Filomat 33:3 (2019), 761–772
https://doi.org/10.2298/FIL1903761W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Differences of Weighted Differentiation Composition Operators From α-Bloch Space to H^∞ Space

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Abstract. This paper characterizes the boundedness and compactness of the differences of weighted differentiation composition operators acting from the α-Bloch space $\mathscr{B}^\alpha$ to the space H^∞ of bounded holomorphic functions on the unit disk $\mathbb{D}$.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all holomorphic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the class of all holomorphic functions from $\mathbb{D}$ in itself, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. Denote by $H^\infty = H^\infty(\mathbb{D})$ the space of all bounded holomorphic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

For $0 < \alpha < \infty$, a holomorphic function f is said to be in the Bloch-type space $\mathscr{B}^\alpha$ or $\alpha-$Bloch space, if

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

The little Bloch-type space $\mathscr{B}_0^\alpha$, consists of all $f \in \mathscr{B}^\alpha$, such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0.$$

As we all know, both $\mathscr{B}^\alpha$ and $\mathscr{B}_0^\alpha$ are Banach spaces under the norm

$$\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \|f\|_\alpha.$$

Moreover, the $\mathscr{B}_0^\alpha$ is the closure of polynomials in $\mathscr{B}^\alpha$. When $0 < \alpha < 1$, $\mathscr{B}^\alpha$ is the analytic Lipschitz space $Lip_{1-\alpha}$, which consists of all $f \in H(\mathbb{D})$ satisfying

$$|f(z) - f(w)| \le C|z-w|^{1-\alpha},$$

2010 *Mathematics Subject Classification*. Primary 47B38; Secondary 30H30, 30H05
Keywords. weighted differentiation composition operators, α-Bloch space, H^∞ space
Received: 09 March 2015; Accepted: 30 April 2019
Communicated by Dragan S. Djordjević
This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11771323; 11371276), Foundation of Henan Educational Committee(18A110023), and the Scientific Research Foundation for Ph.D. of Henan Normal University (No.qd16151).
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for some constant $C > 0$ and all $z, w \in \mathbb{D}$. When $\alpha = 1$, $\mathscr{B}^\alpha$ becomes the classical Bloch space $\mathscr{B}$. When $\alpha > 1$, $\mathscr{B}^\alpha$ is equivalent to the weighted Banach space $H^\infty_{\alpha-1}$, where H^∞_α is the weighted Banach space consisting of all analytic functions f on $\mathbb{D}$ satisfying $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)| < \infty$. We refer the readers to the excellent monograph [1], and the article [21] about the Bloch-type spaces.

Let $\varphi \in S(\mathbb{D})$, $u \in H(\mathbb{D})$, $n \in \mathbb{N}$, we consider the weighted differentiation composition operator $D^n_{\varphi,u}$, $D^n_{\varphi,u}f = u(z)f^{(n)}(\varphi(z))$, which is the product of three operators, the composition operator C_φ, the order n derivative operator D^n, and the multiplication by u operator M_u. If $n = 0$ and $u = 1$, $D^n_{\varphi,u}$ becomes the composition operator C_φ on $H(\mathbb{D})$. If $n = 0$, we get the weighted composition operator uC_φ defined as $uC_\varphi f = uf(\varphi)$. If $u = 1$ and $\varphi(z) = z$, then $D^n_{\varphi,u}$ reduces to the the differentiation operator D^n. The boundedness and compactness of differentiation composition operator between spaces of holomorphic functions have been studied extensively. For example, Hibschweiler and Portnoy [4] studied $C_\varphi D$ between Bergman and Hardy spaces. Wu and Wulan [19] gave a new compactness criterion for $C_\varphi D^n$ on the Bloch space. Recently, the weighted differentiation composition operator between different holomorphic function spaces has also been investigated by several researchers [10, 13, 20].

Motivated by the research in the topological structure of the set $C(H^2)$ of composition operators on H^2 with the operator norm topology, the difference of two composition operators, i.e. an operator of the form $C_\varphi - C_\psi$, where φ, ψ are analytic self-maps of $\mathbb{D}$, was first investigated in the case of H^2 in [15]. Shortly after, the differences of (weighted) composition operators were characterized by many researchers. MacCluer, Ohno and Zhao [11] showed that the compactness of $C_\varphi - C_\psi : H^\infty \to H^\infty$ is equivalent to the compactness of $C_\varphi - C_\psi : \mathscr{B} \to H^\infty$. Also C_φ and C_ψ are in the same path component of the space of composition operators on H^∞ if and only if $C_\varphi - C_\psi : \mathscr{B} \to H^\infty$ is bounded. Hosokawa and Ohno [7] not only provided new results about the boundedness and compactness of the differences of two weighted composition operators from $\mathscr{B}$ to H^∞ on $\mathbb{D}$, but also estimated the essential norms of the differences of two (weighted) composition operators from $\mathscr{B}$ to H^∞. Soon after Song and Zhou [16] improved such characterizations for the high dimensional cases. For further references and details about the difference of two (weighted) composition operators, see [2, 3, 5, 6, 9, 14, 17, 18].

In this paper, our goal is to investigate the boundedness and compactness of the differences of weighted differentiation composition operators from $\mathscr{B}^\alpha$ to H^∞ on $\mathbb{D}$, i.e. $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$, where $u, v \in H(\mathbb{D})$ and $\varphi, \psi \in S(\mathbb{D})$, $n \in \mathbb{N}$.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which varies from one appearance to the next. $A \asymp B$, $A \preceq B$, $A \succeq B$ mean that there exist different positive constants C such that $B/C \le A \le CB$, $A \le CB$, $CB \le A$.

2. Notations and Lemmas

In order to handle the differences of weighted differentiation composition operators we need the pseudo-hyperbolic metric. Recall that, for any a, $z \in \mathbb{D}$, $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of $\mathbb{D}$ which interchanges the origin and a. The pseudo-hyperbolic metric is given by $\rho(z, a) = |\sigma_a(z)|$. Moreover, we have that $\sigma_a'(z) = \frac{|a|^2-1}{(1-\bar{a}z)^2}$.

Our main results are based on the following lemmas.

