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Riemannian Generalized C-spaces with Homogeneous Geodesics

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Abstract. We investigate homogeneous geodesics in a class of homogeneous spaces G/K' called generalized C-spaces. We give necessary conditions so that a G-invariant metric on G/K' is a g.o. metric.

1. Introduction

Let (M, g) be a homogeneous Riemannian manifold, i.e. a connected Riemannian manifold on which the largest connected group G of isometries acts transitively. Then M can be expressed as a homogeneous space $(G/K, g)$, where K is the isotropy group at a fixed pointed o of M, and g is a G-invariant metric. A geodesic $\gamma(t)$ through the origin o of $M = G/K$ is called *homogeneous* if it is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tX)(o), \quad t \in \mathbb{R}, \tag{1}$$

where $\mathfrak{g}$ is the Lie algebra of G and X is a non zero vector of $\mathfrak{g}$.

A homogeneous Riemannian manifold $M = G/K$ is called a *g.o. manifold*, if all geodesics are homogeneous with respect to the largest connected group of isometries $I_o(M)$. A G-invariant metric g on M is called *G-g.o.* if all geodesics are homogeneous with respect to the group $G \subseteq I_o(M)$. Of course a G-g.o. metric is a g.o. metric, but the converse is not true in general. *In this paper we only consider G-g.o. metrics, which we also call them g.o. metrics.*

Naturally reductive spaces, symmetric spaces and weekly symmetric spaces are g.o. spaces ([9], [11], [17], [23]). In [18] O. Kowalski and L. Vanhecke gave an explicit classification of all naturally reductive spaces of dim ≤ 5. In [19] O. Kowalski and L. Vanhecke gave a classification of all g.o. spaces, which are in no way naturally reductive, up to dimension six. In [16] C. Gordon described g.o. spaces which are

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nilmanifolds, and in [21] H. Tamaru classified homogeneous g.o. spaces which are fibered over irreducible symmetric spaces. In [13] and [14] O. Kowalski and Z. Dušek investigated homogeneous geodesics in Heisenberg groups and some H-type groups. Examples of g.o. spaces in dimension seven were obtained by Dušek, O. Kowalski and S. Nikčević in [15]. In [1] the first author and D.V. Alekseevsky classified generalized flag manifolds which are g.o. spaces. Recently, A. Arvanitoyeorgos, Y. Wang and G.S. Zhao classified generalized Wallach spaces and M-spaces which are g.o. spaces ([6], [7], [8]). Also, in [12] Z. Chen and Yu. Nikonorov classified compact simply connected g.o. spaces with two isotropy summands. Finally, the notion of homogeneous geodesics can be extended to geodesics which are orbits of a product of two exponential factors (cf. [4], [5]).

The general problem of classification of compact homogeneous Riemannian manifolds $(M = G/K, g)$ with homogeneous geodesics remains open.

Let G be compact semisimple Lie group. By a *C-subgroup* of G we mean a closed and connected subgroup whose semisimple part coincides with the semisimple part of the centraliser of a toral subgroup of G (c.f. [22], p.13).

Definition 1.1. *Let G be a compact semisimple Lie group and K' be a C-subgroup of G. The homogeneous space G/K' is called a generalized C-space.*

In [22] H.C. Wang introduced and studied M-spaces, which are defined as follows: Let G/K be a generalized flag manifold with $K = C(S) = S \times K_1$, where S is a torus in a compact simple Lie group G and K_1 is the semisimple part of K. Then the *corresponding M-space* is the homogeneous space G/K_1.

It is easy to see that generalized flag manifolds and M-spaces are generalized C-spaces. The classification of homogeneous geodesics in generalized flag manifolds and M-spaces has been done([1], [7], [8]).

The object of this paper is to investigate homogeneous geodesics in generalized C-spaces G/K' associated to a generalized flag manifold G/K with $K_1 \subset K' \subset K$.

