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About Almost Geodesic Curves

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Abstract. We determine in $\mathbb{R}^n$ the form of curves C for which also any image under an $(n-1)$-dimensional algebraic torus is an almost geodesic with respect to an affine connection ∇ with constant coefficients and calculate the components of ∇.

1. Introduction

This paper is a result following on from D. Betten researchs [4] and our papers [1, 8–10].

The geodesics and almost geodesics play an important role in differential geometry. For this reason many geometricians study almost geodesic mappings (see [3], [14], [17]). In [12], [13] almost geodesic curves were considered in generalized Riemannian and Kählerian spaces. E. Beltrami [6] has shown that a differentiable curve is a local geodesic with respect to an affine connection ∇ precisely if it is a solution of an Abelian differential equation with coefficients which are functions of the components of ∇. The investigation with systems of lines of 2-dimensional topological geometries was started in [15]. The explicit calculation of the form of curves C in the n-dimensional real space $\mathbb{R}^n$ which are geodesics or almost geodesics with respect to an affine connection ∇ is not achievable even in the case if the components Γ^h_{ij} of ∇ are constant. But we did it. In [2] the geodesics and special case of almost geodesics were considered. We supposed that with C also all images of C under a real $(n-1)$-dimensional algebraic torus are also geodesics, respectively almost geodesics. This implies that the determination of C becomes an algebraic problem (a problem of polynomial identities). Our model allows you to look at known things globally. In this paper we continue to study almost geodesic curves [7], [16] and here we will consider other case.

We consider a curve C homeomorphic to $\mathbb{R}$ which is a closed subset of $\mathbb{R}^n$ and has the form

$$C = (t, f_2(t), \ldots, f_n(t)), t \in \mathbb{R}, \tag{1}$$

where $f_i(t)\colon \mathbb{R} \to \mathbb{R}$, $i = 2, \ldots, n$, are three times differentiable non-constant functions. The system

$$\mathfrak{X}(C) = \{(t + c_1, b_2 f_2(t) + c_2, \ldots, b_n f_n(t) + c_n), t \in \mathbb{R}\}, \text{where } b_i \neq 0, c_i \in \mathbb{R},$$

is a set of imagines of C.

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If every curve of $\mathfrak{X}(C)$ is a geodesic with respect to an affine connection ∇ with constant coefficients Γ_{ij}^h, then the derivatives $f_i'(t)$ of the functions $f_i(t)$ are solutions of the first order linear ordinary differential equations. If every curve of $\mathfrak{X}(C)$ is an almost geodesic with respect to ∇, then the derivatives $f_i'(t)$ are solutions of harmonic oscillator equations. If $\mathfrak{X}(C)$ consists of Euclidean lines which are geodesics with respect to ∇, then at the most Γ_{11}^1 may be different from 0. In contrast to this if $\mathfrak{X}(C)$ consists of Euclidean lines then there is huge quantity of non-trivial connections ∇ such that the lines of $\mathfrak{X}(C)$ are almost geodesic with respect to ∇.

Since we apply results of differential geometry only for the n-dimensional space $\mathbb{R}^n$, where global coordinates exist and the components Γ_{ij}^h, $h, i, j \in \{1, 2, \dots, n\}$, of any affine connection ∇ can be written in unique way in these coordinates.

Remark 1.1. *If coefficients Γ_{ij}^h of an affine connection ∇ are constants then there exist groups of affine movements.*

Remark 1.2. *It is possible to apply our model for other spaces, because a geodesic and an almost geodesic can be defined in other spaces in the same manner as in $\mathbb{R}^n$ [11].*

2. Almost geodesic curves

Let

$$\ell = (t + c_1, b_2 f_2(t) + c_2, \dots, b_n f_n(t) + c_n),\ t \in \mathbb{R},$$

be a curve of $\mathfrak{X}(C)$. Then

$$\dot\ell = (1, b_2 f_2'(t), \dots, b_n f_n'(t)), \quad \ddot\ell = (0, b_2 f_2''(t), \dots, b_n f_n''(t)).$$

By an *almost geodesic* of an affine connection ∇ we mean a piecewise C^3-curve $\gamma\colon I \to \mathbb{R}^n$ satisfying

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) = \varrho \cdot \dot\gamma + \sigma \cdot \nabla_{\dot\gamma}\dot\gamma,$$

where $\varrho, \sigma\colon I \to \mathbb{R}$ are continuous functions, $I \subset \mathbb{R}$ is an open interval (cf. [16, p. 158], [7, p. 456]).