Lemma 2.1. ([21]) *The following asymptotic relationship holds*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| \asymp \sum_{j=0}^{n-1}|f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)|.$$

Lemma 2.2. ([12]) *For $f \in H^\infty_\alpha$ and $z, w \in \mathbb{D}$, $|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)| \le \|f\|_{H^\infty_\alpha}\rho(z, w)$.*

Remark 2.3. *For more general weights, the result can be found in [9].*

Lemma 2.4. *For $n \in \mathbb{N}$, and $z, w \in \mathbb{D}$, there exists a constant $C > 0$ such that for all $f \in \mathscr{B}^\alpha$,*

$$|(1-|z|^2)^{\alpha+n-1} f^{(n)}(z) - (1-|w|^2)^{\alpha+n-1} f^{(n)}(w)| \leq C\rho(z, w).$$

Proof. From Lemma 2.1 and Lemma 2.2, we get this inequality obviously. □

The following criterion for the compactness is a useful tool and it follows from standard arguments, see Proposition 3.11 of [1].

Lemma 2.5. *Let $u, v \in H(\mathbb{D})$ and $\varphi, \psi \in S(\mathbb{D})$, $n \in \mathbb{N}$. Then $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is compact if and only if $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathscr{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|(D^n_{\varphi,u} - D^n_{\psi,v})f_k\|_\infty \to 0$ as $k \to \infty$.*

Lemma 2.6. (i) *For $z \in \mathbb{D}$ and $a \in \mathbb{D}$ with $a \neq 0$, let*

$$f_a(z) = \frac{1-|a|^2}{(\bar{a})^n \alpha \cdots (\alpha+n-1)(1-\bar{a}z)^\alpha}.$$

Then $f_a(z) \in \mathscr{B}^\alpha$ and

$$f_a^{(n)}(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\alpha+n}}.$$

(ii) *For $z \in \mathbb{D}$ and $a \in \mathbb{D}$ with $a \neq 0$, let*

$$g_a(z) = \sigma_a(z) f_a(z) + \frac{(1-|a|^2) f_a(z)}{\bar{a}(1-\bar{a}z)} - \frac{(1-|a|^2)^2}{(\bar{a})^{n+1}(\alpha+1)\cdots(\alpha+n)(1-\bar{a}z)^{\alpha+1}}.$$

Then $g_a(z) \in \mathscr{B}^\alpha$ and

$$g_a^{(n)}(z) = \sigma_a(z) f_a^{(n)}(z).$$

Proof. (i) Differentiate the function, we get

$$f_a'(z) = \frac{1-|a|^2}{(\bar{a})^{n-1}(\alpha+1)\cdots(\alpha+n-1)(1-\bar{a}z)^{\alpha+1}}$$

and

$$f_a^{(n)}(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\alpha+n}}.$$

Using $|1-\bar{a}z| \geq 1-|a|$, $|1-\bar{a}z| \geq 1-|z|$, we obtain

$$\begin{aligned}
\|f_a\|_\alpha &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f_a'(z)| \\
&= \sup_{z\in\mathbb{D}} \frac{(1-|a|^2)(1-|z|^2)^\alpha}{|a|^{n-1}(\alpha+1)\cdots(\alpha+n-1)|1-\bar{a}z|^{\alpha+1}} \\
&= \sup_{z\in\mathbb{D}} \frac{1}{|a|^{n-1}(\alpha+1)\cdots(\alpha+n-1)} \cdot \frac{1-|a|^2}{|1-\bar{a}z|} \cdot \frac{(1-|z|^2)^\alpha}{|1-\bar{a}z|^\alpha} \\
&\leq \sup_{z\in\mathbb{D}} \frac{1}{|a|^{n-1}(\alpha+1)\cdots(\alpha+n-1)} \cdot \frac{1-|a|^2}{1-|a|} \cdot \frac{(1-|z|^2)^\alpha}{(1-|z|)^\alpha} \\
&< \frac{2^{\alpha+1}}{|a|^{n-1}(\alpha+1)\cdots(\alpha+n-1)} \\
&< \infty.
\end{aligned}$$

Suppose $z = 0$, we have

$$f_a(0) = \frac{1-|a|^2}{(\overline{a})^n \alpha \cdots (\alpha+n-1)}.$$

Thus

$$\|f_a\|_{\mathscr{B}^\alpha} = |f_a(0)| + \|f_a\|_\alpha < \infty.$$

So $f_a(z) \in \mathscr{B}^\alpha$, it follows that there exists a constant $C > 0$ such that $\|f_a\|_{\mathscr{B}^\alpha} \le C$.

(ii) By Leibniz formula, we obtain

$$\begin{aligned}
g_a^{(n)}(z) &= \sum_{k=0}^{n} C_n^k \sigma_a^{(n-k)}(z) f_a^{(k)}(z) + \frac{(1-|a|^2)}{\bar{a}} \sum_{k=0}^{n} C_n^k \frac{(n-k)!(\bar{a})^{n-k}}{(1-\bar{a}z)^{n-k+1}} f_a^{(k)}(z) - \frac{(1-|a|^2)^2}{\bar{a}(1-\bar{a}z)^{\alpha+n+1}} \\
&= \sigma_a(z) f_a^{(n)}(z) + \sum_{k=0}^{n-1} C_n^k \sigma_a^{(n-k)}(z) f_a^{(k)}(z) + \frac{(1-|a|^2)}{\bar{a}} \sum_{k=0}^{n} C_n^k \frac{(n-k)!(\bar{a})^{n-k}}{(1-\bar{a}z)^{n-k+1}} f_a^{(k)}(z) - \frac{(1-|a|^2)^2}{\bar{a}(1-\bar{a}z)^{\alpha+n+1}} \\
&= \sigma_a(z) f_a^{(n)}(z) + \sum_{k=0}^{n-1} C_n^k \frac{(n-k)!(|a|^2-1)(\bar{a})^{n-k-1}}{(1-\bar{a}z)^{n-k+1}} \cdot f_a^{(k)}(z) \\
&\quad + \sum_{k=0}^{n} C_n^k \frac{(n-k)!(\bar{a})^{n-k-1}(1-|a|^2)}{(1-\bar{a}z)^{n-k+1}} \cdot f_a^{(k)}(z) - \frac{(1-|a|^2)^2}{\bar{a}(1-\bar{a}z)^{\alpha+n+1}} \\
&= \sigma_a(z) f_a^{(n)}(z) - \sum_{k=0}^{n-1} C_n^k \frac{(n-k)!(1-|a|^2)(\bar{a})^{n-k-1}}{(1-\bar{a}z)^{n-k+1}} \cdot f_a^{(k)}(z) \\
&\quad + \sum_{k=0}^{n} C_n^k \frac{(n-k)!(\bar{a})^{n-k-1}(1-|a|^2)}{(1-\bar{a}z)^{n-k+1}} \cdot f_a^{(k)}(z) - \frac{(1-|a|^2)^2}{\bar{a}(1-\bar{a}z)^{\alpha+n+1}} \\
&= \sigma_a(z) f_a^{(n)}(z) + \frac{(1-|a|^2) f_a^{(n)}(z)}{\bar{a}(1-\bar{a}z)} - \frac{(1-|a|^2)^2}{\bar{a}(1-\bar{a}z)^{\alpha+n+1}} \\
&= \sigma_a(z) f_a^{(n)}(z) + \frac{(1-|a|^2)^2}{\bar{a}(1-\bar{a}z)^{\alpha+n+1}} - \frac{(1-|a|^2)^2}{\bar{a}(1-\bar{a}z)^{\alpha+n+1}} \\
&= \sigma_a(z) f_a^{(n)}(z),
\end{aligned}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