Let G/K with $K = C(S) = S \times K_1$ be a generalized flag manifold and $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G and K respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an $Ad(K)$-invariant reductive decomposition of the Lie algebra $\mathfrak{g}$, where $\mathfrak{m} \cong T_o(G/K)$. This is orthogonal with respect to $B = -$Killing from on $\mathfrak{g}$. Assume that

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s \tag{2}$$

is a B-orthogonal decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible ad($\mathfrak{k}$)-modules. We choose an intermediate subgroup $K_1 \subset K' \subset K$ such that K_1 is the semisimple part of K'. Then for the corresponding M-space G/K_1 we have that $G/K \subseteq G/K' \subseteq G/K_1$, and we call G/K' a *generalized C-space determined by a generalized flag manifold G/K*.

Lemma 1.2. *Let G/K' be a generalized C-space determined by a generalized flag manifold G/K. Then we have that $K' = S' \times K_1$, where $S' \subseteq S$ a toral subgroup, $K = C(S) = S \times K_1$, S is a torus in G, and K_1 is the semisimple part for both K and K'.*

Let $\mathfrak{s}'$ and $\mathfrak{k}_1$ be the Lie algebras of S' and K_1 respectively. We denote by $\mathfrak{n}$ the tangent space $T_o(G/K')$, where $o = eK'$. Then it follows that $\mathfrak{n} = \mathfrak{s}_1 \oplus \mathfrak{m}$, where $\mathfrak{s}_1$ is the Lie algebra of $S \setminus S'$. A G-invariant metric g on G/K' induces a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ which is $Ad(K')$-invariant. Such an $Ad(K')$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ can be expressed as $\langle x, y \rangle = B(\Lambda x, y)$ $(x, y \in \mathfrak{n})$, where Λ is an $Ad(K')$-equivariant positive definite symmetric operator on $\mathfrak{n}$. Conversely, any such operator Λ determines an $Ad(K')$-invariant scalar product $\langle x, y \rangle = B(\Lambda x, y)$ on $\mathfrak{n}$, which in turn determines a G-invariant Riemannian metric g on $\mathfrak{n}$. We say that Λ is the *operator associated* to the metric g, or simply the *associated operator*.

If a flag manifold G/K has $s \leq 2$ in the decomposition (2), it follows that its second Betti number $b_2(G/K)$ is equal to 1. This implies that there do not exist generalized C-spaces G/K' determined by G/K with $K_1 \subset K' \subset K$. Hence we only consider homogeneous geodesics in generalized C-spaces determined by flag manifolds G/K with $s \geq 3$ in the decomposition (2).

In this paper we investigate homogeneous geodesics in a generalized C-space G/K' which is determined by a generalized flag manifold G/K with G simple Lie group.

The main result is the following:

Theorem 1.3. *Let G/K be a generalized flag manifold for a compact simple Lie group with $s \geq 3$ in the decomposition (2). Let G/K' be a generalized C-space determined by the generalized flag manifold G/K with $K_1 \subset K' \subset K$, $K = C(S) = S \times K_1$, S a torus in G, and K_1 the semisimple part of K. If $(G/K', g)$ is a g.o. space, then*

$$g = \langle \cdot, \cdot \rangle = \Lambda \mid_{\mathfrak{s}_1} + \lambda B(\cdot, \cdot) \mid_{\mathfrak{m}}, \ (\lambda > 0),$$

where Λ is the operator associated to the metric g.

The paper is organized as follows: In Section 1 we recall certain Lie theoretic properties of a generalized flag manifold G/K and generalized C-space G/K'. In Section 2 we recall basic facts about g.o. spaces. In Section 3 we give the proofs of Lemma 1.2 and Theorem 1.3.