Using the components of ∇ the system of differential equations for almost geodesics has the form

$$\dddot\gamma^h + \sum_{i,j,k=1}^{n} (\partial_k\Gamma_{ij}^h + \Gamma_{ij}^\ell\Gamma_{\ell k}^h)\,\dot\gamma^i\dot\gamma^j\dot\gamma^k + 2\sum_{i,j=1}^{n}\Gamma_{ij}^h\ddot\gamma^i\dot\gamma^j + \sum_{i,j=1}^{n}\Gamma_{ij}^h\dot\gamma^i\ddot\gamma^j = \varrho(t)\cdot\dot\gamma^h + \sigma(t)\cdot(\ddot\gamma^h + \sum_{i,j=1}^{n}\Gamma_{ij}^h\dot\gamma^i\dot\gamma^j). \tag{2}$$

A curve ℓ of $\mathfrak{X}(C)$ is an almost geodesic with respect to a connection ∇ with constant coefficients $\{\Gamma_{ij}^h\}$ if and only if according to (2) we have

$$\dddot\ell^h + \sum_{i,j,k=1}^{n}\Gamma_{ij}^\ell\Gamma_{\ell k}^h\,\dot\ell^i\dot\ell^j\dot\ell^k + 2\sum_{i,j=1}^{n}\Gamma_{ij}^h\ddot\ell^i\dot\ell^j + \sum_{i,j=1}^{n}\Gamma_{ij}^h\dot\ell^i\ddot\ell^j = \varrho(t)\cdot\dot\ell^h + \sigma(t)\cdot(\ddot\ell^h + \sum_{i,j=1}^{n}\Gamma_{ij}^h\dot\ell^i\dot\ell^j). \tag{3}$$

We rewrite the formula (3) for $h = 1$ and obtain the function $\varrho(t)$. For $h = 2, \dots, n$ after substitution $\varrho(t)$ in (3) we get

$$b_h f_h'''(t) + \sum_{m=1}^{n}\Gamma_{m1}^h\Gamma_{11}^m + \sum_{i=2,m=1}^{n}(\Gamma_{mi}^h\Gamma_{11}^m + \Gamma_{m1}^h\Gamma_{i1}^m + \Gamma_{m1}^h\Gamma_{1i}^m)\,b_i\,f_i'(t) +$$

$$\sum_{i,j=2,m=1}^{n}(\Gamma_{mi}^h\Gamma_{j1}^m + \Gamma_{mi}^h\Gamma_{1j}^m + \Gamma_{m1}^h\Gamma_{ij}^m)\,b_ib_j\,f_i'(t)\,f_j'(t) +$$

$$\sum_{i,j,k=2,m=1}^{n}\Gamma_{mi}^h\Gamma_{jk}^m\,b_ib_jb_k\,f_i'(t)\,f_j'(t)\,f_k'(t) +$$

$$\sum_{i=2}^{n}(2\Gamma^h_{i1}+\Gamma^h_{1i})\, b_i f_i''(t) + \sum_{i,j=2}^{n}(2\Gamma^h_{ij}+\Gamma^h_{ji})\, b_i b_j f_i''(t) f_j'(t) -$$

$$b_h f_h'(t) \left(\sum_{m=1}^{n} \Gamma^1_{m1}\Gamma^m_{11} + \sum_{i=2,m=1}^{n} (\Gamma^1_{mi}\Gamma^m_{11} + \Gamma^1_{m1}\Gamma^m_{i1} + \Gamma^1_{m1}\Gamma^m_{1i})\, b_i\, f_i'(t) + \right.$$

$$\sum_{i,j=2,m=1}^{n} (\Gamma^1_{mi}\Gamma^m_{j1} + \Gamma^1_{mi}\Gamma^m_{1j} + \Gamma^1_{m1}\Gamma^m_{ij})\, b_i b_j\, f_i'(t)\, f_j'(t) +$$

$$\sum_{i,j,k=2,m=1}^{n} \Gamma^1_{mi}\Gamma^m_{jk}\, b_i b_j b_k\, f_i'(t)\, f_j'(t)\, f_k'(t) +$$

