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Hyperbolic Space Groups with Truncated Simplices as Fundamental Domains

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Abstract. In previous papers there are given numerous cases of groups with truncated simplices as fundamental domains. These groups will be reconsidered here, in order to finish the classification of the supergroups for 44 group series with trunc-simplex domains. There are given 7 new series of groups here, belonging to different families. We also consider 12 new cases of groups of 20 series from families F13 - F32, called maximal series. These results complete the investigation of groups with fundamental trunc-simplices.

1. Introduction

1.1.

Hyperbolic space groups are isometry groups, acting discontinuously on hyperbolic 3-space $\mathbf{H}^3$ with compact fundamental domains. It will be investigated some series of such groups by looking for their fundamental domains. Face pairing identifications on a given polyhedron may give us generators and relations for a space group by the Poincaré Theorem [2].

The simplest fundamental domains are 3-simplices (tetrahedra) and their integer parts. In the process of classifying the fundamental simplices, 64 combinatorially different face pairings of fundamental simplices were determined [6, 19], and also 35 solid transitive non-fundamental simplex identifications [6]. I. K. Zhuk [19] classified Euclidean and hyperbolic fundamental simplices of finite volume up to congruence. An algorithmic procedure was given by E. Molnár and I. Prok [5]. In [6–8] the authors summarized all those results, arranging identified simplices into 32 families. Each of them is characterized by the so-called maximal series of simplex tilings, i.e. maximal symmetry groups with smallest fundamental domains. Some complete cases of supergroups with fundamental truncated simplices (shorty trunc-simplices) are discussed in [3, 9–17]. Investigation of such hyperbolic space groups, especially with fundamental domains of truncated simplices found applications in newer packing and covering problems, e.g. in [18].

In the first 12 families from the above 32 ones, appear 44 series of basic simplices, while the remaining 20 families contain only maximal series of the simplices. Possible supergroups of the mentioned 44 group series, that have trunc-simplices as fundamental domains, will be reconsidered in this paper. Together with these new results, there are also presented 12 new series in maximal families 13 - 32, as it will be formulated in Theorem 1.1.

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1.2.

For a given fundamental simplex and a given equivalence class of edges, the sum of dihedral angles is always $2\pi/\nu$, with natural ν. That is the reason why we shall have parameters for equivalence classes of edges. In [8] there is given space of realization for every value of parameters of simplices investigated here. In all cases of the investigated simplices, if parameters are large enough, the simplex is always realizable in hyperbolic space with vertices out of the absolute, but the fine discussion is not detailed here, see e.g. [8].

If the vertices are outer ones, the simplex is not compact and it is possible to truncate it with polar planes of the vertices. The new compact polyhedron obtained in that way is the fundamental domain of a larger group. It has new triangular faces whose pairing gives us new generators. Dihedral angles around the new edges are $\pi/2$, i.e., there are four congruent trunc-simplices around them in the fundamental space filling.

1.3.

We use the generalized Poincarè theorem [4] to obtain generators and relations for a space group G with a combinatorially given polyhedron P as a fundamental domain, in the way briefly described below. In this paper such polyhedra will be trunc-simplices.

It is necessary to consider all *face pairing identifications* of such domains. These are isometries which generate a group G and induce subdivision of directed edge segments of P into equivalence classes, such that an edge segment does not contain two G-equivalent points in its interior. The Poincarè algorithm gives us for each edge segment class one cycle transformation of the form $c = g_1 g_2 \dots g_r$, where g_i, $i = 1, 2, \cdots r$ are face pairing identifications. Each of these transformations will be a rotation of order ν, so the cycle relations are of the form $(g_1 g_2 \dots g_r)^\nu = 1$. The Poincarè theorem guarantees us that these cycle relations, together with relations $g_i^2 = 1$ to the occasional involutive generators $g_i = g_i^{-1}$, form a complete set of defining relations for G. Details will be given at our examples.

1.4.

In order to obtain all possibilities of the face pairings for the new triangular faces of trunc-simplices, we have to consider the stabilizer groups of the corresponding vertex figures. Case-by-case analysis of the orbits and their symmetries for the whole tessellation in the polar plane of vertex figure will be the way to get all possible face pairings. Besides that, it was also established that the possible number of different cases can be 1, 2, 4, 8 or 16($= 4 \times 4$).

After reconsidering supergroups for all investigated groups in [3, 9–17] using this method, we have found some new cases. In this way investigation of trunc-simplex families, for general case of parameters, has been completed.

In this paper there are also considered 12 new maximal group series with trunc-simplex domains. These results together with those in previous papers are completing this part of investigation.

Total numbers of hyperbolic space group series with trunc-simplex fundamental domains are given in the following summarizing theorem. In Table data for each family are indicated separately.

Theorem 1.1. *There are 187 group extension, making totally 212 hyperbolic space group series with truncated simplices as fundamental domains.*

Family	F1	F2	F3	F4	F5	F6	F7
References	[3]	[17]	[13, 16]	[13, 16]	-	[13, 16]	[9, 10]
Number of simplex series	7	3	2	6	0	4	4
Number of group extension series	13	13	7	38	-	24	7
Number of trunc-simplex series	13	13	6	68	-	32	7

Family	F8	F9	F10	F11	F12	F13 - F32	Σ
References	[12]	[15]	[15]	[9, 10]	[10, 14]	[9–12, 14, 15]	
Number of simplex series	6	2	3	4	3	20	**64**
Number of group extension series	24	8	6	7	6	34	**187**
Number of trunc-simplex series	24	8	6	7	6	22	**212**

Notations used here are according to those given in [8], but may differ of the notations in original papers.

