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Infinitesimal Rotary Transformation

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Abstract. The paper is devoted to further study of a certain type of infinitesimal transformations of two-dimensional (pseudo-) Riemannian spaces, which are called rotary. An infinitesimal transformation is called rotary if it maps any geodesic on (pseudo-) Riemannian space onto an isoperimetric extremal of rotation in their principal parts on (pseudo-) Riemannian space. We study basic equations of the infinitesimal rotary transformations in detail and obtain the simpler fundamental equations of these transformations.

1. Introduction

The paper concerns with a study of infinitesimal rotary transformations [8], for other exmaples of infinitesimal transformations see also [2, 12, 15–20]. The rotary diffeomorphism and rotary transformations of two-dimensional Riemannian spaces were first introduced by Leiko [4, 5, 7, 11]. He defined the term of the rotary diffeomorphism under which any geodesic is mapped onto isoperimetric extremal of rotation, and he obtained fundamental equation for this task, see also [9].

Results about the isoperimetric extremals of rotation have physical application, i.e. in the theory of gravitational fields, for example see [3, 6, 7, 10].

In this paper, we study above mentioned rotary transformations of two-dimensional (pseudo-) Riemannian spaces and obtain new fundamental equations in a simpler form.

2. Basic definition of infinitesimal rotary transformation

In this section, we are going to define the term of the infinitesimal rotary transformation of two-dimensional (pseudo-) Riemannian space V_2. The study will be based on well-known facts valid in n-dimensional (pseudo-) Riemannian spaces V_n.

Let us consider an n-dimensional (pseudo-) Riemannian space V_n, where the object of the Levi-Civita connection ∇ is given. We denote $x = (x^1, x^2, \dots, x^n)$ a coordinate system on the space V_n. Here and further we suppose that $n \geq 2$.

A curve ℓ in the space V_n given by the equations $x = x(t)$ is said to be *geodesic* if its tangent vector $\lambda \equiv dx(t)/dt$ is recurrent along it.

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A curve ℓ is a geodesic if and only if $\nabla_t \lambda = \rho(t)\lambda$, which can be rewritten into a coordinate form

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}(x(t))\lambda^\alpha\lambda^\beta = \rho(t)\lambda^h, \tag{1}$$

where Γ^h_{ij} are components of the connection ∇, and $\rho(t)$ is a function of parameter t.

Let us consider a smooth curve $\bar{\ell}(t)$ given by the equations $x^h = \bar{x}^h(t)$ on the two-dimensional (pseudo-) Riemannian space V_2 with the metric tensor g and the Gaussian curvature K of a constant sign. We fix the ends of this curve at the points $p_0 = \bar{\ell}(t_0)$ and $p_1 = \bar{\ell}(t_1)$ and consider following functionals of length and rotation

$$s[\bar{\ell}] = \int_{t_0}^{t_1} \sqrt{|\bar{\lambda}|}\, dt \quad \text{and} \quad \theta[\bar{\ell}] = \int_{t_0}^{t_l} k(t)\, dt;$$

where $k \geq 0$ is the Frenet curvature of the curve $\bar{\ell}$ and $\bar{\lambda}$ is a tangent vector of the curve $\bar{\ell}$. Then, extremals of variational problem $\theta[\bar{\ell}]$ and $s[\bar{\ell}] = \text{const}$ are called *isoperimetric extremals of rotation* (IER), see [12, p. 405]; for properly Riemannian spaces see [4–6, 9, 11].

If the $s[\ell]$ is the minimal distance, then the geodesic going through the points p_0 and p_1 is the unique solution of this problem. In this case, we talk about the trivial isoperimetric extremal of rotation.

A curve $\bar{\ell}$ is an isoperimetric extremal of rotation if and only if the following equation holds $\nabla_s \lambda = cK \cdot F\lambda$, and in the coordinate form

$$\frac{d\bar{\lambda}^h}{dt} + \Gamma^h_{\alpha\beta}(\bar{x})\bar{\lambda}^\alpha\bar{\lambda}^\beta = cK(\bar{x}) \cdot F^h_\alpha(\bar{x})\bar{\lambda}^\alpha, \tag{2}$$

where c is a constant, $K(\bar{x})$ is the Gaussian curvature, $\bar{\lambda}$ is a tangent vector, and $F(\bar{x})$ is an affinor, tensor field of type $\binom{1}{1}$, see [1, 14], which satisfies the following conditions

$$F^2 = -e \cdot Id, \quad g(X, FX) = 0, \quad \nabla F = 0.$$

In this case, parameter t is a length of the curve and $\bar{\lambda}$ is a unit tangent vector.