$$\left. \sum_{i=2}^{n}(2\Gamma^1_{i1}+\Gamma^1_{1i})\, b_i f_i''(t) + \sum_{i,j=2}^{n}(2\Gamma^1_{ij}+\Gamma^1_{ji})\, b_i b_j f_i''(t) f_j'(t) \right) =$$

$$\sigma(t)\cdot\left(\Gamma^h_{11} - \Gamma^1_{11} + 2\sum_{i=2}^{n}(\Gamma^h_{1i}+\Gamma^h_{i1}-\Gamma^1_{1i}-\Gamma^1_{i1}) b_i\, f_i'(t) + \sum_{i,j=2}^{n}(\Gamma^h_{ij}-\Gamma^1_{ij})\, b_i b_j\, f_i'(t)\, f_j'(t)\right). \quad (4)$$

One can determine σ only if not all coefficients in (4) are zero. In [2] we treated the case that for $h \geq 2$ one has

$$\Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i},\ i \geq 1, \text{ and } \Gamma^h_{ij} = \Gamma^1_{ij} \text{ for all } i, j \geq 2.$$

Now let an α and i_0, j_0 such that for these indices we have

$$\Gamma^\alpha_{11} \neq \Gamma^1_{11} \text{ or } \Gamma^\alpha_{i_0 1} + \Gamma^\alpha_{1 i_0} \neq \Gamma^1_{i_0 1} + \Gamma^1_{1 i_0}, \text{ or } \Gamma^\alpha_{i_0 j_0} \neq \Gamma^1_{i_0 j_0},\ i_0, j_0 \geq 2. \quad (5)$$

In this case the coefficient of σ is not identically zero, and we can compute σ. Putting the expression of σ into relation (4) we obtain

$$\left((\Gamma^\alpha_{11}-\Gamma^1_{11}) + 2(\Gamma^\alpha_{1i_0}+\Gamma^\alpha_{i_01}-\Gamma^1_{1i_0}-\Gamma^1_{i_01}) f_{i_0}' b_{i_0} + (\Gamma^\alpha_{i_0j_0}-\Gamma^1_{i_0j_0}) f_{i_0}' f_{j_0}' b_{i_0} b_{j_0}\right)\cdot$$

$$\left(T_{h111} + \left(f_h''' - T_{1111} f_h'\right) b_h + \sum_{i=2}^{n}\left(S_{h11i} f_i' + (2\Gamma^h_{i1}+\Gamma^h_{1i}) f_i''\right) b_i - \right.$$

$$\sum_{i=2}^{n}\left(S_{1i11} f_i' + (2\Gamma^1_{i1}+\Gamma^1_{1i}) f_i''\right) f_h' b_h b_i + \sum_{i,j=2}^{n}\left(S_{hij1} f_i' f_j' + (2\Gamma^h_{ij}+\Gamma^h_{ji}) f_i'' f_j'\right) b_i b_j -$$

$$\left.\sum_{i,j=2}^{n}\left(S_{11ij} f_i' f_j' + (2\Gamma^1_{ij}+\Gamma^1_{ji}) f_i'' f_j'\right) f_h' b_h b_i b_j + \sum_{i,j,k=2}^{n} T_{hijk} f_i' f_j' f_k' b_i b_j b_k - \sum_{i,j,k=2}^{n} T_{1ijk} f_h' f_i' f_j' f_k' b_h b_i b_j b_k\right) =$$

$$\left((\Gamma^h_{11}-\Gamma^1_{11}) + 2\sum_{i=2}^{n}(\Gamma^h_{1i}+\Gamma^h_{i1}-\Gamma^1_{1i}-\Gamma^1_{i1}) f_i' b_i + \sum_{i,j=2}^{n}(\Gamma^h_{ij}-\Gamma^1_{ij}) f_i' f_j' b_i b_j\right)\cdot$$

$$\left(T_{\alpha111} + \left(f_\alpha''' - T_{1111} f_\alpha'\right) b_\alpha + \left(S_{\alpha11i_0} f_{i_0}' + (2\Gamma^\alpha_{i_01}+\Gamma^\alpha_{1i_0}) f_{i_0}''\right) b_{i_0} - \left(S_{1i_011} f_{i_0}' + (2\Gamma^1_{i_01}+\Gamma^1_{1i_0}) f_{i_0}''\right) f_\alpha' b_h b_{i_0} + \right.$$