Using the facts that $|\sigma_a(z)| \le 1$, $|1-\bar{a}z| \ge 1-|a|$ and $|1-\bar{a}z| \ge 1-|z|$, we obtain

$$\begin{aligned}
\|g_a\|_\alpha &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |g_a'(z)| \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\sigma_a(z) f_a'(z)| \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \left| \frac{\sigma_a(z)(1-|a|^2)}{(\bar{a})^{n-1}(\alpha+1)\cdots(\alpha+n-1)(1-\bar{a}z)^{\alpha+1}} \right| \\
&= \sup_{z\in\mathbb{D}} \frac{|\sigma_a(z)|}{|a|^{n-1}(\alpha+1)\cdots(\alpha+n-1)} \cdot \frac{(1-|a|^2)}{|1-\bar{a}z|} \cdot \frac{(1-|z|^2)^\alpha}{|1-\bar{a}z|^\alpha} \\
&< \frac{2^{\alpha+1}}{|a|^{n-1}(\alpha+1)\cdots(\alpha+n-1)} \\
&< \infty.
\end{aligned}$$

Taking $z = 0$, we have

$$g_a(0) = a f_a(0) - \frac{(1-|a|^2)}{\bar{a}} f_a(0) - \frac{(1-|a|^2)^2}{(\bar{a})^{n+1}(\alpha+1)\cdots(\alpha+n)}.$$

Thus

$$\|g_a\|_{\mathscr{B}^\alpha} = |g_a(0)| + \|g_a\|_\alpha < \infty.$$

So $g_a(z) \in \mathscr{B}^\alpha$, in other words there exists a constant $C > 0$ such that $\|g_a\|_\alpha \le C$. □

In order to state our main results conveniently, we define some sets as follows.

$$I_1(z) = \frac{u(z)}{(1-|\varphi(z)|^2)^{\alpha+n-1}},\ I_2(z) = \frac{v(z)}{(1-|\psi(z)|^2)^{\alpha+n-1}},$$
$$\Gamma_\varphi = \{\{z_j\} \subset \mathbb{D} : |\varphi(z_j)| \to 1\},\ \Gamma_\psi = \{\{z_j\} \subset \mathbb{D} : |\psi(z_j)| \to 1\},$$
$$G_{u,\varphi} = \{\{z_j\} \in \Gamma_\varphi : I_1(z_j) \nrightarrow 0\},\ G_{v,\psi} = \{\{z_j\} \in \Gamma_\psi : I_2(z_j) \nrightarrow 0\}.$$

3. The boundedness of $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$

Theorem 3.1. *Let $u, v \in H(\mathbb{D})$ and $\varphi, \psi \in S(\mathbb{D})$, $n \in \mathbb{N}$. Then the following statements are equivalent:*
(i) *$D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is bounded.*
(ii)

$$\sup_{z\in\mathbb{D}} |I_1(z)|\rho(\varphi(z), \psi(z)) < \infty, \tag{1}$$

$$\sup_{z\in\mathbb{D}} |I_1(z) - I_2(z)| < \infty. \tag{2}$$

(iii) *Condition* (2) *and*

$$\sup_{z\in\mathbb{D}} |I_2(z)|\rho(\varphi(z), \psi(z)) < \infty. \tag{3}$$

Proof. (i) ⇒ (ii). Suppose that $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is bounded. We choose the test function $k_\omega(z) = (z - \psi(\omega))^{n+1}/(n+1)!$. Since

$$\|k_\omega\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |k'_\omega(z)| = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \Big|\frac{(z-\psi(\omega))^n}{n!}\Big| \le \frac{2^n}{n!} < \infty$$

yields $k_\omega(z) \in \mathscr{B}^\alpha$, meanwhile, $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is bounded, it follows that

$$\infty > \|(D^n_{\varphi,u} - D^n_{\psi,v})k_\omega\|_\infty \ge |u(\omega)(\varphi(\omega) - \psi(\omega))|. \tag{4}$$

If $\varphi(\omega) = 0$, (4) shows

$$\infty > |u(\omega)\psi(\omega)| = |I_1(\omega)|\rho(\varphi(\omega), \psi(\omega)). \tag{5}$$

Next, we consider another case $\varphi(\omega) \neq 0$. For $a \in \mathbb{D}$ with $a \neq 0$, set

$$f_a(z) = \frac{1-|a|^2}{(\bar{a})^n \alpha \cdots (\alpha+n-1)(1-\bar{a}z)^\alpha}$$

and

$$g_a(z) = \sigma_a(z) f_a(z) + \frac{(1-|a|^2) f_a(z)}{\bar{a}(1-\bar{a}z)} - \frac{(1-|a|^2)^2}{(\bar{a})^{n+1}(\alpha+1)\cdots(\alpha+n)(1-\bar{a}z)^{\alpha+1}}.$$