1.5.

The existence proof for some families from 13-32 are missing yet. This needs careful computations, illustrated in [8], and also in [11] for e.g. Zhuk's families F26, F30.

So for a while, simplices in fibred spaces $\mathbf{H}^2 \times \mathbf{R}$ and more $\widetilde{\mathbf{SL}_2\mathbf{R}}$ cannot be excluded yet, although $\mathbf{H}^3$ seems to be very probable. We intend to turn back to this problem later.

2. Truncated simplices from non-maximal simplex schemes

2.1. Simplex T_{62} from Family F1

The trunc-simplices to the simplices from family F1 were considered in [3]. To the simplex T_{62} (Fig. 1) in Family 1 we can give 4 face pairings of the new triangular faces of trunc-simplex O_{62}, and no more. After searching for more symmetries of the vertex figure tilling and its hyperbolic plane group with fundamental domain $F^0 = P_{A_0}$ and signature $\Gamma^0 = uu\times\times$ by [1], we find additional symmetry by half-turn indicated as O_{62}^2. For completeness we repeat all domains in Fig. 1.

The group for T_{62} is

$$\Gamma(T_{62}, 6u) = (z_1, z_2 - (z_2^{-1}z_1^2z_2^2z_1^{-1})^u = 1, u \geq 1),$$

while the supergroups with fundamental trunc-simplex O_{62} are

$$\Gamma(O_{62}^1, 6u) = \Gamma(T_{62}, 6u) \text{ and } (\bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3 - \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = \bar{m}_3^2 = \bar{m}_0z_2\bar{m}_0z_2^{-1} =$$
$$= \bar{m}_0z_1^{-1}\bar{m}_2z_1 = \bar{m}_1z_2\bar{m}_2z_2^{-1} = \bar{m}_1z_2^{-1}\bar{m}_3z_2 = \bar{m}_1z_1\bar{m}_2z_1^{-1} = \bar{m}_3z_1\bar{m}_3z_1^{-1} = 1, u \geq 2),$$

$$\Gamma(O_{62}^2, 6u) = \Gamma(T_{62}, 6u) \text{ and } (\bar{h}_0, \bar{h}_1, \bar{h}_2, \bar{h}_3 - \bar{h}_0^2 = \bar{h}_1^2 = \bar{h}_2^2 = \bar{h}_3^2 =$$
$$= \bar{h}_0z_1^{-1}\bar{h}_2z_1 = \bar{h}_0z_2\bar{h}_2z_2 = \bar{h}_1z_2^{-1}\bar{h}_3z_2 = \bar{h}_2z_1^{-1}\bar{h}_1z_2 = \bar{h}_3z_1\bar{h}_3z_1 = 1, u \geq 2),$$

$$\Gamma(O_{62}^3, 6u) = \Gamma(T_{62}, 6u) \text{ and } (\bar{g}_1, \bar{g}_2 - \bar{g}_2z_1\bar{g}_1z_2 = \bar{g}_1z_1\bar{g}_1^{-1}z_2 = \bar{g}_2z_1^{-1}\bar{g}_2z_2^{-1} = 1, u \geq 2),$$

$$\Gamma(O_{62}^4, 6u) = \Gamma(T_{62}, 6u) \text{ and } (\bar{s}_1, \bar{s}_2 - \bar{s}_1z_2\bar{s}_2z_1 = \bar{s}_1z_1\bar{s}_1^{-1}z_2^{-1} = \bar{s}_2z_1^{-1}\bar{s}_2z_1^{-1} = \bar{s}_2z_2^{-1}\bar{s}_2z_2^{-1} = 1, u \geq 2).$$

2.2. Simplex T_{36} from Family F11

In [9, 10] there has been considered simplex T_{36} from family F11 (Fig. 2). There wasn't obtained jet any symmetry of the fundamental domain for the stabilizer subgroup of vertex figure and its hyperbolic plane group $22uvv$ ([1]). So, there was given only the trivial supergroup with fundamental trunc-simplex O_{36}, i.e. the group with plane reflections as new generators. But after reconsidering symmetries of the vertex figure tiling and recomposing its fundamental domain, we find new point reflection face pairings (Fig. 2), giving O_{36}^2.

The group for T_{36} is

$$\Gamma(T_{36}, 2u, 5v) = (r_2, r_3, s - (r_2r_3)^u = (s^2r_2sr_3)^v = 1, u \geq 2, v \geq 1, 2u \neq 5v)$$

and the supergroups for O_{36} are

$$\Gamma(O_{36}^1, 2u, 5v) = \Gamma(T_{36}, 2u, 5v) \text{ and } (\bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3 - \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = \bar{m}_3^2 = \bar{m}_0r_2\bar{m}_1r_2 =$$
$$= \bar{m}_0r_3\bar{m}_1r_3 = \bar{m}_0s^{-1}\bar{m}_3s = \bar{m}_2r_3\bar{m}_2r_3 = \bar{m}_3r_2\bar{m}_3r_2 = \bar{m}_1s\bar{m}_2s^{-1} = \bar{m}_2s\bar{m}_3s^{-1} = 1),$$

$$\Gamma(O_{36}^2, 2u, 5v) = \Gamma(T_{36}, 2u, 5v) \text{ and } (\bar{g}_1, \bar{g}_2 - r_3\bar{g}_2r_2\bar{g}_2^{-1} = s\bar{g}_1s\bar{g}_2 = s\bar{g}_2^{-1}s\bar{g}_2^{-1} = r_2\bar{g}_1r_3\bar{g}_1 = 1).$$

