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Rotary Mappings of Spaces with Affine Connection

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Abstract. This paper concerns with rotary mappings of two-dimensional spaces with an affine connection onto (pseudo-) Riemannian spaces. The results obtained in the theory of rotary mappings are further developed. We prove that any (pseudo-) Riemannian space admits rotary mapping. There are also presented certain properties from which yields the existence of these rotary mappings.

1. Introduction

Special diffeomorphisms, for which any special curve maps onto a special curve, were studied in many works. These are for example geodesic, holomorphically projective, F-planar, almost geodesic, and other mappings, i.e. see [1–7, 11–13, 15–22, 24–32, 34, 35, 37].

Our work is devoted to a certain question about rotary mappings, for which any geodesic is mapped onto an isoperimetric extremal of rotation, i.e. [12–14, 16, 18, 23, 33].

Questions about isoperimetric extremals of rotation and rotary mappings had been studied by S. G. Leiko. He was the first one to introduce terms of isoperimetric extremals of rotation and rotary mappings [12–14, 16, 18].

Equations of these extremals of rotation were later specified in work [24]. Another contribution to this topic can be found in [4], where authors refined requirements for spaces which admit rotary mapping.

Above mentioned results Leiko obtained in his works have their application in the theory of gravitation fields, see [11, 15, 17]. In addition, he continued the research with Vinnik cooperation [35].

Leiko [12] found a necessary condition for the existence of the rotary mapping of two-dimensional Riemannian spaces $\mathbb{V}_2$, which is the existence of vector field θ that satisfies the following necessary condition

$$\nabla_X\theta = (\mathcal{A}(X) + \nabla_X K/K) \cdot \theta + \nu \cdot X \tag{1}$$

for any tangent vector X, where ∇ is the Levi-Civita connection, K is the Gaussian curvature, $\mathcal{A}$ is a linear form for which $\mathcal{A}(X) = g(X, \theta)$, g is a metric tensor, and ν is a function on $\mathbb{V}_2$.

In [4] Chudá, Mikeš and Sochor stated that for any two-dimmensional (pseudo-) Riemannian space $\mathbb{V}_2$ where exist vector fields satisfying the conditions (1) it is possible to construct the space with affine connection $\mathbb{A}_2$ which admits rotary mapping onto $\mathbb{V}_2$.

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In papers [12, 13, 16, 18] Leiko claims that from the equations (1) it yields spaces $\mathbb{V}_2$ are isometric with surfaces of revolution.

In the presented paper we are going to prove that the above mentioned statement is not valid, i.e. the following theorem holds

Theorem 1.1. *There exists a (pseudo-) Riemannian space* $\mathbb{V}_2$ *which is not isometric with surface of revolution and where exists the vector field satisfying equations* (1).

This result is in the shorter form presented in [23]. Further, we analyse gained results in more detail.

2. On isopetrimetric extremal of rotation and rotary mapping

Isoperimetric extremals of rotation were first introduced in [12] by Leiko. The term was defined on two-dimensional Riemannian spaces $\mathbb{V}_2$ and surfaces $\mathcal{S}_2$ with a metric g as follows.

A curve ℓ: $x = x(t)$ on surface or on two-dimensional Riemannian space is called the *isoperimetric extremal of rotation* if ℓ is the extremal of functionals $\theta[\ell]$ and $s[\ell] = \text{const}$ with fixed ends.

Here

$$\theta[\ell] = \int_{t_0}^{t_1} k(t)\ \mathrm{d}t \quad \text{and} \quad s[\ell] = \int_{t_0}^{t_1} |\lambda|\ \mathrm{d}t,$$

where $k(t)$ is the curvature and $|\lambda|$ is the length of the tangent vector λ of ℓ.

Later, it was proved by Leiko [12, 16] that a curve ℓ is an isoperimetric extremal of rotation if and only if its Frenet curvature k and Gaussian curvature K are proportional $k = c \cdot K$, where c is a constant. For $c = 0$ we get a geodesic.