For Riemannian manifold V_2 is $e = +1$ and for pseudo-Riemannian manifold V_2 is $e = -1$. The tensor F is uniquely defined (with respect to the sign) as follows

$$F^h_j = g^{hi}\varepsilon_{ij}, \qquad \varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g^2_{12}|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An *infinitesimal transformation* of a (pseudo-) Riemannian space V_n is given with respect to the coordinates in this manner

$$\bar{x}^h = x^h + \varepsilon\, \xi^h(x), \tag{3}$$

where x^h are the coordinates of a certain point in V_n and $\bar{x}^h$ are the coordinates of its image under the infinitesimal transformation, ε is an infinitesimal parameter not depending on x^h, and ξ^h is a displacement vector, see [2, 12].

If a certain object $\mathcal{A}$ of the space V_n depends on $x \in V_n$ but also on the infinitesimal parameter ε, then the *principal part* of the object $\mathcal{A}$ is $\underset{0}{\mathcal{A}}(x) + \underset{1}{\mathcal{A}}(x)\varepsilon$ in the expansion of series with respect to the infinitesimal parameter ε

$$\mathcal{A}(x, \varepsilon) = \underset{0}{\mathcal{A}}(x) + \underset{1}{\mathcal{A}}(x)\varepsilon + \underset{2}{\mathcal{A}}(x)\varepsilon^2 + \dots.$$

For our purposes the curves obtained by the infinitesimal transformation of geodesics satisfy the equations of isoperimetric extremals of rotation (2) under the condition, that we dropped the terms containing higher powers of the infinitesimal parameter ε, i.e. $\varepsilon^2, \varepsilon^3, \dots$.

Definition 2.1. An infinitesimal transformation of the two-dimensional (pseudo-) Riemannian space V_2 is called *rotary* if it maps any geodesic of the space V_2 onto an isoperimetric extremal of rotation in their principal parts.

3. Basic equations of infinitesimal rotary transformations

We prove the following theorem.

Theorem 3.1. *A differential operator $X = \xi^\alpha(x)\partial_\alpha$ $(\partial_\alpha = \partial/\partial x^\alpha)$ determines an infinitesimal rotary transformation of (pseudo-) Riemannian space V_2 if and only if X satisfies*

$$L_\xi \Gamma^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h g_{ij}, \quad \theta^h_{,i} = \theta^h(\theta_i + K_i/K) + \nu\delta^h_i, \tag{4}$$

where ψ_i is a covector, δ^h_i is the Kronecker delta, θ^h is a vector field, g is a metric tensor, K is the Gaussian curvature, and L_ξ is the Lie derivative with respect to ξ.

Proof. Let us consider an infinitesimal rotary transformation of (pseudo-) Riemannian space V_2 determined by the equations (3). Furthermore, let ℓ be a geodesic of the space V_2 given by the equations $x^h = x^h(t)$. Further, let ℓ satisfy the equations (1). The curve $\bar{\ell}$ which corresponds to the curve ℓ under the infinitesimal rotary transformation (3) has the following equations

$$\bar{x}^h(t) = x^h(t) + \varepsilon\, \xi^h(x(t)). \tag{5}$$

The infinitesimal transformation (3) is rotary if $\bar{\ell}$ is an isoperimetric extremal of rotation in its principal parts. Therefore, the equations $\bar{x}(t)$ given by (5) satisfy in the principal part equations (2) which could be written like follows

$$\frac{d\bar{\lambda}^h(t)}{dt} + \Gamma^h_{\alpha\beta}(\bar{x}(t))\bar{\lambda}^\alpha(t)\bar{\lambda}^\beta(t) = cK(\bar{x}(t)) \cdot F^h_\alpha(\bar{x}(t))\bar{\lambda}^\alpha(t). \tag{6}$$

Next, we shall find the objects involved in the equations (6). The tangent vector $\bar{\lambda}^h(t)$ we receive after derivation of equations (5)

$$\bar{\lambda}^h(t) \equiv \frac{d\bar{x}^h(t)}{dt} = \frac{dx^h(t)}{dt} + \varepsilon\frac{\partial\xi^h(x(t))}{\partial x^\gamma}\frac{dx^\gamma(t)}{dt} = \lambda^h(t) + \varepsilon\lambda^\gamma(t)\partial_\gamma\xi^h(x(t)).$$