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2. Generalized flag manifolds and generalized *C*-spaces

Let $G/K = G/C(S)$ be a generalized flag manifold, where G is a compact semisimple Lie group and S is a torus in G, here $C(S)$ denotes the centralizer of S in G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G and K respectively, and $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition with respect to $B = -$Killing form on $\mathfrak{g}$ with $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. Let T be a maximal torus of G containing S. Then this is a maximal torus in K. Let $\mathfrak{a}$ be the Lie algebra of T and $\mathfrak{a}^{\mathbb{C}}$ its complexification. Then $\mathfrak{a}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let R be a root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$ and $\Pi = \{\alpha_1, \ldots, \alpha_l\}, (l = \dim_{\mathbb{C}} \mathfrak{a}^{\mathbb{C}})$ be a system of simple roots of R, and $\{\Lambda_1, \ldots, \Lambda_l\}$ be the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π, that is $\frac{2B(\Lambda_i, \alpha_j)}{B(\alpha_j, \alpha_j)} = \delta_{ij}, (1 \leq i, j \leq l)$. We can identify $(\mathfrak{a}^{\mathbb{C}})^*$ with $\mathfrak{a}^{\mathbb{C}}$ as follows: For every $\alpha \in (\mathfrak{a}^{\mathbb{C}})^*$ it corresponds to $h_\alpha \in \mathfrak{a}^{\mathbb{C}}$ by the equation $B(H, h_\alpha) = \alpha(H)$ for all $H \in \mathfrak{a}^{\mathbb{C}}$. Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\mathbb{C}}_\alpha \tag{3}$$

be the root space decomposition, where

$$\mathfrak{g}^{\mathbb{C}}_\alpha = \{X \in \mathfrak{g}^{\mathbb{C}} : [H, X] = \alpha(H)X \ \ \forall H \in \mathfrak{a}^{\mathbb{C}}\}. \tag{4}$$

Since $\mathfrak{k}^{\mathbb{C}}$ contains $\mathfrak{a}^{\mathbb{C}}$, there is a subset R_K of R such that $\mathfrak{k}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}^{\mathbb{C}}_\alpha$. We choose a system of simple roots Π_K of R_K and a system of simple roots Π of R so that $\Pi_K \subset \Pi$. We choose an ordering in R^+. Then there is a natural ordering in R^+_K, so that $R^+_K \subset R^+$. Set $R_M = R \setminus R_K$ (complementary roots). Then $\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathfrak{g}^{\mathbb{C}}$.

Definition 2.1. *An invariant ordering R^+_M in R_M is a choice of a subset $R^+_M \subset R_M$ such that*
(i) $R = R_K \sqcup R^+_M \sqcup R^-_M$, where $R^-_M = \{-\alpha : \alpha \in R^+_M\}$,
(ii) If $\alpha, \beta \in R^+_M$ and $\alpha + \beta \in R_M$, we have $\alpha + \beta \in R^+_M$,
(iii) If $\alpha \in R^+_M, \beta \in R^+_K$ and $\alpha + \beta \in R$, we have $\alpha + \beta \in R^+_M$.
For any $\alpha, \beta \in R^+_M$ we define $\alpha > \beta$ if and only if $\alpha - \beta \in R^+_M$.

We choose a Weyl basis $\{E_\alpha, H_\alpha : \alpha \in R\}$ in $\mathfrak{g}^{\mathbb{C}}$ with $B(E_\alpha, E_{-\alpha}) = 1, [E_\alpha, E_{-\alpha}] = H_\alpha$ and

$$[E_\alpha, E_\beta] = \begin{cases} 0, & if \ \ \alpha + \beta \notin R \ and \ \alpha + \beta \neq 0, \\ N_{\alpha,\beta} E_{\alpha+\beta}, & if \ \ \alpha + \beta \in R, \end{cases} \tag{5}$$

where the structural constants $N_{\alpha,\beta}$ $(\neq 0)$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ and $N_{\beta,\alpha} = -N_{\alpha,\beta}$. Then we have that