The equations of the isoperimetric extremal of rotation were simplified by Mikeš, Stepanova and Sochor [24] to $\nabla_s\lambda = c \cdot K \cdot F\lambda$, where c is a constant, s is the arc length, F is a tensor $\binom{1}{1}$ which satisfies the conditions

$$F^2 = -e \cdot Id, \quad g(X, FX) = 0, \quad \nabla F = 0.$$

For Riemannian manifold $\mathbb{V}_2$ is $e = +1$ and F is a *complex structure* and for pseudo-Riemannian manifold is $e = -1$ and F is a *product structure*. This tensor F is uniquely defined (with the respect to the sign) with using the skew-symmetric and covariantly constant discriminant tensor ε_{ij}, which is defined

$$F^h_j = g^{hi}\varepsilon_{ij}, \qquad \varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g^2_{12}|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In [12] there was introduced the term of rotary diffeomorphism between the two-dimensional Riemannian spaces $\mathbb{V}_2$ and the surfaces $\mathcal{S}_2$ with the metric g.

A diffeomorphism between two-dimensional (pseudo-) Riemannian manifolds $\mathbb{V}_2$ and $\bar{\mathbb{V}}_2$ is called *rotary* if any geodesic on $\bar{\mathbb{V}}_2$ is mapped onto isoperimetric extremal of rotation on $\mathbb{V}_2$.

This definition which was formulated by Leiko [12] was later generalized as follows, see [4].

A diffeomorphism f: $\mathbb{V}_2 \to \bar{\mathbb{A}}_2$ is called *rotary mapping* if any geodesic on manifold $\bar{\mathbb{A}}_2$ with affine connection $\bar{\nabla}$ is mapped onto isoperimetric extremal of rotation on two-dimensional (pseudo-) Riemanninan manifold $\mathbb{V}_2$.

If the definition was formulated the other way around: A diffeomorphism between two-dimensional (pseudo-) Riemannian manifolds $\mathbb{V}_2$ and $\bar{\mathbb{V}}_2$ is called *rotary* if any isoperimetric extremal of rotation on $\mathbb{V}_2$ is mapped onto geodesic on $\bar{\mathbb{V}}_2$, then this mapping would be a geodesic mapping.

Later, some new properties were proved, see [4]: When $\mathbb{V}_2$ admits rotary mapping f onto $\bar{\mathbb{A}}_2$ then if $\mathbb{V}_2$ and $\bar{\mathbb{A}}_2$ in common coordinate system belong differentiability class C^2 and C^1, respectively, then Gaussian curvature K on $\mathbb{V}_2$ is differentiable. As a result authors formulated new theorem: Rotary diffeomorphism $\mathbb{V}_2 \to \bar{\mathbb{A}}_2$ does not exist if Gaussian curvature $K \notin C^1$.

Chudá, Mikeš and Sochor [4] also proved that (pseudo-) Riemannian manifold $\mathbb{V}_2$ admits rotary mapping onto $\bar{\mathbb{A}}_2$ if and only if in $\mathbb{V}_2$ holds equation

$$\theta^h_{,j} = \theta^h(\theta_j + \partial_j \ln |K|) + \nu\, \delta^h_j, \tag{2}$$

where $\theta_i = g_{i\alpha}\theta^\alpha$, ν is a function on $\mathbb{V}_2$ and vector field θ^h is a special case of torse-forming field. Here and after comma denotes covariant derivative respective connection ∇, and $\partial_1 = \partial/\partial x^i$.

3. Contra example of spaces admitting rotary mappings

A necessary condition for a (pseudo-) Riemannian space $\mathbb{V}_2$ to admit rotary mapping onto a manifold $\bar{\mathbb{A}}_2$ is existence of a vector field that satisfies condition (1). Apparently, this vector field is a special type of torse-forming vector field which was defined by K. Yano [36], see also [19].

A vector field ξ is called *torse-forming* if for any tangent vector X holds

$$\nabla_X \xi = a(X) \cdot \xi + \nu\, X,$$

where a is a linear form and ν is a function.

Riemannian spaces $\mathbb{V}_n$ where these vector fields exist are characterized with a metric it the following form

$$ds^2 = (dx^1)^2 + f(x^1, \dots, x^n)\, d\tilde{s}^2,$$

where $d\tilde{s}^2$ is a metric of the $(n-1)$ dimensional (pseudo-) Riemannian space $\tilde{\mathbb{V}}_{n-1}$ and f is a function of all variables.