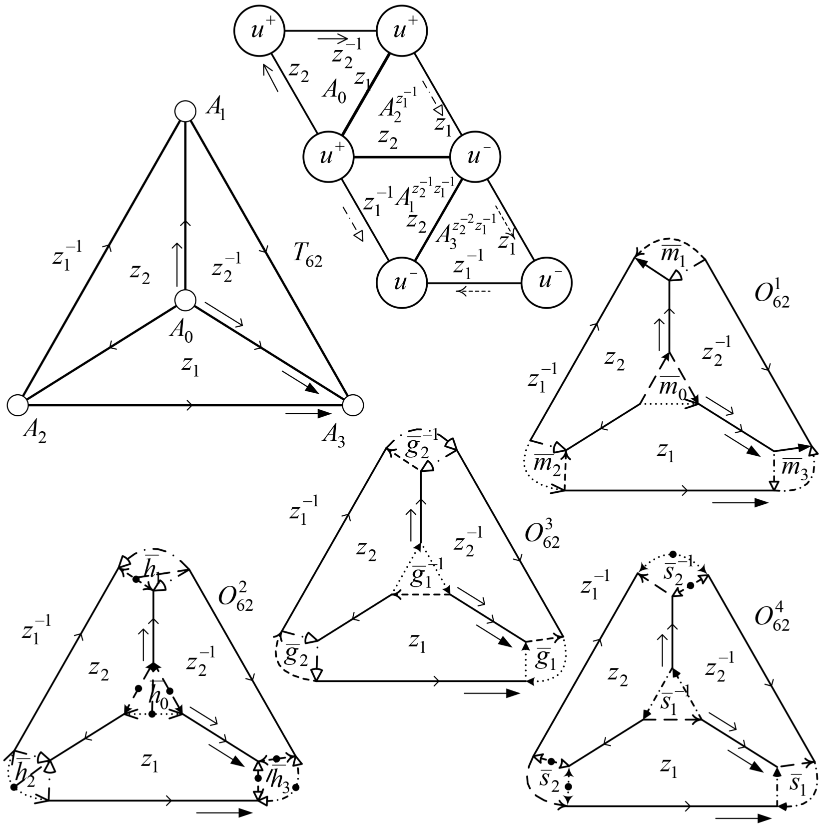

Figure 1: Simplex T_{62}, its vertex figure and trunc-simplices O_{62}^i, $i = \{1, 2, 3, 4\}$

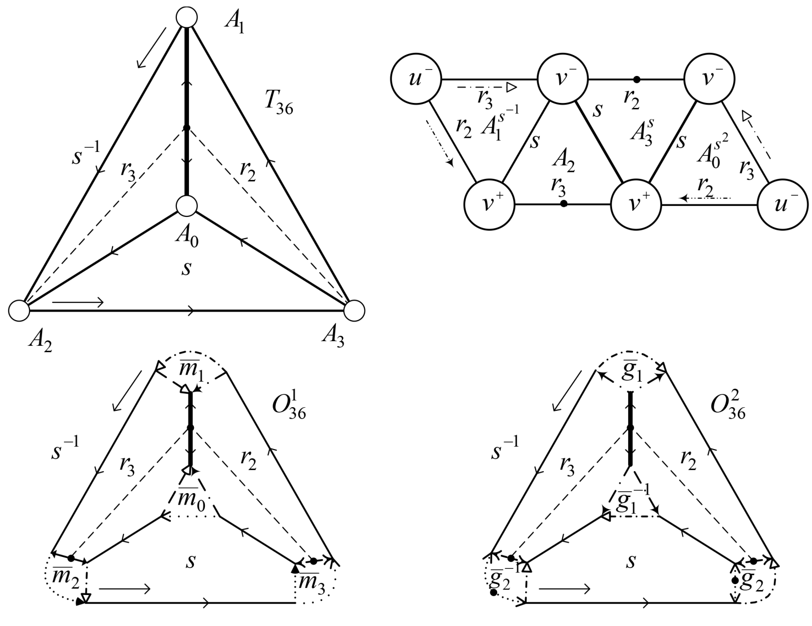

Figure 2: Simplex T_{36}, its vertex figure and trunc-simplices O_{36}^1, O_{36}^2

2.3. *Simplices T_{17} and T_{38} from Family F4*

The supergroups of the groups with fundamental simplices T_{17} and T_{38} (Fig. 3) from family F4 were investigated in [13, 16]. For these simplices there are two classes of equivalence for vertices $\{A_0, A_1\}$, $\{A_2, A_3\}$. It means that truncating vertices in different classes are independent. That is the reason to give for these truncating only the extensions to the original groups

$$\Gamma(T_{17}, 2u, 4v, 2w) = (r_0, r_1, r_2, r_3 - (r_2r_3)^u = (r_1r_2r_0r_3)^v = (r_0r_1)^w = r_0^2 = r_1^2 = r_2^2 = r_3^2 = 1,\ 2 \le u \ne w, 1 \le v),$$

$$\Gamma(T_{38}, 2u, 4v, 2w) = (r_0, r_1, z - (zz)^u = (r_1zr_0z)^v = (r_0r_1)^w = r_0^2 = r_1^2 = 1,\ 2 \le u \ne w, 1 \le v).$$