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}^{\mathbb{C}}_\alpha \oplus \sum_{\alpha \in R_M} \mathfrak{g}^{\mathbb{C}}_\alpha, \tag{6}$$

and $\{E_\alpha : \alpha \in R_M\}$ is a basis of $\mathfrak{m}^{\mathbb{C}}$. It is well known that

$$\mathfrak{g}_u = \sum_{\alpha \in R^+} \mathbb{R}\sqrt{-1}H_\alpha \oplus \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha), \tag{7}$$

where $A_\alpha = E_\alpha - E_{-\alpha}, B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$ $(\alpha \in R^+)$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$. Hence we can identify $\mathfrak{g}$ with $\mathfrak{g}_u$. In fact $\mathfrak{g} = \mathfrak{g}_u$ is the fixed point set of the conjugation $X + \sqrt{-1}Y \mapsto \overline{X + \sqrt{-1}Y} = X - \sqrt{-1}Y$ in $\mathfrak{g}^{\mathbb{C}}$ so that $\overline{E_\alpha} = -E_{-\alpha}$. Hence $\mathfrak{k} = \sum_{\alpha \in R^+} \mathbb{R}\sqrt{-1}H_\alpha \oplus \sum_{\alpha \in R_K^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$. We set $R_M^+ = R^+ \setminus R_K^+$. Then

$$\mathfrak{m} = \sum_{\alpha \in R_M^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha). \tag{8}$$

The next lemma gives us information about the Lie algebra structure of $\mathfrak{g}$.

Lemma 2.2. *The Lie bracket among the elements of* $\{A_\alpha, B_\alpha, \sqrt{-1}H_\beta : \alpha \in R^+, \beta \in \Pi\}$ *of* $\mathfrak{g}$ *are given by*

$$\begin{aligned}
&[\sqrt{-1}H_\alpha, A_\beta] = \beta(H_\alpha)B_\beta, && [A_\alpha, A_\beta] = N_{\alpha,\beta}A_{\alpha+\beta} + N_{-\alpha,\beta}A_{\alpha-\beta}\ (\alpha \neq \beta),\\
&[\sqrt{-1}H_\alpha, B_\beta] = -\beta(H_\alpha)A_\beta, && [B_\alpha, B_\beta] = -N_{\alpha,\beta}A_{\alpha+\beta} - N_{\alpha,-\beta}A_{\alpha-\beta}\ (\alpha \neq \beta),\\
&[A_\alpha, B_\alpha] = 2\sqrt{-1}H_\alpha, && [A_\alpha, B_\beta] = N_{\alpha,\beta}B_{\alpha+\beta} + N_{\alpha,-\beta}B_{\alpha-\beta}\ (\alpha \neq \beta),
\end{aligned}$$

where $N_{\alpha,\beta}$ *are the structural constants and* $\alpha + \beta,\ \alpha - \beta$ *are roots in (5).*

An important invariant of a generalized flag manifold G/K is the set $R_{\mathfrak{t}}$ of $\mathfrak{t}$-roots. Their importance arises from the fact that the knowledge of $R_{\mathfrak{t}}$ gives us crucial information about the decomposition of the isotropy representation of the flag manifold G/K.

From now on we fix a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r, \phi_1, \ldots, \phi_k\}$ of R, so that $\Pi_K = \{\phi_1, \ldots, \phi_k\}$ is a basis of the root system R_K and $\Pi_M = \Pi \setminus \Pi_K = \{\alpha_1, \ldots, \alpha_r\}$ $(r + k = l)$.

We consider the decomposition $R = R_K \cup R_M$ and let

$$\mathfrak{t} = \mathfrak{z}(\mathfrak{k}^{\mathbb{C}}) \cap \sqrt{-1}\mathfrak{a} = \{X \in \sqrt{-1}\mathfrak{a} : \phi(X) = 0, \text{ for all } \phi \in R_K\}, \tag{9}$$

where $\mathfrak{z}(\mathfrak{k}^{\mathbb{C}})$ is the center of $\mathfrak{k}^{\mathbb{C}}$. Consider the restriction map $\kappa : (\mathfrak{a}^{\mathbb{C}})^* \to \mathfrak{t}^*$ defined by $\kappa(\alpha) = \alpha\,|_{\mathfrak{t}}$, and set $R_{\mathfrak{t}} = \kappa(R) = \kappa(R_M)$. Note that $\kappa(R_K) = 0$ and $\kappa(0) = 0$.