In our case, we suppose that the metric of the two-dimensional (pseudo-) Riemannian space $\mathbb{V}_2$ has the following form

$$ds^2 = \left(dx^1\right)^2 + f(x^1, x^2) \cdot \left(dx^2\right)^2. \tag{3}$$

It is known that this form of the metric always exists in any (pseudo-) Riemannian space $\mathbb{V}_2$. This coordinate system is called the semi-geodesic coordinate system, see [9].

In case the function f is a function of the variable x^1 then the space $\mathbb{V}_2$ is isometric with a surface of revolution. Also, let us suppose that the component θ^2 in this coordinate system is vanishing.

Now, we can compute non vanishing Christoffel symbols of the first and the second kind

$$\Gamma_{122} = \Gamma_{212} = 1/2\, f_1, \quad \Gamma_{221} = -1/2\, f_1, \quad \Gamma_{222} = 1/2\, f_2, \quad \text{and}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = 1/2\, \frac{f_1}{f}, \quad \Gamma^1_{22} = -1/2\, f_1, \quad \Gamma^2_{22} = 1/2\, \frac{f_2}{f},$$

here and further we denote $f_i = \partial_i f$, $\partial_i \equiv \partial/\partial x^i$ and analogically $f_{ij} = \partial_{ij} f$.

We use a well known formula to calculate the Gaussian curvature K of the surface $\mathbb{V}_2$

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2},$$

where

$$R_{hijk} = g_{h\alpha} R^\alpha_{ijk} \quad \text{and} \quad R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k} \tag{4}$$

are the components of the Riemannian tensors. Because $R_{1212} = R^1_{212} \cdot g_{11}$ from (4) it follows that

$$R_{1212} = R^1_{212} \cdot 1 = \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{21} + \Gamma^\alpha_{22}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{21}\Gamma^1_{\alpha 2} = -1/2 f_{11} + 1/4\, \frac{f_1^2}{f}$$

therefore

$$K = -\frac{f_{11}}{2f} + \left(\frac{f_1}{2f}\right)^2.$$

To simplify this relation we use substitution $F = f_1/f$ and thus we obtain

$$K = -1/2\,F_1 - 1/4\,F^2, \tag{5}$$

where similarly as above $F_1 = \partial_1 F$.

We can rewrite fundamental equation (2) in the following form

$$\theta^h_{,i} = \theta^h(\theta_i + \partial_i \ln|K|) + \nu\,\delta^h_i$$

and after lowering indices we get

$$\theta_{h,i} = \theta_h(\theta_i + \partial_i K/K) + \nu g_{hi}, \tag{6}$$

where K is the Gaussian curvature of the space $\mathbb{V}_2$ and $\theta_i = g_{i\alpha}\theta^i$. From it follows $\theta_1 = \theta^1$, and additionally in chosen coordinate system holds $\theta_2 = 0$.

For indices $(hi) = (12)$ from (6) and after lowering indices we obtain

$$\partial_2\theta_1 = \theta_1 \cdot \partial_2 K/K$$

and after integration we get

$$\theta_1 = \varkappa(x^1)K$$

where $\varkappa$ is a function of variable x^1. Evidently, for $(hi) = (21)$ formula (6) is identity and for $(hi) = (11)$ and (22) we get following equations

$$\nu = \theta_{11} - \theta_1^2 - \theta_1\,\partial_i K/K \quad \text{and} \quad \nu = \frac{1}{2}\,\theta_1 \cdot f_1/f.$$

We merge these formulas and obtain following equation

$$\frac{\varkappa'}{\varkappa} - \varkappa \cdot K = \frac{1}{2} \cdot \frac{f_1}{f}. \tag{7}$$

Therefore from (7) and (5) we get the equation

$$F' = -\frac{1}{2}\,F^2 + \frac{1}{\varkappa}\,F - 2 \cdot \frac{\varkappa'}{\varkappa^2}$$

which is a differential equation called Riccati equation, see [8]. Here, symbol " ′ " denotes a derivative with respect to variable x^1 and in these formulas x^2 is a parameter.

We use special substitution $F = 2 \cdot \dfrac{u'}{u}$ therefore we get a linear differential equation of the second order respective the unknown function u

$$u'' = \frac{1}{\varkappa}\,u' - \frac{\varkappa'}{\varkappa^2}\,u. \tag{8}$$

The general solution of equation (8) can be written in the following form

$$u = C_1 u_1(x^1) + C_2 u_2(x^1)$$

where $C_1 = C_1(x^2)$ and $C_2 = C_2(x^2)$.