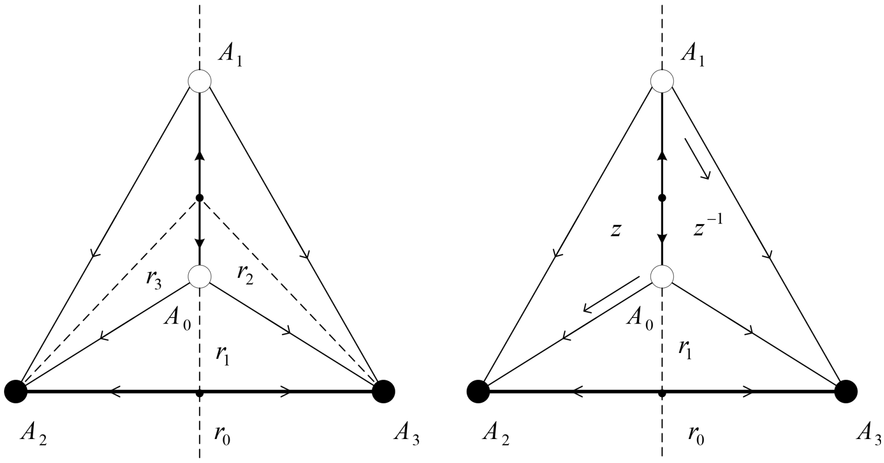

Figure 3: Simplices T_{17} and T_{38}

In [13, 16] are considered group extensions for T_{17} and T_{38} after truncating. For both simplex-series if conditions $\frac{1}{2} + \frac{1}{2u} + \frac{1}{2v} < 1$, $\frac{1}{2} + \frac{1}{2v} + \frac{1}{2w} < 1$ are satisfied, vertices from both edge class $\{A_0, A_1\}$ and $\{A_2, A_3\}$ are outer. Reconsidering is giving four extensions for each of edge classes, of both trunc-simplices. Here is given complete list of these extensions.

- For O_{17}, class of vertices A_0, A_1:
  $\bar{m}_0^2 = \bar{m}_1^2 = (\bar{m}_0r_1)^2 = (\bar{m}_1r_0)^2 = \bar{m}_0r_2\bar{m}_1r_2 = \bar{m}_0r_3\bar{m}_1r_3 = 1$; $\bar{h}_0^2 = \bar{h}_1^2 = (\bar{h}_0r_1)^2 = \bar{h}_0r_2\bar{h}_1r_3 = (\bar{h}_1r_0)^2 = 1$; $\bar{z}_1r_3\bar{z}_1r_2 = \bar{z}_1r_0\bar{z}_1^{-1}r_1 = 1$; $\bar{s}_1r_0\bar{s}_1^{-1}r_1 = (\bar{s}_1r_2)^2 = (\bar{s}_1r_3)^2 = 1$.
- For O_{17}, class of vertices A_2, A_3:
  $\bar{m}_2^2 = \bar{m}_3^2 = (\bar{m}_2r_3)^2 = (\bar{m}_3r_2)^2 = \bar{m}_2r_1\bar{m}_3r_1 = \bar{m}_2r_0\bar{m}_3r_0 = 1$; $\bar{h}_2^2 = \bar{h}_3^2 = (\bar{h}_2r_3)^2 = \bar{h}_2r_1\bar{h}_3r_0 = (\bar{h}_3r_2)^2 = 1$; $\bar{z}_2r_1\bar{z}_2r_0 = \bar{z}_2r_2\bar{z}_2^{-1}r_3 = 1$; $\bar{s}_2r_2\bar{s}_2^{-1}r_3 = (\bar{s}_2r_0)^2 = (\bar{s}_2r_1)^2 = 1$.
- For O_{38}, class of vertices A_0, A_1:
  $\bar{m}_0^2 = \bar{m}_1^2 = (\bar{m}_0r_1)^2 = (\bar{m}_1r_0)^2 = \bar{m}_1z\bar{m}_0z^{-1} = \bar{m}_0z\bar{m}_1z^{-1} = 1$; $\bar{h}_0^2 = \bar{h}_1^2 = (\bar{h}_0r_1)^2 = \bar{h}_0z\bar{h}_1z = (\bar{h}_1r_0)^2 = 1$; $(\bar{z}_1z)^2 = (\bar{z}_1z^{-1})^2 = \bar{z}_1r_0\bar{z}_1^{-1}r_1 = 1$; $\bar{s}_1r_0\bar{s}_1^{-1}r_1 = \bar{s}_1z\bar{s}_1z^{-1} = 1$.
- For O_{38}, class of vertices A_2, A_3:
  $\bar{m}_2^2 = \bar{m}_3^2 = \bar{m}_2r_1\bar{m}_3r_1 = \bar{m}_2r_0\bar{m}_3r_0 = \bar{m}_3z\bar{m}_2z^{-1} = 1$; $\bar{h}_2^2 = \bar{h}_3^2 = \bar{h}_2r_1\bar{h}_3r_0 = \bar{h}_3z\bar{h}_2z^{-1} = 1$; $(\bar{z}_2z)^2 = \bar{z}_2r_0\bar{z}_2r_1 = 1$; $(\bar{s}_2r_1)^2 = (\bar{s}_2r_0)^2 = \bar{s}_2z\bar{s}_2z = 1$.

