2,\dots,k$ is positive or negative definite.*

(ii) *the set of vectors* $\nabla_x f_1(\bar{u},\bar{v}),\ldots,\nabla_x f_k(\bar{u},\bar{v})$ *is linearly independent,*

(iii) $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{xx}f_i\bar{q}_i \notin span\{\nabla_x f_i+\nabla_{q_i}g_i-\nabla_x g_i,\nabla_x f_i,\ i=1,2,\ldots,k\}\setminus\{0\}$,

(iv) $\bar{q}_i\neq 0$, *for some* $i\in\{1,2,\ldots,k\}$ *imply that* $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{xx}f_i\bar{q}_i\neq 0$,

(v) $\sum_{i=1}^{k}\bar{\lambda}_i g_i(\bar{u},\bar{v},0)=\sum_{i=1}^{k}\bar{\lambda}_i h_i(\bar{u},\bar{v},0)$, $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x g_i(\bar{u},\bar{v},0)=0$, $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{q_i}g_i(\bar{u},\bar{v},0)=0$, $\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y g_i(\bar{u},\bar{v},0)=\sum_{i=1}^{k}\bar{\lambda}_i\nabla_{p_i}h_i(\bar{u},\bar{v},0)$ *and*

(vi) K *is a closed convex pointed cone with* $R_+^k\subseteq K$.

Then,

(I) *there exists* $\bar{z}\in D$ *such that* $(\bar{u},\bar{v},\bar{\lambda},\bar{z},\bar{p}=0)$ *is feasible for* $(PP)_{\bar{\lambda}}$, *and*

(II) $S(\bar{u},\bar{v},\bar{\lambda},\bar{p})=T(\bar{u},\bar{v},\bar{\lambda},\bar{q})$.

Furthermore, if the hypotheses of Theorem 3.1. *are satisfied for all feasible solutions of* $(PP)_{\bar{\lambda}}$ *and (DP), then* $(\bar{u},\bar{v},\bar{\lambda},\bar{z},\bar{p}=0)$ *is an efficient solution for* $(PP)_{\bar{\lambda}}$.

4. Self-duality

A mathematical programming problem is said to be self-dual if it is formally identical with its dual i.e. the dual can be recast in the form of the primal. Mond and Cottle [17] observed that the symmetric dual programs of Dantzig et al. [5] are self duals if $H(x,y)$ is skew symmetric and gave self duality results. In general (PP) and (DP) are not self dual without an added restriction on f, g and h. For the programs (PP) and (DP), self duality exists under the following assumptions:
(i) $m=n$, (ii) $C_1=C_2$, (iii) $D=E$, (iv) the vector functions $f:R^n\times R^m\to R^k$ and $g:R^n\times R^m\times R^n\to R^k$ to be skew symmetric, i.e.,$f_i(x,y)=-f_i(y,x)$ and $g_i(u,v,q_i)=-g_i(v,u,q_i)$, $i\in\{1,2,\ldots,k\}$.
Now recasting the dual problem (DP) as a minimization problem:
(DP1) K-minimize

$$\Big(-f(u,v)+S(v|D)e_k+u^T\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+\nabla_{q_i}g_i(u,v,q_i))e_k$$
$$-\sum_{i=1}^{k}\lambda_i g_i(u,v,q_i)e_k+\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{q_i}g_i(u,v,q_i))e_k\Big)$$
$$\text{subject to }\sum_{i=1}^{k}\lambda_i(\nabla_x f_i(u,v)+w+\nabla_{q_i}g_i(u,v,q_i))\in C_1^*,$$
$$w\in E$$
$$\lambda^T e_k=1$$
$$\lambda\in\text{int }K^*,\ v\in C_2$$

Now f and g are skew symmetric,
i.e., $\nabla_x f_i(u,v)=-\nabla_y f_i(v,u)$ and $\nabla_{q_i}g_i(u,v,q_i)=-\nabla_{q_i}g_i(v,u,q_i)$ for $i=1,\ldots,k$. Therefore, the problem (DP1)

reduces to, K-minimize

$$\Big(f(v,u)+S(v|E)e_k-u^T\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(v,u)+\nabla_{q_i}g_i(v,u,q_i))e_k$$
$$+\sum_{i=1}^{k}\lambda_i g_i(v,u,q_i)e_k-\sum_{i=1}^{k}\lambda_i(q_i^T\nabla_{q_i}g_i(v,u,q_i))e_k\Big)$$
$$\text{subject to } -\sum_{i=1}^{k}\lambda_i(\nabla_y f_i(v,u)-w+\nabla_{q_i}g_i(v,u,q_i))\in C_2^*,$$
$$w\in D$$
$$\lambda^T e_k=1$$
$$\lambda\in \operatorname{int}K^*,\ v\in C_1$$

This shows that (DP1) is formally identical to (PP), that is, the objective and the constraint functions are identical. Hence (PP) is self dual. Consequently, the feasibility of (x,y,λ,z,p) for (PP) implies the feasibility of (y,x,λ,z,p) for (DP) and conversely.

5. Special Cases

In all these cases, if $\sum_{i=1}^{k}\lambda_i h_i(x,y,p_i)=\sum_{i=1}^{k}\lambda_i\frac{1}{2}p_i\nabla_{yy}f_i(x,y)p_i$ and $\sum_{i=1}^{k}\lambda_i g_i(u,v,q_i)=\sum_{i=1}^{k}\lambda_i\frac{1}{2}q_i\nabla_{xx}f_i(u,v)q_i$.

(i) If $E=\{0\}$ and $D=\{0\}$, then our problems (PP) and (DP) become the problems studied in Gupta and Kailey [11].

(ii) For $K=R_+^k,\ C_1=R_+^n,\ C_2=R_+^m,\ k=1,\ q_i=q,\ p_i=p,\ E=\{By: y^TBy\leqq 1\},\ D=\{Cx: x^TCx\leqq 1\}$, where B and C are positive semidefinite matrices, $(x^TBx)^{\frac{1}{2}}=S(x|E)$ and $(y^TCy)^{\frac{1}{2}}=S(y|D)$, (PP) and (DP) reduce to the problems considered in Ahmad and Hussain [1].

(iii) If $K=R_+^k,\ C_1=R_+^n,\ C_2=R_+^m,\ k=1,\ q_i=q,\ p_i=p$, then our problems (PP) and (DP) reduce to the programs studied in Yang et al. [26].

(iv) The cases given in Gupta and Kailey [11] can also be extracted from our problems.

6. Conclusions

A new pair of multiobjective higher-order symmetric dual programs involving support functions over arbitrary cones has been formulated. We have given an example of a non trivial function to show the existence of higher-order K-η-convex functions. Weak, strong and converse duality theorems under higher-order K-η-convexity assumptions have also been established. It is to be noted that some of the known results, including Ahmad and Hussain [1], Gupta and Kailey [11] and Yang et al. [26], are special cases of our study. This work can be further extended to study mixed symmetric higher-order nondifferentiable multiobjective dual programs over arbitrary cones.

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