The elements of $R_{\mathfrak{t}}$ are called $\mathfrak{t}$-*roots*. For an invariant ordering $R_M^+ = R^+ \setminus R_K^+$ in R_M, we set $R_{\mathfrak{t}}^+ = \kappa(R_M^+)$ and $R_{\mathfrak{t}}^- = -R_{\mathfrak{t}}^+ = \{-\xi : \xi \in R_{\mathfrak{t}}^+\}$. It is obvious that $R_{\mathfrak{t}}^- = \kappa(R_M^-)$, thus the splitting $R_{\mathfrak{t}} = R_{\mathfrak{t}}^- \cup R_{\mathfrak{t}}^+$ defines an ordering in $R_{\mathfrak{t}}$. A $\mathfrak{t}$-root $\xi \in R_{\mathfrak{t}}^+$ (respectively $\xi \in R_{\mathfrak{t}}^-$) will be called *positive* (respectively negative). A $\mathfrak{t}$-root is called *simple* if it is not a sum of two positive $\mathfrak{t}$-roots. The set $\Pi_{\mathfrak{t}}$ of all simple $\mathfrak{t}$-roots is called a $\mathfrak{t}$-*basis* of $\mathfrak{t}^*$, in the sense that any $\mathfrak{t}$-root can be written as a linear combination of its elements with integer coefficients of the same sign.

Definition 2.3 ([1]). *(1) Two* $\mathfrak{t}$-*roots* $\xi, \eta \in R_{\mathfrak{t}}$ *are called adjacent if one of the following occurs:*
(i) If η *is a multiple of* ξ*, then* $\eta \neq \pm 2\xi$ *and* $\xi \neq \pm 2\eta$*.*
(ii) If η *is not a multiple of* ξ*, then* $\xi + \eta \in R_{\mathfrak{t}}$ *or* $\xi - \eta \in R_{\mathfrak{t}}$*.*
(2) Two $\mathfrak{t}$-*roots* $\xi, \eta \in R_{\mathfrak{t}}$ *are called connected if there is a chain of* $\mathfrak{t}$-*roots*

$$\xi = \xi_1, \xi_2, \ldots, \xi_k = \eta$$

such that ξ_i, ξ_{i+1} *are adjacent* $(i = 1, \ldots, k - 1)$*.*

We remark that ξ and $\pm\xi$ are connected, and if $\xi, 2\xi$ are the only positive $\mathfrak{t}$-roots, then these are not connected. We define the relation

$$\xi \sim \eta \Leftrightarrow \xi, \eta \text{ are connected.} \tag{10}$$

One can easily check that this is an equivalence relation. Let R^i be the equivalent classes consisting of mutually connected $\mathfrak{t}$-roots. Then the set $R_{\mathfrak{t}}$ is decomposed into a disjoint union

$$R_{\mathfrak{t}} = R^1 \cup \cdots \cup R^r. \tag{11}$$

Definition 2.4. *The set of* $\mathfrak{t}$*-roots* $R_{\mathfrak{t}}$ *is called connected if* $r = 1$.

By a result in [1] if G is simple, then for $s \geq 3$ in the decomposition (2), the set of $\mathfrak{t}$-roots is connected.

Proposition 2.5 ([3]). *There is one-to-one correspondence between* $\mathfrak{t}$*-roots and complex irreducible* $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$*-submodules* $\mathfrak{m}_\xi$ *of* $\mathfrak{m}^{\mathbb{C}}$. *This correspondence is given by*

$$R_{\mathfrak{t}} \ni \xi \leftrightarrow \mathfrak{m}_\xi = \sum_{\alpha \in R_M : \kappa(\alpha) = \xi} \mathbb{C}E_\alpha.$$

Thus $\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_{\mathfrak{t}}} \mathfrak{m}_\xi$. *Moreover, these submodules are inequivalent as* $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$*-modules.*