Remark 2.1. *Since truncations of vertices in different equivalence classes are independent, we can combine appropriate group extensions. It means, if there is p different extensions for the first equivalence classes and q extensions for the second one, then total number of different group series for truncated simplex is pq. Similarly, if there are three equivalence classes for vertices with resp. p, q, r group extensions, then total number of group series is pqr.*

There is also possibility to truncate vertices only in some of the equivalence classes, but that is not the topic of this paper.

So, for the both simplices T_{17} and T_{38} there are 16 possibilities to create supergroups with fundamental trunc-simplices.

In Fig. 4 there are given vertex figures of T_{17} and T_{38} resp., for classes of vertices $\{A_0, A_1\}$ and $\{A_2, A_3\}$, whose hyperbolic plane groups (by [1]) are $22uv$, $22vw$, and $22uv$, $vw\times$, resp.

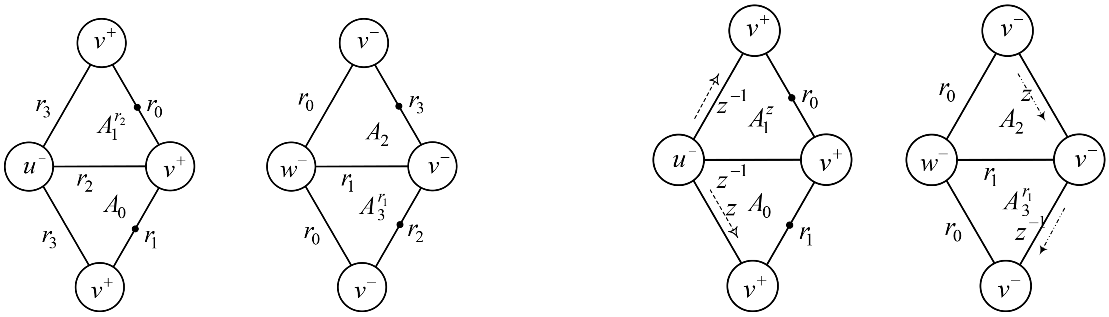

Figure 4: Vertex figures of simplices T_{17} and T_{38}

2.4. Simplex T_{53} from Family F7

By printing mistake in [9] are omitted results for T_{53} from family F7. There are only given in [10]. Here, the simplex T_{53}, its vertex figure with hyperbolic plane group $22uv\times$ (by [1]) and the trunc-simplices O^1_{53}, O^2_{53} are given if Fig. 5. The group for T_{53} from [8] is

$$\Gamma(T_{53}, 2u, 10v) = (r_0, r_1, z - (zz)^u = (r_1 z r_0 r_1 z^{-1} r_0 z r_1 r_0 z^{-1})^v = 1, u \geq 2, v \geq 1, 2u \neq 10v).$$

One fundamental domain of vertex figure is

$$P_{A_2} := T_{A_2} \cup T_{A_3}^{z^{-1}} \cup T_{A_0}^{r_1 z^{-1}} \cup T_{A_1}^{r_0 z^{-1}},$$

and the groups for O^1_{53}, O^2_{53} are

$$\Gamma(O^1_{53}, 2u, 10v) = \Gamma(T_{53}, 2u, 10v) \text{ and } (\bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3 - \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = \bar{m}_3^2 = \bar{m}_0 r_1 \bar{m}_3 r_1 =$$
$$= \bar{m}_0 z \bar{m}_1 z^{-1} = \bar{m}_0 z^{-1} \bar{m}_1 z = \bar{m}_1 r_0 \bar{m}_3 r_0 = \bar{m}_3 z \bar{m}_2 z^{-1} = \bar{m}_2 r_0 \bar{m}_2 r_0 = \bar{m}_2 r_1 \bar{m}_2 r_1 = 1),$$

$$\Gamma(O^2_{53}, 2u, 10v) = \Gamma(T_{53}, 2u, 10v) \text{ and } (\bar{h}_2, \bar{h}_3, \bar{s} - \bar{h}_2^2 = \bar{h}_3^2 = \bar{s} z \bar{s} z^{-1} = \bar{s} r_0 \bar{h}_3 r_1 = \bar{h}_2 z^{-1} \bar{h}_3 z = \bar{h}_2 r_1 \bar{h}_2 r_0 = 1).$$

3. Truncated simplices from maximal group series

In each of families F13 - F32 there is a single simplex series and all of them are the maximal ones for general parameters. In the previous papers [9–12, 14, 15] there are investigated simplices from families F21, F23, F25, F26, F27, F29, F30, F32. So, it remains to investigate 12 families. Data for these families are given in [8]. In all here considered cases, for all equivalence classes of vertices, only trivial extensions are possible. So, for each of the trunc-simplices there is only one group series and the number of group extensions is equal to number of equivalence classes of vertices. Note that situation is similar for previously considered trunc-simplices from maximal group series, except in cases of Zhuk's simplices from families