By Lemma 2.6, $f_a(z), g_a(z) \in \mathscr{B}^\alpha$. Fix $\omega \in \mathbb{D}$ with $\varphi(\omega) \neq 0$, we get

$$\begin{aligned}
\infty &> \|(D^n_{\varphi,u} - D^n_{\psi,v}) f_{\varphi(\omega)}\|_\infty \\
&\ge |u(\omega) f^{(n)}_{\varphi(\omega)}(\varphi(\omega)) - v(\omega) f^{(n)}_{\varphi(\omega)}(\psi(\omega))| \\
&= \Big|I_1(\omega) - \frac{v(\omega)(1-|\varphi(\omega)|^2)}{(1-\overline{\varphi(\omega)}\psi(\omega))^{n+\alpha}}\Big| \\
&\ge |I_1(\omega)| - |I_2(\omega)| \frac{(1-|\psi(\omega)|^2)^{\alpha+n-1}(1-|\varphi(\omega)|^2)}{(1-\overline{\varphi(\omega)}\psi(\omega))^{n+\alpha}}
\end{aligned} \tag{6}$$

and

$$\begin{aligned}\infty &> \|(D^n_{\varphi,u}-D^n_{\psi,v})g_{\varphi(\omega)}\|_\infty\\ &\geq |u(\omega)g^{(n)}_{\varphi(\omega)}(\varphi(\omega))-v(\omega)g^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &= |v(\omega)\sigma_{\varphi(\omega)}(\psi(\omega))f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &= |\frac{v(\omega)(1-|\varphi(\omega)|^2)}{(1-\overline{\varphi(\omega)}\psi(\omega))^{n+\alpha}}|\rho(\varphi(\omega),\psi(\omega))\\ &= |I_2(\omega)|\frac{(1-|\psi(\omega)|^2)^{\alpha+n-1}(1-|\varphi(\omega)|^2)}{(1-\overline{\varphi(\omega)}\psi(\omega))^{n+\alpha}}\rho(\varphi(\omega),\psi(\omega)).\end{aligned}\tag{7}$$

Multiplying (6) by $\rho(\varphi(\omega),\psi(\omega))$, then adding (7) gives for all $\omega\in\mathbb{D}$ with $\varphi(\omega)\neq 0$

$$|I_1(\omega)|\rho(\varphi(\omega),\psi(\omega))|<\infty.\tag{8}$$

Therefore, by (5) and (8), condition (1) holds. If we change $\psi(\omega)$ into $\varphi(\omega)$ for the function $k_\omega(z)$, $\varphi(\omega)$ into $\psi(\omega)$ for the functions $f_{\varphi(\omega)}(z)$, $g_{\varphi(\omega)}(z)$, we can show that (3) holds.

To prove (2), using function $f_a(z)$, by Lemma 2.6, we have

$$\begin{aligned}\infty &> \|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\varphi(\omega)}\|_\infty\\ &\geq |u(\omega)f^{(n)}_{\varphi(\omega)}(\varphi(\omega))-v(\omega)f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &= |I_1(\omega)-\frac{v(\omega)}{(1-|\psi(\omega)|^2)^{\alpha+n-1}}(1-|\psi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &= |I_1(\omega)-I_2(\omega)(1-|\psi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &= |I_1(\omega)-I_2(\omega)+I_2(\omega)-I_2(\omega)(1-|\psi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &= |I_1(\omega)-I_2(\omega)+I_2(\omega)\frac{(1-|\varphi(\omega)|^2)^{\alpha+n-1}}{(1-|\varphi(\omega)|^2)^{\alpha+n-1}}-I_2(\omega)(1-|\psi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &= |I_1(\omega)-I_2(\omega)+I_2(\omega)(1-|\varphi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\varphi(\omega))-I_2(\omega)(1-|\psi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &\geq |I_1(\omega)-I_2(\omega)|-|I_2(\omega)|\cdot|(1-|\varphi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\varphi(\omega))-(1-|\psi(\omega)|^2)^{\alpha+n-1}f^{(n)}_{\varphi(\omega)}(\psi(\omega))|\\ &\geq |I_1(\omega)-I_2(\omega)|-C|I_2(\omega)|\rho(\varphi(\omega),\psi(\omega)).\end{aligned}\tag{9}$$

(9) and (3) guarantee

$$|I_1(\omega)-I_2(\omega)|<\infty,\quad \text{for all }\ \omega\in\mathbb{D}\ \text{ with }\ \varphi(\omega)\neq 0.\tag{10}$$

If $\varphi(\omega)=0$ and $1>|\psi(\omega)|\geq\frac{1}{2}$, then $\rho(\varphi(\omega),\psi(\omega))=|\psi(\omega)|\geq\frac{1}{2}$. By conditions (1) and (3), we can deduce directly

$$\begin{aligned}\frac{|I_1(\omega)-I_2(\omega)|}{2} &\leq |I_1(\omega)-I_2(\omega)|\rho(\varphi(\omega),\psi(\omega))\\ &\leq |I_1(\omega)|\rho(\varphi(\omega),\psi(\omega))+|I_2(\omega)|\rho(\varphi(\omega),\psi(\omega))\\ &< \infty.\end{aligned}\tag{11}$$

Let $f(z)=\frac{z^n}{n!}$. Since Taylor expansion, $1-(1-|\psi(\omega)|^2)^{n+\alpha-1}\le C|\psi(\omega)|$. If $\varphi(\omega)=0$ and $|\psi(\omega)|<\frac{1}{2}$, then

$$\begin{aligned}
\infty &> \|(D^n_{\varphi,u}-D^n_{\psi,v})f\|_\infty\\
&\ge |u(\omega)f^{(n)}(\varphi(\omega))-v(\omega)f^{(n)}(\psi(\omega))|\\
&= |u(\omega)-v(\omega)|\\
&\ge |I_1(\omega)-I_2(\omega)|-|I_2(\omega)|(1-(1-|\psi(\omega)|^2)^{n+\alpha-1})\\
&\ge |I_1(\omega)-I_2(\omega)|-C|I_2(\omega)\psi(\omega)|\\
&= |I_1(\omega)-I_2(\omega)|-C|I_2(\omega)|\rho(\varphi(\omega),\psi(\omega)).
\end{aligned}$$

Applying the above inequality with (3), we obtain

$$|I_1(\omega)-I_2(\omega)|<\infty. \tag{12}$$

Thus by (10), (11) and (12), we conclude that (2) holds for all $\omega\in\mathbb{D}$.