Since the complex conjugation $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, $X + \sqrt{-1}Y \mapsto X - \sqrt{-1}Y$ $(X, Y \in \mathfrak{g})$ of $\mathfrak{g}^{\mathbb{C}}$ with respect to the compact real form $\mathfrak{g}$ interchanges the root spaces, i.e. $\tau(E_\alpha) = E_{-\alpha}$ and $\tau(E_{-\alpha}) = E_\alpha$, a decomposition of the real $Ad(K)$-module $\mathfrak{m} = (\mathfrak{m}^{\mathbb{C}})^\tau$ into real irreducible $Ad(K)$-submodule is given by

$$\mathfrak{m} = \sum_{\xi \in R_{\mathfrak{t}}^+ = \kappa(R_M^+)} (\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau, \tag{12}$$

where V^τ denotes the set of fixed points of the complex conjugation τ in a vector subspace $V \subset \mathfrak{g}^{\mathbb{C}}$. If, for simplicity, we set $R_{\mathfrak{t}}^+ = \{\xi_1, \ldots, \xi_s\}$, then according to (12) each real irreducible $Ad(K)$-submodule $\mathfrak{m}_i = (\mathfrak{m}_{\xi_i} \oplus \mathfrak{m}_{-\xi_i})^\tau$ $(1 \leq i \leq s)$ corresponding to the positive $\mathfrak{t}$-roots ξ_i, is given by

$$\mathfrak{m}_i = \sum_{\alpha \in R_M^+ : \kappa(\alpha) = \xi_i} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha). \tag{13}$$

In [22] the author gave the definition of C-spaces as following:

Definition 2.6 ([22]). *Simply-connected and compact homogeneous complex manifolds are called* C*-spaces.*

It is well known that each closed homogeneous complex manifold is analytically homeomorphic with a complex coset space([22], p.3). In order to describe C-spaces definitely we should give the definition of a C-subgroup of a compact semi-simple Lie group.

Definition 2.7 ([22]). *Let* G *be a compact semi-simple Lie group. By a* C*-subgroup of* G*, we mean a closed and connected subgroup whose semi-simple part coincides with the semi-simple part of the centraliser of a toral subgroup of* G.

The following two theorems are very important for us to understand a C-space.

Theorem 2.8 ([22]). *Each* C*-space is homeomorphic with a coset space* G/K'*, where* G *denotes a compact semi-simple Lie group, and* K' *is a* C*-subgroup of* G.

Theorem 2.9 ([22]). *Let* K' *be a* C*-subgroup of a simply connected compact semi-simple Lie group* G*. If* G/K' *is even dimension, then* G/K' *has a homogeneous complex structure, or in other words,* G/K' *is homeomorphic with a* C*-space.*

In fact, any generalized C-space G/K' has a relation with some generalized flag manifold G/K and corresponding M-space G/K_1.

Definition 2.10. *We call that a generalized* C*-space* G/K' *is determined by a generalized flag manifold* G/K*, if* $K_1 \subseteq K' \subseteq K$*, where* $K = C(S) = S \times K_1$*, and* S *is a torus in* G *and* K_1 *is the semi-simple part of both* K *and* K'.

3. Riemannian g.o. spaces

Let $(M = G/K, g)$ be a homogeneous Riemannian manifold with G a compact connected semisimple Lie group. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition.

Definition 3.1. *A nonzero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve (1) is a geodesic.*

Lemma 3.2 ([19]). *A nonzero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0 \tag{14}$$

for all $Y \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection into $\mathfrak{m}$.

A useful description characterization of g.o. spaces is the following:

Proposition 3.3 ([1]). *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold. Then $(M = G/K, g)$ is a g.o. space if and only if for every $x \in \mathfrak{m}$ there exists an $a(x) \in \mathfrak{k}$ such that*

$$[a(x) + x, \Lambda x] \in \mathfrak{k}. \tag{15}$$

For later use we recall the following:

Proposition 3.4. ([2, Proposition 5]) *Let $(M = G/H, g)$ be a compact g.o. space with associated operator Λ. Let $X, Y \in \mathfrak{m}$ be eigenvectors of Λ with different eigenvalues of λ, μ. Then*

$$[X, Y] = \frac{\lambda}{\lambda - \mu}[h, X] + \frac{\mu}{\lambda - \mu}[h, Y] \tag{16}$$

for some $h \in \mathfrak{h}$.