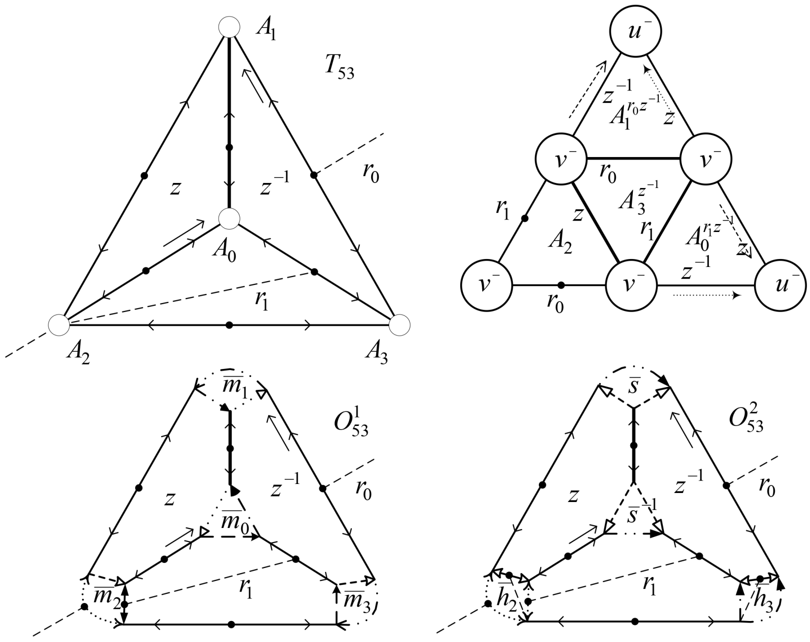

Figure 5: Simplex T_{53}, its vertex figure and trunc-simplices O^1_{53}, O^2_{53}

F26 and F30 considered in [11]. There, in class of vertex $\{A_1\}$ there are two different group extensions and so, two different groups for each of these trunc-simplices.

Families F13, F14, F15 (with trunc-simplices O_1, O_2, O_3, respectively):

$$\Gamma(O_1, 2a, 2b, 2c, 2d, 2e, 2f) = (m_0, m_1, m_2, m_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0m_1)^a = (m_1m_2)^b = (m_2m_0)^c = (m_2m_3)^d =$$
$$= (m_0m_3)^e = (m_1m_3)^f = (\bar{m}_i m_j)^2 = m_k^2 = \bar{m}_l^2 = 1,\ i, j, k, l \in \{0, 1, 2, 3\}, i \neq j,$$
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1, \frac{1}{a} + \frac{1}{e} + \frac{1}{f} < 1, \frac{1}{c} + \frac{1}{d} + \frac{1}{e} < 1, \frac{1}{b} + \frac{1}{d} + \frac{1}{f} < 1,\ 2 \leq a, b, c, d, e, f)$$

$$\Gamma(O_2, 4a, 4b, 2c, 2d, 2e) = (m_0, r_1, m_2, m_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0r_1m_0r_1)^a = (m_2r_1m_3r_1)^b = (m_2m_3)^c = (m_0m_3)^d$$
$$= (m_0m_2)^e = (\bar{m}_i f_j)^2 = f_k^2 = \bar{m}_l^2 = 1,\ i, j, k, l \in \{0, 1, 2, 3\}, i \neq j,\ f_0 \equiv m_0, f_1 \equiv r_1, f_2 \equiv m_2, f_3 \equiv m_3,$$
$$\frac{1}{b} + \frac{1}{c} < 1, \frac{1}{c} + \frac{1}{d} + \frac{1}{e} < 1, \frac{1}{a} + \frac{1}{b} + \frac{1}{d} + \frac{1}{e} < 1,\ 1 \leq a, b, 2 \leq c, d, e)$$

$$\Gamma(O_3, 2a, 6b, 4c, 4d) = (m_0, m_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0m_1)^a = (m_1r_2r_3m_1r_3r_2)^b = (m_0r_2m_0r_2)^c = (m_0r_3m_0r_3)^d$$
$$= \bar{m}_0r_2\bar{m}_0r_2 = \bar{m}_0r_3\bar{m}_0r_3 = \bar{m}_1r_3\bar{m}_2r_3 = \bar{m}_1r_2\bar{m}_3r_2 = (\bar{m}_i m_0)^2 = (\bar{m}_j m_1)^2 = m_0^2 = m_1^2 = r_2^2 = r_3^2 = \bar{m}_k^2 = 1,$$
$$i \in \{1, 2, 3\}, j \in \{0, 2, 3\}, k \in \{0, 1, 2, 3\},\ \frac{2}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} < 3,\ 2 \leq a, 2 \leq b, 1 \leq c \leq d)$$