$$\left(S_{\alpha i_0 j_0 1} f_{i_0}' + (2\Gamma^\alpha_{i_0j_0}+\Gamma^\alpha_{j_0i_0}) f_{i_0}''\right) f_{j_0}' b_{i_0} b_{j_0} - \left(S_{11i_0j_0} f_{i_0}' + (2\Gamma^1_{i_0j_0}+\Gamma^1_{j_0i_0}) f_{i_0}''\right) f_\alpha' f_{j_0}' b_\alpha b_{i_0} b_{j_0} +$$

$$\left.\sum_{k=2}^{n} T_{\alpha i_0 j_0 k} f_{i_0}' f_{j_0}' f_k' b_{i_0} b_{j_0} b_k - \sum_{k=2}^{n} T_{1 i_0 j_0 k} f_\alpha' f_{i_0}' f_{j_0}' f_k' b_\alpha b_{i_0} b_{j_0} b_k\right), \quad (6)$$

where

$$S_{ABCD} \overset{def}{=} \sum_{m=1}^{n} \Big(\Gamma^A_{mD}\Gamma^m_{BC} + \Gamma^A_{mB}(\Gamma^m_{DC} + \Gamma^m_{CD})\Big),$$

$$T_{ABCD} \overset{def}{=} \sum_{m=1}^{n} \Gamma^A_{mB}\Gamma^m_{CD}.$$

If $n = 2$, then $h = \alpha = 2$ and from (6) we obtain that any plane curve of the system $\mathfrak{X}(C)$, where C has the form (1), is an almost geodesic if the affine connection Γ^h_{ij} satisfies the conditions (5). Hence we assume $n \geq 3$.

Now we consider the first case, when

$$\Gamma^h_{11} = \Gamma^1_{11} \text{ for all } 2 \leq h \leq n \text{ and } \Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i} \text{ for all } 2 \leq i \leq n, \tag{7}$$

but there exists an α and i_0, j_0 such that

$$\Gamma^\alpha_{i_0 j_0} \neq \Gamma^1_{i_0 j_0}. \tag{8}$$

Writing a system of equations and conditions which follow from (6) and using linear independence functions we get differential equations. Integrating them (see. [5]) we obtain the following

Theorem 2.1. *Let C be a curve of the form* (1) *and ∇ be a connection with constant coefficients $\{\Gamma^h_{ij}\}$ satisfying relations* (7), (8).
Then any curve ℓ of $\mathfrak{X}(C)$ is almost geodesic with respect to ∇ if and only if ℓ is represented by the functions f_h, f_α, f_{i_0} having the following forms

* $f_h(t) = \hat{C}_h e^{\lambda^h_1 t} + \hat{D}_h e^{\lambda^h_2 t}$, *where $\hat{C}_h, \hat{D}_h \in \mathbb{R}$ are not both zero and $a_h^2 - 4c_h > 0$,*
* $f_h(t) = (\tilde{C}_h t + \tilde{D}_h)e^{\frac{-a_h}{2}t}$, *where $\tilde{C}_h, \tilde{D}_h \in \mathbb{R}$ are not both zero and $a_h^2 - 4c_h = 0$,*
* $f_h(t) = e^{-a_h t/2}\Big(\bar{C}_h cos\frac{\sqrt{a_h^2-4c_h}}{2}t + \bar{D}_h sin\frac{\sqrt{a_h^2-4c_h}}{2}t\Big)$, *where $\bar{C}_h, \bar{D}_h \in \mathbb{R}$ are not both zero and $a_h^2 - 4c_h < 0$*

with

$$a_h = 2\Gamma^h_{h1} + \Gamma^h_{1h}, \quad c_h = S_{h11h} - T_{1111},$$

$$\lambda^h_1 = \frac{-a_h - \sqrt{a_h^2 - 4c_h}}{2}, \quad \lambda^h_2 = \frac{-a_h + \sqrt{a_h^2 - 4c_h}}{2};$$