Proposition 3.5. ([20, Corollary 4]) *The inner product $\langle \cdot, \cdot \rangle$, generating the metric of a geodesic orbit Riemannian space $(G/H, g)$, is not only $Ad(H)$-invariant but also $Ad(N_G(H_0))$-invariant, where $N_G(H_0)$ is the normalizer of the unit component H_0 of the group H in G.*

The above proposition will be crucial for simplifying metrics, which are g.o..

4. Proof of Lemma 1 and Theorem 1

Let G/K be a generalized flag manifold with $K = C(S) = S \times K_1$, where S is a torus in the simple compact Lie group G and K_1 is the semisimple part of K. Let G/K' be a given generalized C-space determined by a generalized flag manifold G/K.

Proof of Lemma 1.2. Since K' is a C-subgroup of G, it follows that K' is connected. This implies that K' can be decomposed into a product $K' = G_1 \times R_1$ of a maximal semi-simple subgroup G_1 and the radical R_1, such that $G_1 \cap R_1$ is a discrete subgroup of K'. Since K_1 is a maximal semi-simple subgroup of K, therefore K_1 is a maximal semi-simple subgroup of K', that is $K' = K_1 \times R_1$. Since $K' \subseteq K$, we obtain that $R_1 \subseteq S$, that is $R_1 = S'$. This completes the proof. □

Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}_1$, $\mathfrak{s}$ and $\mathfrak{s}'$ be the Lie algebras of G, K, K_1, S and S' respectively, and let $\mathfrak{s}_1 = \mathfrak{s} \setminus \mathfrak{s}'$. Let $B = -$Killing form on $\mathfrak{g}^{\mathbb{C}}$. Assume that $\mathfrak{m} = T_o(G/K)$. Then the module $\mathfrak{m}$ decomposes into a direct sum of $Ad(K)$-invariant irreducible submodules pairwise orthogonal with respect to B (cf. (2)). Let $\langle \cdot, \cdot \rangle = B(\Lambda \cdot, \cdot)$ be an Ad(K)-invariant scalar product on $\mathfrak{m}$, where Λ is the associated operator. Therefore, G-invariant metrics on G/K which are Ad(K)-invariant are defined by

$$\langle \cdot, \cdot \rangle = \lambda_1 B(\cdot, \cdot)|_{\mathfrak{m}_1} + \cdots + \lambda_s B(\cdot, \cdot)|_{\mathfrak{m}_s}. \tag{17}$$

Proof of Theorem 1.3. For a given generalized C-space G/K', K' is connected and $K \subset N_G(K')$, where $N_G(K')$ is the normalizer of the group K' in G. Indeed, at the Lie algebra level we have that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{k}'] \subset \mathfrak{m}$, and $[\mathfrak{k}, \mathfrak{k}'] = [\mathfrak{s} \oplus \mathfrak{k}_1, \mathfrak{k}'] = ([\mathfrak{s}, \mathfrak{k}'] + [\mathfrak{k}_1, \mathfrak{k}']) \subset \mathfrak{k}'$. So, Proposition 3.5 implies that if the metric defined on G/K' is a g.o. metric, then it is not only $Ad(K')$-invariant but also $Ad(K)$-invariant. Also, the eigenspaces of Λ are $Ad(K)$-invariant. Therefore if the metric defined on G/K_1 is a g.o. metric, it reduces to

$$\langle \cdot, \cdot \rangle = \Lambda|_{\mathfrak{s}_1} + \lambda_1 B(\cdot, \cdot)|_{\mathfrak{m}_1} + \cdots + \lambda_s B(\cdot, \cdot)|_{\mathfrak{m}_s}. \tag{18}$$

Let $R_{\mathfrak{t}}^+ = \{\xi_1, \ldots, \xi_s\}$ be the set of positive $\mathfrak{t}$-roots of the generalized flag manifold G/K with $s \geq 3$, so $R_{\mathfrak{t}}^+$ is connected. Then for any $\xi, \eta \in R_{\mathfrak{t}}^+$ there exists (without loss of generality) a chain of positive $\mathfrak{t}$-roots

$$\xi = \zeta_1, \zeta_2, \ldots, \zeta_k = \eta, \tag{19}$$

where ζ_i, ζ_{i+1} are adjacent $(i = 1, \ldots, k-1)$.