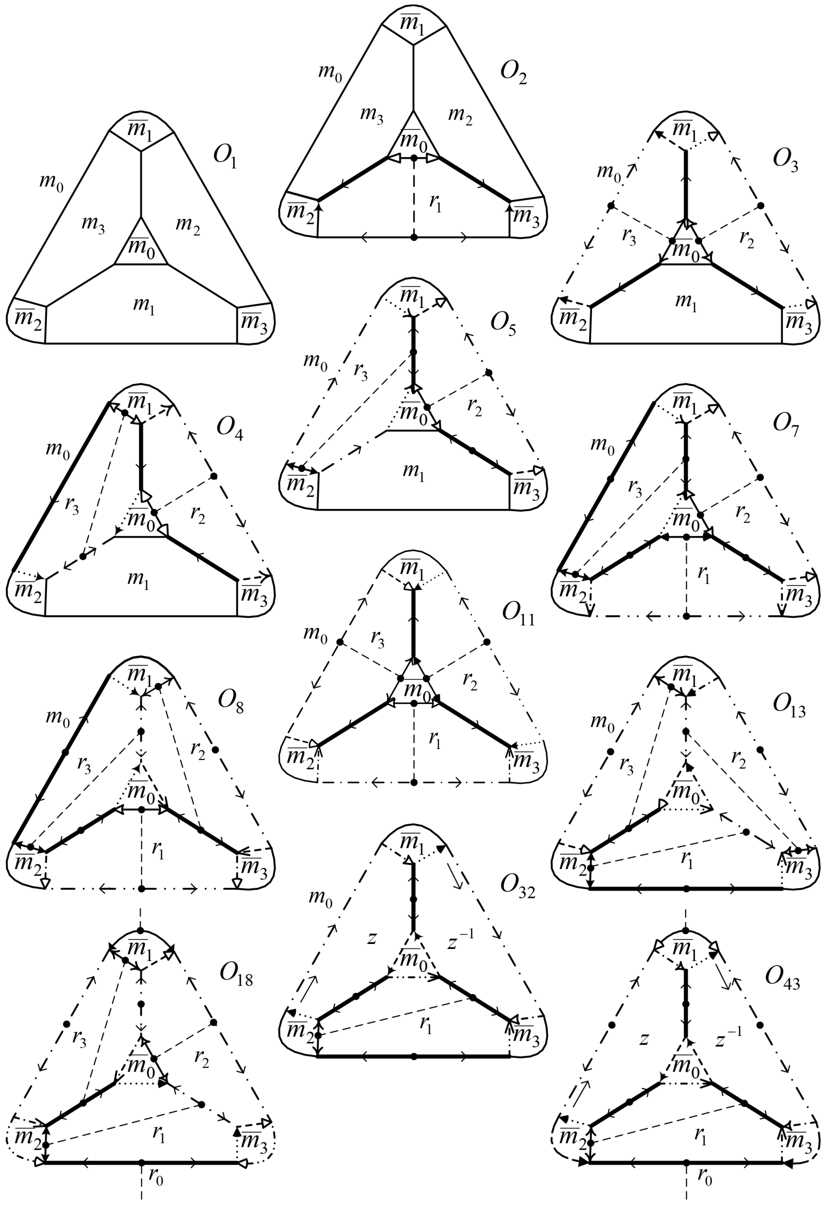

Figure 6: Trunc-simplices from maximal group series

Families F16, F17, F18 (with trunc-simplices O_4, O_5, O_7, respectively):

$$\Gamma(O_4, 2a, 6b, 4c, 4d) = (m_0, m_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0m_1)^a = (m_1r_2r_3m_0r_3r_2)^b = (m_1r_3m_1r_3)^c =$$
$$= (m_0r_2m_0r_2)^d = \bar{m}_0r_2\bar{m}_0r_2 = \bar{m}_0r_3\bar{m}_2r_3 = \bar{m}_1r_3\bar{m}_1r_3 = \bar{m}_1r_2\bar{m}_3r_2 = (\bar{m}_im_0)^2 = (\bar{m}_jm_1)^2 = m_0^2 = m_1^2 = r_2^2 = r_3^2$$
$$= \bar{m}_k^2 = 1,\ i \in \{1,2,3\}, j \in \{0,2,3\}, k \in \{0,1,2,3\},\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 2, \frac{1}{a} + \frac{1}{b} + \frac{1}{d} < 2,\ 2 \le a, 1 \le b, 1 \le c \le d)$$

$$\Gamma(O_5, 2a, 8b, 4c, 4d) = (m_0, m_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0m_1)^a = (m_1r_2r_3r_2m_1r_2r_3r_2)^b = (m_0r_2m_0r_2)^c =$$
$$= (m_0r_3m_1r_3)^d = \bar{m}_0r_2\bar{m}_0r_2 = \bar{m}_0r_3\bar{m}_1r_3 = \bar{m}_1r_2\bar{m}_3r_2 = \bar{m}_2r_3\bar{m}_2r_3 = (\bar{m}_im_0)^2 = (\bar{m}_jm_1)^2 = m_0^2 = m_1^2 = r_2^2 = r_3^2$$
$$= \bar{m}_k^2 = 1,\ i \in \{1,2,3\}, j \in \{0,2,3\}, k \in \{0,1,2,3\},\ \frac{1}{a} + \frac{1}{d} < 1, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} < 3,\ 2 \le a, 1 \le b, 1 \le c, 1 \le d)$$

$$\Gamma(O_7, 4a, 16b, 4c) = (m_0, r_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0r_1m_0r_1)^a = (m_0r_3r_1r_2r_3r_2r_1r_3m_0r_3r_1r_2r_3r_2r_1r_3)^b =$$
$$= (m_0r_2m_0r_2)^c = \bar{m}_0r_1\bar{m}_0r_1 = \bar{m}_0r_2\bar{m}_0r_2 = \bar{m}_0r_3\bar{m}_1r_3 = \bar{m}_1r_2\bar{m}_3r_2 = \bar{m}_2r_3\bar{m}_2r_3 = \bar{m}_2r_1\bar{m}_3r_1 = (\bar{m}_im_0)^2 = m_0^2 =$$
$$= r_1^2 = r_2^2 = r_3^2 = \bar{m}_k^2 = 1,\ i \in \{1,2,3\}, k \in \{0,1,2,3\},\ 1 \le a, b, c)$$