* $f_\alpha(t) = C_\alpha t^2 + D_\alpha t + E$, *where $C_\alpha, D_\alpha, E \in \mathbb{R}$, C_α, D_α are not both zero and $\gamma_\alpha = 0$,*
* $f_\alpha(t) = \hat{C}_\alpha e^{\sqrt{-\gamma_\alpha}\, t} - \hat{D}_\alpha e^{-\sqrt{-\gamma_\alpha}\, t}$, *where $\hat{C}_\alpha, \hat{D}_\alpha \in \mathbb{R}$ are not both zero and $\gamma_\alpha < 0$,*
* $f_\alpha(t) = \hat{C}_\alpha \sin(\sqrt{\gamma_\alpha}\, t) - \hat{D}_\alpha \cos(\sqrt{\gamma_\alpha}\, t)$, *where $\hat{C}_\alpha, \hat{D}_\alpha \in \mathbb{R}$ are not both zero and $\gamma_\alpha > 0$*

with

$$\gamma_\alpha = \frac{(\Gamma^\alpha_{i_0 j_0} - \Gamma^1_{i_0 j_0})(T_{h\alpha ij} + T_{h\alpha ji} + T_{hi\alpha j} + T_{hij\alpha} + T_{hj\alpha i} + T_{hji\alpha})}{\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji}} - T_{1111};$$

* $f_{i_0}(t) = \hat{C}_{i_0} e^{\lambda^{i_0}_1 t} + \hat{D}_{i_0} e^{\lambda^{i_0}_2 t}$, *where $\hat{C}_{i_0}, \hat{D}_{i_0} \in \mathbb{R}$ are not both zero and $a_{i_0}^2 - 4c_{i_0} > 0$,*
* $f_{i_0}(t) = (\tilde{C}_{i_0} t + \tilde{D}_{i_0})e^{\frac{-a_{i_0}}{2}t}$, *where $\tilde{C}_{i_0}, \tilde{D}_{i_0} \in \mathbb{R}$ are not both zero and $a_{i_0}^2 - 4c_{i_0} = 0$,*
* $f_{i_0}(t) = e^{-a_{i_0} t/2}\Big(\bar{C}_{i_0} cos\frac{\sqrt{a_{i_0}^2-4c_{i_0}}}{2}t + \bar{D}_{i_0} sin\frac{\sqrt{a_{i_0}^2-4c_{i_0}}}{2}t\Big)$, *where $\bar{C}_{i_0}, \bar{D}_{i_0} \in \mathbb{R}$ are not both zero and $a_{i_0}^2 - 4c_{i_0} < 0$*

with

$$a_{i_0} = 2\Gamma^{i_0}_{i_0 1} + \Gamma^{i_0}_{1 i_0},$$

$$c_{i_0} = \frac{(\Gamma^{i_0}_{i_0 j_0} - \Gamma^{1}_{i_0 j_0})(T_{hi_0 ij} + T_{hi_0 ji} + T_{hii_0 j} + T_{hiji_0} + T_{hji_0 i} + T_{hjii_0})}{\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji}} + S_{i_0 11 i_0} - T_{1111},$$

$$\lambda^{i_0}_1 = \frac{-a_{i_0} - \sqrt{a^2_{i_0} - 4c_{i_0}}}{2}, \quad \lambda^{i_0}_2 = \frac{-a_{i_0} + \sqrt{a^2_{i_0} - 4c_{i_0}}}{2}.$$

The components $\{\Gamma^h_{ij}\}$ *of affine connection* ∇ *satisfy the following relations*

$$2\Gamma^1_{ij} + \Gamma^1_{ji} = 0, \quad 2\Gamma^\alpha_{i_0 j_0} + \Gamma^\alpha_{j_0 i_0} = 0, \quad 2\Gamma^h_{ij} + \Gamma^h_{ji} = 0, \quad 2\Gamma^{i_0}_{i_0 i} + \Gamma^{i_0}_{ii_0} = 0,$$

$$\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^k_{ij} - \Gamma^k_{ji} = 0 \text{ for } k = i_0, \alpha, \quad 2\Gamma^k_{ik} + \Gamma^k_{ki} - 2\Gamma^1_{i1} - \Gamma^1_{1i} = 0 \text{ for } k = i_0, h,$$