We define the subset $\{\mathfrak{m}_{i_1}, \mathfrak{m}_{i_2}, \ldots, \mathfrak{m}_{i_k}\}$ of $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_s\}$ by

$$\mathfrak{m}_{i_q} = \sum_{\alpha \in R_M^+ : \kappa(\alpha) = \zeta_q} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha), \ (q = 1, \ldots, k). \tag{20}$$

If $\gamma_1 = k\gamma_2$ for $\gamma_1, \gamma_2 \in R_{\mathfrak{t}}^+$ and $(k \geq 2)$, we always have that either $\gamma_1 + \gamma_2 \in R_{\mathfrak{t}}$ or $\gamma_1 - \gamma_2 \in R_{\mathfrak{t}}$. Also, since ζ_q, ζ_{q+1} $(q = 1, \ldots, k-1)$ are adjacent, it follows that either $\zeta_q + \zeta_{q+1} \in R_{\mathfrak{t}}$ or $\zeta_{q+1} - \zeta_q \in R_{\mathfrak{t}}$. Therefore, we have

$$[\mathfrak{m}_{i_q}, \mathfrak{m}_{i_{q+1}}] \subseteq \left(\sum_{\alpha \in R_M^+ : \kappa(\alpha) = \zeta_q + \zeta_{q+1}} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \right) \oplus \left(\sum_{\alpha \in R_M : \kappa(\alpha) = \zeta_{q+1} - \zeta_q} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \right).$$

If $\zeta_q + \zeta_{q+1} \notin R_{\mathfrak{t}}$, then $\sum_{\alpha \in R_M^+ : \kappa(\alpha) = \zeta_{q+1} + \zeta_q} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) = \{0\}$. If $\zeta_{q+1} - \zeta_q \notin R_{\mathfrak{t}}$, then $\sum_{\alpha \in R_M : \kappa(\alpha) = \zeta_{q+1} - \zeta_q} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) = \{0\}$. Since $\zeta_q + \zeta_{q+1} \neq \pm\zeta_q$, $\zeta_q + \zeta_{q+1} \neq \pm\zeta_{q+1}$ and $\zeta_q - \zeta_{q+1} \neq \pm\zeta_q$, $\zeta_q - \zeta_{q+1} \neq \pm\zeta_{q+1}$, it follows that

$$\left(\sum_{\alpha \in R_M^+ : \kappa(\alpha) = \zeta_q + \zeta_{q+1}} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \oplus \sum_{\alpha \in R_M : \kappa(\alpha) = \zeta_{q+1} - \zeta_q} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \right) \cap \left(\mathfrak{m}_{i_q} \oplus \mathfrak{m}_{i_{q+1}} \right) = \{0\}.$$

Therefore we get that

$$[\mathfrak{m}_{i_q}, \mathfrak{m}_{i_{q+1}}] \cap (\mathfrak{m}_{i_q} \oplus \mathfrak{m}_{i_{q+1}}) = \{0\}. \tag{21}$$

Also, since ζ_q, ζ_{q+1} are adjacent $(q = 1, \ldots, k-1)$ in (19), there exist $X \in \mathfrak{m}_{i_q}, Y \in \mathfrak{m}_{i_{q+1}}$ eigenvectors of Λ such that $[X, Y] \neq 0$. If we had that $\lambda_{i_q} \neq \lambda_{i_{q+1}}$, then Proposition 3.4 implies that $[X, Y] \subset \mathfrak{m}_{i_q} \oplus \mathfrak{m}_{i_{q+1}}$, which contradicts (21), hence $\lambda_{i_q} = \lambda_{i_{q+1}}, (q = 1, \ldots, k-1)$. Since this is true for any $\xi, \eta \in R_{\mathfrak{t}}^+$ we obtain that $\lambda_1 = \lambda_2 = \cdots = \lambda_s$, and the conclusion follows. □

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