(ii) ⇒ (iii). Suppose that the conditions (1) and (2) hold. Then

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}|I_2(z)|\rho(\varphi(z),\psi(z))\\
\le\ &\sup_{z\in\mathbb{D}}|I_1(z)\rho(\varphi(z),\psi(z))-I_1(z)\rho(\varphi(z),\psi(z))+I_2(z)\rho(\varphi(z),\psi(z))|\\
\le\ &\sup_{z\in\mathbb{D}}|I_1(z)-I_2(z)|\rho(\varphi(z),\psi(z))+\sup_{z\in\mathbb{D}}|I_1(z)|\rho(\varphi(z),\psi(z))\\
\le\ &\sup_{z\in\mathbb{D}}|I_1(z)-I_2(z)|+\sup_{z\in\mathbb{D}}|I_1(z)|\rho(\varphi(z),\psi(z))\\
<\ &\infty.
\end{aligned}$$

Thus (3) follows.

(iii) ⇒ (i). For $\forall f\in\mathscr{B}^\alpha$ with $\|f\|_\alpha\le1$, by Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned}
&\|(D^n_{\varphi,u}-D^n_{\psi,v})f\|_\infty\\
\le\ &\sup_{z\in\mathbb{D}}|u(z)f^{(n)}(\varphi(z))-v(z)f^{(n)}(\psi(z))|\\
=\ &\sup_{z\in\mathbb{D}}|\frac{u(z)(1-|\varphi(z)|^2)^{\alpha+n-1}}{(1-|\varphi(z)|^2)^{\alpha+n-1}}f^{(n)}(\varphi(z))-\frac{v(z)(1-|\psi(z)|^2)^{\alpha+n-1}}{(1-|\psi(z)|^2)^{\alpha+n-1}}f^{(n)}(\psi(z))|\\
=\ &\sup_{z\in\mathbb{D}}|I_1(z)(1-|\varphi(z)|^2)^{\alpha+n-1}f^{(n)}(\varphi(z))-I_2(z)(1-|\psi(z)|^2)^{\alpha+n-1}f^{(n)}(\psi(z))|\\
=\ &\sup_{z\in\mathbb{D}}|I_1(z)(1-|\varphi(z)|^2)^{\alpha+n-1}f^{(n)}(\varphi(z))-I_2(z)(1-|\varphi(z)|^2)^{\alpha+n-1}f^{(n)}(\varphi(z))\\
&+I_2(z)(1-|\varphi(z)|^2)^{\alpha+n-1}f^{(n)}(\varphi(z))-I_2(z)(1-|\psi(z)|^2)^{\alpha+n-1}f^{(n)}(\psi(z))|\\
\le\ &\sup_{z\in\mathbb{D}}|I_1(z)-I_2(z)|(1-|\varphi(z)|^2)^{\alpha+n-1}|f^{(n)}(\varphi(z))|\\
&+\sup_{z\in\mathbb{D}}|I_2(z)||(1-|\varphi(z)|^2)^{\alpha+n-1}f^{(n)}(\varphi(z))-(1-|\psi(z)|^2)^{\alpha+n-1}f^{(n)}(\psi(z))|\\
\le\ &\sup_{z\in\mathbb{D}}|I_1(z)-I_2(z)|\|f\|_\alpha+C\sup_{z\in\mathbb{D}}|I_2(z)|\rho(\varphi(z),\psi(z))\\
\le\ &\sup_{z\in\mathbb{D}}|I_1(z)-I_2(z)|+C\sup_{z\in\mathbb{D}}|I_2(z)|\rho(\varphi(z),\psi(z))\\
<\ &\infty.
\end{aligned}$$

Therefore, $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}^\alpha\to H^\infty$ is bounded. □

Corollary 3.2. *Let $u\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$, $n\in\mathbb{N}$. Then $D^n_{\varphi,u}:\mathscr{B}^\alpha\to H^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}|I_1(z)|<\infty.$$

Remark 3.3. *In fact, if* $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ *is bounded, choosing the test function* $f(z) = \frac{z^n}{n!}$, *then there exists a constant* $C > 0$ *such that*

$$\sup_{z\in\mathbb{D}} |u(z)| < C.$$

Remark 3.4. *While preparing the revisions, we found Theorem 3.1 has been obtained by Liang in [8]. Here we only use different methods.*

4. The compactness of $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$

Theorem 4.1. *Let* $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$. *Then* $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ *is compact if and only if* $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} |I_1(z)| = 0. \tag{13}$$

Proof. Suppose that $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ is bounded and (13) holds. To establish the assertion, it suffices, in view of Lemma 2.5, to show that for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathscr{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|D^n_{\varphi,u} f_k\|_\infty \to 0$ as $k \to \infty$.
Without loss of generality, we assume that $\|f_k\|_\alpha \leq 1$. (13) implies that, for any $\varepsilon > 0$, there exists $r \in (0, 1)$, such that when $r < |\varphi(z)| < 1$, we have

$$|I_1(z)| = \frac{|u(z)|}{(1 - |\varphi(z)|^2)^{\alpha+n-1}} < \varepsilon.$$

On the other hand, Lemma 2.1 gives

$$\sup_{r<|\varphi(z)|<1} u(z)|f_k^{(n)}(\varphi(z))| \leq \sup_{r<|\varphi(z)|<1} \frac{u(z)}{(1 - |\varphi(z)|^2)^{\alpha+n-1}} \|f_k\|_\alpha < \varepsilon. \tag{14}$$

Since $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ is bounded, Corollary 3.2 states that

$$\sup_{z\in\mathbb{D}} |I_1(z)| < \infty.$$

Also, since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $f_k^{(n)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Therefore, there exists $N \in \mathbb{N}$, such that $k > N$ implies that

$$\sup_{|\varphi(z)|\leq r} u(z)|f_k^{(n)}(\varphi(z))| < \varepsilon \sup_{|\varphi(z)|\leq r} u(z) < C\varepsilon. \tag{15}$$

By (14) and (15),

$$\begin{aligned} \|D^n_{\varphi,u} f_k\|_\infty &= \sup_{z\in\mathbb{D}} u(z)|f_k^{(n)}(\varphi(z))| \\ &= \sup_{|\varphi(z)|\leq r} u(z)|f_k^{(n)}(\varphi(z))| + \sup_{r<|\varphi(z)|<1} u(z)|f_k^{(n)}(\varphi(z))| \\ &\leq (C + 1)\varepsilon. \end{aligned}$$

It follows that the operator $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ is compact.