$$(\Gamma^1_{ij_0} + \Gamma^1_{j_0 i} - \Gamma^h_{ij_0} - \Gamma^h_{j_0 i})S_{1i_0 11} = 0, \quad (\Gamma^1_{ij_0} + \Gamma^1_{j_0 i} - \Gamma^h_{ij_0} - \Gamma^h_{j_0 i})(2\Gamma^1_{i_0 1} + \Gamma^1_{1i_0}) = 0,$$

$$(\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji})(2\Gamma^\alpha_{i_0 1} + \Gamma^\alpha_{1i_0}) = 0,$$

$$(\Gamma^\alpha_{i_0 j_0} - \Gamma^1_{i_0 j_0})(S_{hij1} + S_{hji1}) + (\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji})T_{\alpha 111} = 0,$$

$$(\Gamma^\alpha_{i_0 j_0} - \Gamma^1_{i_0 j_0})(T_{h111} + S_{hi_0 j_0 1} + S_{hj_0 i_0 1}) + (\Gamma^1_{i_0 j_0} + \Gamma^1_{j_0 i_0} - \Gamma^h_{i_0 j_0} - \Gamma^h_{j_0 i_0})T_{\alpha 111} = 0,$$

$$(\Gamma^\alpha_{i_0 j_0} - \Gamma^1_{i_0 j_0})(S_{kki1} + S_{kik1} - S_{1i11}) + (\Gamma^1_{ik} + \Gamma^1_{ki} - \Gamma^k_{ik} - \Gamma^k_{ki})T_{\alpha 111} = 0 \text{ for } k = i_0, h,$$

$$(\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji})T_{1i_0 j_0 k} + (\Gamma^1_{ik} + \Gamma^1_{ki} - \Gamma^h_{ik} - \Gamma^h_{ki})T_{1i_0 j_0 j} + (\Gamma^1_{jk} + \Gamma^1_{kj} - \Gamma^h_{jk} - \Gamma^h_{kj})T_{1i_0 j_0 i} = 0,$$

$$S_{i11h} = 2\Gamma^i_{h1} + \Gamma^i_{1h} = 0 \text{ if } f'_h \neq D_h e^{\frac{-a_h}{2}t}, \qquad S_{i11h} = \frac{1}{2}a_h(2\Gamma^i_{h1} + \Gamma^i_{1h}) \text{ if } f'_h = D_h e^{\frac{-a_h}{2}t},$$

$$(\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji})(T_{\alpha i_0 j_0 \alpha} - S_{11i_0 j_0}) + (\Gamma^1_{i\alpha} + \Gamma^1_{\alpha i} - \Gamma^h_{i\alpha} - \Gamma^h_{\alpha i})T_{\alpha i_0 j_0 j} + (\Gamma^1_{j\alpha} + \Gamma^1_{\alpha j} - \Gamma^h_{j\alpha} - \Gamma^h_{\alpha j})T_{\alpha i_0 j_0 i} = 0,$$

$$(\Gamma^1_{i_0 j_0} - \Gamma^\alpha_{i_0 j_0})(T_{1i_0 j_0 i} + T_{1i_0 ij_0} + T_{1j_0 i_0 i} + T_{1j_0 ii_0} + T_{1ii_0 j_0} + T_{1ij_0 i_0}) + (\Gamma^1_{ki} + \Gamma^1_{ik} - \Gamma^h_{ki} - \Gamma^h_{ik})S_{\alpha i_0 j_0 1} = 0 \text{ for } k = j_0, h,$$

$$(\Gamma^\alpha_{i_0 j_0} - \Gamma^1_{i_0 j_0})(T_{hi_0 ij} + T_{hi_0 ji} + T_{hii_0 j} + T_{hiji_0} + T_{hji_0 i} + T_{hjii_0}) + (\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji})S_{\alpha 11 i_0} = 0,$$

$$T_{i_0 111} - S_{1j_0 11} + S_{i_0 i_0 j_0 1} + S_{i_0 j_0 i_0 1} = 0,$$

$$T_{\alpha i_0 ij} + T_{\alpha i_0 ji} + T_{\alpha ii_0 j} + T_{\alpha iji_0} + T_{\alpha ji_0 i} + T_{\alpha jii_0} = 0,$$

$$T_{kkij} + T_{kkji} + T_{kikj} + T_{kijk} + T_{kjki} + T_{kjik} - S_{11ij} - S_{11jl} = 0 \text{ for } k = i_0, h, \alpha.$$

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