Let us suppose that $\mathcal{U}(x^1)$ is a particular solution of differential equation (8). We put this solution into differential equation (8) and then we obtain

$$\varkappa' = -\frac{\mathcal{U}''}{\mathcal{U}}\,\varkappa^2 + \frac{\mathcal{U}'}{\mathcal{U}}\,\varkappa.$$

It is a differential equation of Bernoulli type, from which we can get inhomogeneous linear differential equation using substitution $v = \dfrac{1}{\varkappa}$

$$v' = -\frac{\mathcal{U}'}{\mathcal{U}}\,v + \frac{\mathcal{U}''}{\mathcal{U}}.$$

This equation can be solved using method of variation of parameters, from which we obtain $v = \dfrac{\mathcal{U}'}{\mathcal{U}}$ therefore $\varkappa = \dfrac{\mathcal{U}}{\mathcal{U}'}$. From it follows that one from the solutions of the equation (8) with a priori given $\varkappa(x^1)$ is

$$u = \mathrm{e}^{\int 1/\varkappa\,\mathrm{d}x^1}.$$

If the functions $\mathcal{U}$ and $\mathcal{V}$ are two solution of the differential equation (8) it is possible to form their Wronskian

$$W = \begin{vmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{U}' & \mathcal{V}' \end{vmatrix} = \mathcal{U}\mathcal{V}' - \mathcal{V}\mathcal{U}'.$$

Then after differentiating W and using (8) for $\mathcal{U}$ and $\mathcal{V}$ we get

$$W' = \begin{vmatrix} \mathcal{U}' & \mathcal{V}' \\ \mathcal{U}' & \mathcal{V}' \end{vmatrix} + \begin{vmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{U}'' & \mathcal{V}'' \end{vmatrix} = \begin{vmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{U}'/\varkappa + \mathcal{U}\cdot\varkappa'/\varkappa^2 & \mathcal{V}'/\varkappa + \mathcal{V}\cdot\varkappa'/\varkappa^2 \end{vmatrix} = \frac{1}{\varkappa}\,W.$$

Because $W' = \dfrac{1}{\varkappa}\,W$ we get this relation

$$W = C_1 \cdot \mathrm{e}^{\int 1/\varkappa \mathrm{d}x^1}, \tag{9}$$

where C_1 is a constant of integration.

Because $\dfrac{1}{\varkappa} = \dfrac{\mathcal{U}'}{\mathcal{U}}$ then $\displaystyle\int \frac{1}{\varkappa}\,\mathrm{d}x^1 = \ln|\mathcal{U}|$ and from (9) we obtain

$$\mathcal{U}\mathcal{V}' - \mathcal{V}\mathcal{U}' = C_1 \cdot \mathrm{e}^{\ln|\mathcal{U}|},$$

therefore we get a linear inhomogeneous differential equation $\mathcal{V}' = \dfrac{\mathcal{U}'}{\mathcal{U}}\,\mathcal{V} + C_1$.

Firstly, we solve related homogeneous equation $\mathcal{V}' = \dfrac{\mathcal{U}'}{\mathcal{U}}\,\mathcal{V}$ and we get the solution $\mathcal{V} = C\cdot\mathcal{U}$, where C is a constant of integration.

Secondly, using the method of variation of parameters, we suppose that C is a function of the variable x^1 and then we obtain $C = \displaystyle\int \frac{C_1}{\mathcal{U}}\,\mathrm{d}x^1$ thus the other partial solution of (8) is

$$\mathcal{V} = C_1 \cdot \mathcal{U} \cdot \int \frac{1}{\mathcal{U}}\,\mathrm{d}x^1 + C_2,$$

where C_2 is a constant of integration. As above $C_1 = C_1(x^2)$ and $C_2 = C_2(x^2)$.

In conclusion, if the certain particular solution of the equation (8) is known it is possible to find the other particular solution, therefore, the general solution of this equation. From this follows that the vector field θ which satisfies the conditions (6) always exists. In general case, the Riemannian space $\mathbb{V}_2$ given by the metric in the form (3) is not a surface of revolution, therefore, the Theorem 1.1 from the Introduction is valid.

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