To prove the converse, assume that $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ is compact. Then it is obvious that $D^n_{\varphi,u} : \mathscr{B}^\alpha \to H^\infty$ is bounded. Let z_k be a sequence in $\mathbb{D}$ such that $\varphi(z_k) \to 1$ as $k \to \infty$. If we choose test function

$$f_k(z) = \frac{1 - |\varphi(z_k)|^2}{(\overline{\varphi(z_k)})^n \alpha \cdots (\alpha + n - 1)(1 - z\overline{\varphi(z_k)})^\alpha},$$

since $|1-z\overline{\varphi(z_k)}| \geq 1-|z|$, clearly f_k converges to 0 uniformly on $\mathbb{D}$ as $k \to \infty$. Hence f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$. Lemma 2.5 implies

$$0 \leftarrow \|D^n_{\varphi,u} f_k\|_\infty \quad \geq \quad |u(z_k)||f_k^{(n)}(\varphi(z_k))| = \frac{|u(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n-1}}, \text{ as } k \to \infty.$$

Since $z_k \in \mathbb{D}$ is arbitrary, (13) follows. □

Theorem 4.2. *Let $u, v \in H(\mathbb{D})$ and $\varphi, \psi \in S(\mathbb{D})$, $n \in \mathbb{N}$. Suppose that $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is bounded and that neither of $D^n_{\varphi,u}, D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is compact. Then $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is compact if and only if* (a), (b), (c) *and* (d) *hold:*

(a) $G_{u,\varphi} = G_{v,\psi}$,

(b) $\lim_{j\to\infty} |I_1(z_j)|\rho(\varphi(z_j), \psi(z_j)) = 0, \forall z_j \in \Gamma_\varphi \bigcap \Gamma_\psi$,

(c) $\lim_{j\to\infty} |I_2(z_j)|\rho(\varphi(z_j), \psi(z_j)) = 0, \forall z_j \in \Gamma_\varphi \bigcap \Gamma_\psi$,

(d) $\lim_{j\to\infty} |I_1(z_j) - I_2(z_j)| = 0, \forall z_j \in \Gamma_\varphi \bigcap \Gamma_\psi$.

Proof. Sufficiency. Suppose that the four conditions hold. If $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B}^\alpha \to H^\infty$ is not compact, via Theorem 4.1, there exists a bounded sequence $\{f_j\} \in \mathscr{B}^\alpha$ such that $\|f_j\|_{\mathscr{B}^\alpha} \leqslant 1$ and converges to 0 uniformly on every compact subset of $\mathbb{D}$. However $\|(D^n_{\varphi,u} - D^n_{\psi,v})f_j\|_\infty \nrightarrow 0$ as $j \to \infty$. Then for $\forall\ \varepsilon > 0$, there exists $K > 0$, such that when $j > K$, $\|(D^n_{\varphi,u} - D^n_{\psi,v})f_j\|_\infty > \varepsilon$. Obviously there exists $\{z_j\} \subset \mathbb{D}$ such that

$$\begin{aligned}
& \|(D^n_{\varphi,u} - D^n_{\psi,v})f_j\|_\infty \\
= \ & |\frac{u(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1}}{(1-|\varphi(z_j)|^2)^{\alpha+n-1}} f_j^{(n)}(\varphi(z_j)) - \frac{v(z_j)(1-|\psi(z_j)|^2)^{\alpha+n-1}}{(1-|\psi(z_j)|^2)^{\alpha+n-1}} f_j^{(n)}(\psi(z_j))| \\
= \ & |I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\varphi(z_j)) - I_2(z_j)(1-|\psi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\psi(z_j))| \\
> \ & \varepsilon.
\end{aligned} \tag{16}$$

This implies that either $|\varphi(z_j)|$ or $|\psi(z_j)|$ tends to 1. In order to prove this, assume that $|\varphi(z_j)| \to 1$. Let $\omega \in \overline{\mathbb{D}}$ be a limit point of $|\psi(z_j)|$. Passing to a subsequence, if necessary, we may assume that $|\psi(z_j)| \to \omega$.
If $|\omega| < 1$, then $\{z_j\} \not\subseteq \Gamma_\varphi \bigcap \Gamma_\psi$. Since $G_{u,\varphi} \subset \Gamma_\varphi \bigcap \Gamma_\psi$, $\{z_j\} \not\subseteq G_{u,\varphi}$. By the definition of $G_{u,\varphi}$, clearly, $|I_1(z_j)| \to 0$. Moreover, by Cauchy's estimate, $|\omega| < 1$ yields $|f_j^{(n)}(\varphi(z_j))| \to 0$ as $j \to \infty$. Therefore

$$|I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\varphi(z_j))| \quad \to \quad 0, \text{ as } j \to \infty. \tag{17}$$

By (a), we have $\{z_j\} \not\subseteq G_{v,\psi}$. Using Cauchy's estimate again, it follows that

$$|I_2(z_j)(1-|\psi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\psi(z_j))| \quad \to \quad 0, \text{ as } j \to \infty. \tag{18}$$