Families F19, F20, F22 (with trunc-simplices O_8, O_{11}, O_{13}, respectively):

$$\Gamma(O_8, 4a, 12b, 8c) = (m_0, r_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0r_1m_0r_1)^a = (m_0r_3r_1r_2r_1r_3m_0r_3r_1r_2r_1r_3)^b =$$
$$= (m_0r_2r_3r_2m_0r_2r_3r_2)^c = \bar{m}_0r_1\bar{m}_0r_1 = \bar{m}_0r_3\bar{m}_1r_3 = \bar{m}_0r_2\bar{m}_3r_2 = \bar{m}_1r_2\bar{m}_1r_2 = \bar{m}_2r_3\bar{m}_2r_3 = \bar{m}_2r_1\bar{m}_3r_1 = (\bar{m}_im_0)^2 =$$
$$m_0^2 = r_1^2 = r_2^2 = r_3^2 = \bar{m}_k^2 = 1,\ i \in \{1,2,3\}, k \in \{0,1,2,3\},\ 1 \le a, b, c)$$

$$\Gamma(O_{11}, 4a, 4b, 4c, 3d) = (m_0, r_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0r_1m_0r_1)^a = (m_0r_3m_0r_3)^b = (m_0r_2m_0r_2)^c =$$
$$= (r_1r_2r_3)^d = \bar{m}_0r_1\bar{m}_0r_1 = \bar{m}_0r_2\bar{m}_0r_2 = \bar{m}_0r_3\bar{m}_0r_3 = \bar{m}_1r_3\bar{m}_2r_3 = \bar{m}_1r_2\bar{m}_3r_2 = \bar{m}_2r_1\bar{m}_3r_1 = (\bar{m}_im_0)^2 =$$
$$= m_0^2 = r_1^2 = r_2^2 = r_3^2 = \bar{m}_k^2 = 1,\ i \in \{1,2,3\}, k \in \{0,1,2,3\},\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{2}{d} < 3,\ 1 \le a \le b \le c, 3 \le d)$$

$$\Gamma(O_{13}, 8a, 8b, 8c) = (m_0, r_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0r_1r_3r_1)^{2a} = (m_0r_2r_1r_2)^{2b} = (m_0r_3r_2r_3)^{2c} =$$
$$= \bar{m}_0r_2\bar{m}_1r_2 = \bar{m}_0r_3\bar{m}_2r_3 = \bar{m}_0r_1\bar{m}_3r_1 = \bar{m}_1r_3\bar{m}_1r_3 = \bar{m}_2r_1\bar{m}_2r_1 = \bar{m}_3r_2\bar{m}_3r_2 = (\bar{m}_im_0)^2 =$$
$$= m_0^2 = r_1^2 = r_2^2 = r_3^2 = \bar{m}_k^2 = 1,\ i \in \{1,2,3\}, k \in \{0,1,2,3\},\ 1 \le a \le b \le c)$$

Families F24, F28, F31 (with trunc-simplices O_{18}, O_{32}, O_{43}, respectively):

$$\Gamma(O_{18}, 4a, 8b) = (r_0, r_1, r_2, r_3, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(r_0r_1r_3r_1)^a = (r_1r_2r_3r_0r_2r_0r_3r_2)^b = \bar{m}_0r_2\bar{m}_0r_2 = \bar{m}_0r_3\bar{m}_2r_3 =$$
$$= \bar{m}_0r_1\bar{m}_3r_1 = \bar{m}_1r_0\bar{m}_1r_0 = \bar{m}_1r_3\bar{m}_1r_3 = \bar{m}_1r_2\bar{m}_3r_2 = \bar{m}_2r_1\bar{m}_2r_1 = \bar{m}_2r_0\bar{m}_3r_0 =$$
$$= r_0^2 = r_1^2 = r_2^2 = r_3^2 = \bar{m}_k^2 = 1,\ k \in \{0,1,2,3\},\ \frac{1}{a} + \frac{1}{b} < 2,\ 1 \le a, b)$$

$$\Gamma(O_{32}, 4a, 16b) = (m_0, r_1, z, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(m_0 z^{-1} m_0 z)^a = (m_0 r_1 z^{-2} r_1 z^2 r_1)^{2b} = \bar{m}_0 z \bar{m}_0 z^{-1} = \bar{m}_0 r_1 \bar{m}_3 r_1 =$$
$$= \bar{m}_1 z \bar{m}_2 z^{-1} = \bar{m}_1 z^{-1} \bar{m}_3 z = \bar{m}_2 r_1 \bar{m}_2 r_1 = (\bar{m}_i m_0)^2 = m_0^2 = r_1^2 = \bar{m}_k^2 = 1,\ i \in \{1,2,3\}, k \in \{0,1,2,3\},$$
$$\frac{2}{a} + \frac{1}{b} < 4,\ 1 \le a, 1 \le b)$$

$$\Gamma(O_{43}, 4a, 8b) = (r_0, r_1, z, \bar{m}_0, \bar{m}_1, \bar{m}_2, \bar{m}_3, -(z r_0 z r_0)^a = (r_0 r_1 z^{-2} r_1 z^2 r_1)^b = \bar{m}_0 z \bar{m}_0 z^{-1} = \bar{m}_0 r_1 \bar{m}_3 r_1 =$$
$$= \bar{m}_1 r_0 \bar{m}_1 r_0 = \bar{m}_1 z \bar{m}_2 z^{-1} = \bar{m}_1 z^{-1} \bar{m}_3 z = \bar{m}_2 r_1 \bar{m}_2 r_1 = \bar{m}_2 r_0 \bar{m}_3 r_0 = r_0^2 = r_1^2 = \bar{m}_k^2 = 1,$$
$$k \in \{0,1,2,3\},\ \frac{1}{a} + \frac{1}{b} < 2,\ 1 \le a, 1 \le b)$$

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