Combining (17) and (18), we get a contradiction to (16). Thus $|\omega|$ can only be 1. Hence $|\varphi(z_j)|, |\psi(z_j)|$ tend to 1, and so $\{z_j\} \subset \Gamma_\varphi \bigcap \Gamma_\psi$. The assumptions ($b$) and ($d$) imply that

$$\begin{aligned}
& |I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\varphi(z_j)) - I_2(z_j)(1-|\psi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\psi(z_j))| \\
= \ & |I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\varphi(z_j)) - I_2(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\varphi(z_j)) \\
& + I_2(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\varphi(z_j)) - I_2(z_j)(1-|\psi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\psi(z_j))| \\
\leq \ & |I_1(z_j) - I_2(z_j)|(1-|\varphi(z_j)|^2)^{\alpha+n-1}|f_j^{(n)}(\varphi(z_j))| \\
& + |I_2(z_j)||(1-|\varphi(z)|^2)^{\alpha+n-1} f_j^{(n)}(\varphi(z_j)) - (1-|\psi(z_j)|^2)^{\alpha+n-1} f_j^{(n)}(\psi(z_j))| \\
\leq \ & |I_1(z) - I_2(z_j)|\|f_j\|_\alpha + C|I_2(z_j)|\rho(\varphi(z_j), \psi(z_j)) \\
\leq \ & |I_1(z_j) - I_2(z_j)| + C|I_1(z_j)|\rho(\varphi(z_j), \psi(z_j)) \to 0, \text{ as } j \to \infty.
\end{aligned}$$

We arrive at a contradiction to (16) again. So under the assumption and conditions $(a)-(d)$, $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}^\alpha\to H^\infty$ is compact.

Necessity. If $D^n_{\varphi,u}:\mathscr{B}^\alpha\to H^\infty$ is not compact, by Theorem 4.1, there exists a sequence $\{z_j\}\in G_{u,\varphi}$ with $|\varphi(z_j)|\to 1$ such that $|I_1(z_j)|\nrightarrow 0$. For $\omega_j=\varphi(z_j)$, define f_{ω_j}, g_{ω_j} as in Lemma 2.6. For $|1-z\overline{\omega_j}|\geq 1-|z|$, it is easy to check that f_{ω_j}, g_{ω_j} converge to 0 uniformly on every compact subset of $\mathbb{D}$ as $j\to\infty$. Since $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}^\alpha\to H^\infty$ is compact, by Lemma 2.5, as $j\to\infty$, we obtain

$$\begin{aligned} 0 &\leftarrow \|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\omega_j}\|_\infty \\ &\geq |u(z_j)f^{(n)}_{\omega_j}(\omega_j)-v(z_j)f^{(n)}_{\omega_j}(\psi(z_j))| \\ &= |\frac{u(z_j)}{(1-|\omega_j|^2)^{\alpha+n-1}}-\frac{v(z_j)(1-|\omega_j|^2)}{(1-\overline{\omega_j}\psi(z_j))^{n+\alpha}}| \\ &= |I_1(z_j)-\frac{v(z_j)(1-|\omega_j|^2)}{(1-\overline{\omega_j}\psi(z_j))^{n+\alpha}}| \\ &\geq |I_1(z_j)|-|I_2(z_j)|\frac{(1-|\psi(z_j)|)^{\alpha+n-1}(1-|\omega_j|^2)}{(1-\overline{\omega_j}\psi(z_j))^{n+\alpha}} \end{aligned} \tag{19}$$

and

$$\begin{aligned} 0 &\leftarrow \|(D^n_{\varphi,u}-D^n_{\psi,v})g_{\omega_j}\|_\infty \\ &\geq |u(z_j)g^{(n)}_{\omega_j}(\omega_j)-v(z_j)g^{(n)}_{\omega_j}(\psi(z_j))| \\ &= |v(z_j)\sigma_{\omega_j}(\psi(z_j))f^{(n)}_{\omega_j}(\psi(z_j))| \\ &= |I_2(z_j)|\frac{(1-|\psi(z_j)|)^{\alpha+n-1}(1-|\omega_j|^2)}{(1-\overline{\omega_j}\psi(z_j))^{n+\alpha}}\rho(\omega_j,\psi(z_j)). \end{aligned} \tag{20}$$

Multiplying (19) by $\rho(\omega_j,\psi(z_j))$, and combining it with (20), we find

$$\lim_{j\to\infty}|I_1(z_j)|\rho(\omega_j,\psi(z_j))=0. \tag{21}$$

Since $|I_1(z_j)|\nrightarrow 0$, we see that

$$\rho(\omega_j,\psi(z_j))=0 \tag{22}$$

as $j\to\infty$. Because $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}^\alpha\to H^\infty$ is bounded, by (22) and (3), we have

$$\lim_{j\to\infty}|I_2(z_j)|\rho(\omega_j,\psi(z_j))=0. \tag{23}$$

In addition, we know

$$0\leftarrow\|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\omega_j}\|_\infty\geq(|I_1(z_j)-I_2(z_j)|-|I_2(z_j)\rho(\omega_j,\psi(z_j))|) \tag{24}$$

as $j\to\infty$. Hence by (23) and (24), we get

$$\lim_{j\to\infty}|I_1(z_j)-I_2(z_j)|=0. \tag{25}$$

Hence, from (22) and (25), we have $G_{u,\varphi}\subseteq G_{v,\psi}$. Similar to the above proof, we conclude that $G_{u,\varphi}\supseteq G_{v,\psi}$. Therefore $G_{u,\varphi}=G_{v,\psi}$. Meanwhile, (b), (c) and (d) can be got from (21), (23) and (25), respectively, where $\{z_j\}\subset\Gamma_\varphi\bigcap\Gamma_\psi$ with $|I_1(z_j)|\nrightarrow 0$.

Next, for $\forall\{z_j\}\subset\Gamma_\varphi\bigcap\Gamma_\psi$ with $|I_1(z_j)|\to 0$ as $j\to\infty$, we will prove (b), (c) and (d). First we can easily get

$$\lim_{j\to\infty}|I_1(z_j)|\rho(\varphi(z_j),\psi(z_j))=0. \tag{26}$$

On the other hand, using $f_{\psi(z_j)}$ which defined as Lemma 2.6, for $|1-z\overline{\psi(z_j)}|\geq 1-|z|$ and $\{z_j\}\subset\Gamma_\varphi\bigcap\Gamma_\psi$, it is easy to check that $f_{\psi(z_j)}$ converge to 0 uniformly on every compact subset of $\mathbb{D}$ as $j\to\infty$. Lemma 2.5 implies that

$$\begin{aligned}
0 &\leftarrow \|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\psi(z_j)}\|_\infty\\
&\geq |u(z_j)f^{(n)}_{\psi(z_j)}(\varphi(z_j))-v(z_j)f^{(n)}_{\psi(z_j)}(\psi(z_j))|\\
&= |\frac{u(z_j)}{(1-|\varphi(z_j)|^2)^{\alpha+n-1}}(1-|\varphi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\varphi(z_j))-I_2(z_j)|\\
&= |I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\varphi(z_j))-I_2(z_j)|\\
&= |I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\varphi(z_j))-I_2(z_j)+I_1(z_j)-I_1(z_j)|\\
&= |I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\varphi(z_j))-I_2(z_j)+I_1(z_j)-I_1(z_j)\frac{(1-|\psi(z_j)|^2)^{\alpha+n-1}}{(1-|\psi(z_j)|^2)^{\alpha+n-1}}|\\
&= |I_1(z_j)(1-|\varphi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\varphi(z_j))-I_2(z_j)+I_1(z_j)-I_1(z_j)(1-|\psi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\psi(z_j))|\\
&\geq |I_1(z_j)-I_2(z_j)|-|I_1(z_j)||(1-|\varphi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\varphi(z_j))-(1-|\psi(z_j)|^2)^{\alpha+n-1}f^{(n)}_{\psi(z_j)}(\psi(z_j))|\\
&\geq |I_1(z_j)-I_2(z_j)|-|I_1(z_j)|\rho(\varphi(z_j),\psi(z_j)) \;\text{ as }\; j\to\infty.
\end{aligned}$$

By (26), clearly we can obtain

$$\lim_{j\to\infty}|I_1(z_j)-I_2(z_j)|=0.$$

So, for $\forall\{z_j\}\subset\Gamma_\varphi\bigcap\Gamma_\psi$ with $|I_1(z_j)|\to 0$ as $j\to\infty$, $|I_2(z_j)|$ converges to 0 as $j\to\infty$, too. Therefore

$$\lim_{j\to\infty}|I_2(z_j)|\rho(\varphi(z_j),\psi(z_j))=0.$$

The theorem is established. □

Acknowledgement. The authors are very grateful to the referee for helpful comments and suggestions.

References

[1] C. C. Cowen, B. D. MacCluer, Composition operators on spaces of analytic functions, CRC Press, Studies in Ad-vanced Mathematics, CRC, Boca Raton, FL, 1995.

[2] Z. S. Fang, Z. H. Zhou, Differences of composition operators on the space of bounded analytic functions in the Polydisc, Abstract and Applied Analysis 2008 (2008) 10p., Article ID 983132.

[3] K. Heller, B. D. MacCluer, R. J. Weir, Compact differences of composition operators in several variables, Integral Equations and Operator Theory 69 (2011) 247–268.

[4] R. A. Hibschweiler, N. Portnoy, Composition followed by differentiation between Bergman and Hardy spaces, The Rocky Mountain Journal of Mathematics 35 (2009) 843–855.

[5] T. Hosokawa, Differences of weighted composition operators on the Bloch spaces, Complex Analysis and Operator Theory 3 (2009) 847–866.

[6] T. Hosokawa, K. Izuchi, Essential norms of differences of composition operators on H^∞, Journal of the Mathematical Society of Japan 57 (2005) 669–690.

[7] T. Hosokawa, S. Ohno, Differences of weighted composition operators acting from bloch space to H^∞, Transactions of the American Mathematical Society 363 (2011) 5321–5340.

[8] Y. Liang, New characterizations for differences of weighted differentiation composition operators from a Bloch-type space to a weighted-type space. Periodica Mathematica Hungarica. Journal of the Jnos Bolyai Mathematical Society 77 (2018) 119–138.

[9] M. Lindström, E. Wolf, Essential norm of the difference of weighted composition operators, Monatshefte für Mathematik 153 (2008) 133–143.

[10] J. Liu, Z. Lou, Ajay K. Sharma, Weighted Differentiation Composition Operators to Bloch-Type Spaces, Abstract and Applied Analysis 2013 (2013) 9p, Article ID 151929.

[11] B. D. MacCluer, S. Ohno, R. Zhao, Topological structure of the space of composition operators on H^∞, Integral Equations and Operator Theory 40 (2001) 481–494.

[12] P. J. Nieminen, Compact differences of composition operators on Bloch and Lipschitz spaces, Computational Methods and Function Theory 7 (2007) 325–344.

[13] S. Stević, Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces, Applied Mathematics and Computation 211 (2009) 3634–3641.

[14] S. Stević, Essential norm of differences of weighted composition operators between weighted-type spaces on the unit ball, Applied Mathematics and Computation 217 (2010) 1811–1824.

[15] J. H. Shapiro, C. Sundberg, Isolation amongst the composition operators, Pacific Journal of Mathematics 145 (1990) 117–152.

[16] X. Song, Z. Zhou, Differences of weighted composition operators from Bloch space to H^∞ on the unit ball, Journal of Mathematical Analysis and Applications 401 (2013) 447–457.

[17] E. Wolf, Differences of composition operators between weighted Banach spaces of holomorphic functions in the unit polydisk, Results in Mathematics 51 (2008) 361–372.

[18] E. Wolf, Differences of weighted composition operators between weighted Banach spaces of holomorphic functions and weighted Bloch type spaces, Cubo. A Mathematical Journal 12 (2010) 19–27.

[19] Y. Wu, H. Wulan, Products of differentiation and composition operators on the Bloch space, Collectanea Mathematica 63 (2012) 93–107.

[20] Y. Yu, Y. Liu, Weighted differentiation composition operators from H^∞ to Zygmund spaces, Integral Transforms and Special Functions 22 (2011) 507–520.

[21] K. H. Zhu, Bloch type spaces of analytic functions, The Rocky Mountain Journal of Mathematics 23 (1993) 1143–1177.