Also, for the connection Γ and the structure F we get

$$\Gamma^h_{\alpha\beta}(\bar{x}) = \Gamma^h_{\alpha\beta} + \varepsilon\frac{\partial\Gamma^h_{\alpha\beta}}{\partial x^\gamma}\xi^\gamma + \boxed{\varepsilon^2} \quad \text{and} \quad F^h_\alpha(\bar{x}) = F^h_\alpha + \varepsilon\frac{\partial F^h_\alpha}{\partial x^\gamma}\xi^\gamma + \boxed{\varepsilon^2}.$$

Furthermore, for the Gaussian curvature K it holds

$$K(\bar{x}) = K + \varepsilon\frac{\partial K}{\partial x^\gamma}\xi^\gamma + \boxed{\varepsilon^2},$$

And finally, we expand the function ρ and the constant c like follows

$$\rho(t) = \rho_0(t) + \varepsilon\rho_1(t) + \boxed{\varepsilon^2}, \quad \text{and} \quad c = c_0 + \varepsilon c_1 + \boxed{\varepsilon^2}.$$

Let us remind, that here and after $\boxed{\varepsilon^2}$ stands for the terms containing higher powers of the infinitesimal parameter ε, which will be dropped later.

Now we substitute above mentioned expressions into the equation (6) and we get

$$\begin{aligned}
&\tfrac{d\lambda^h}{dt} + \varepsilon\left(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta + \tfrac{d\lambda^\alpha}{dt}\partial_\alpha\xi^h\right) + \\
&+(\Gamma^h_{\alpha\beta} + \varepsilon\partial_\gamma\Gamma^h_{\alpha\beta}\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta) = \\
&= (c_0 + \varepsilon c_1 + \boxed{\varepsilon^2})(F^h_\alpha + \varepsilon\partial_\gamma F^h_\alpha\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(K + \varepsilon\partial_\gamma K\xi^\gamma + \boxed{\varepsilon^2}).
\end{aligned}$$

Since we know that the curve ℓ is a geodesic, we use equations (1) to eliminate $\frac{d\lambda^h}{dt}$ from the expression above

$$\begin{aligned} &-\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \lambda^h(\rho_0 + \varepsilon\rho_1 + \boxed{\varepsilon^2}) + \\ &+\varepsilon\Big(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta + \partial_\alpha\xi^h\big(- \Gamma^\alpha_{\beta\gamma}\lambda^\beta\lambda^\gamma + \lambda^\alpha(\rho_0 + \varepsilon\rho_1 + \boxed{\varepsilon^2})\big)\Big) + \\ &+(\Gamma^h_{\alpha\beta} + \varepsilon\partial_\gamma\Gamma^h_{\alpha\beta}\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta) = \\ &= (c_0 + \varepsilon c_1 + \boxed{\varepsilon^2})(F^h_\alpha + \varepsilon\partial_\gamma F^h_\alpha\xi^\gamma + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(K + \varepsilon\partial_\gamma K\xi^\gamma + \boxed{\varepsilon^2}). \end{aligned}$$

The constant term (not depending on ε) and the linear term (with respect to ε) from above mentioned equation vanishes, in which case we receive two following equations, the first one

$$\rho_0\lambda^h = c_0F^h_\alpha\lambda^\alpha K, \tag{7}$$

and the second one

$$\begin{aligned} &\lambda^\alpha\lambda^\beta(\partial_{\alpha\beta}\xi^h - \Gamma^\gamma_{\beta\alpha}\partial_\gamma\xi^h + \Gamma^h_{\alpha\gamma}\partial_\beta\xi^\gamma + \Gamma^h_{\gamma\beta}\partial_\alpha\xi^\gamma + \partial_\gamma\Gamma^h_{\alpha\beta}\xi^\gamma) + \rho_1\lambda^h + \rho_0\lambda^\alpha = \\ &\qquad = c_0(KF^h_\alpha\lambda^\gamma\partial_\gamma\xi^\alpha + K\lambda^\alpha\partial_\gamma F^h_\alpha\xi^\gamma + \lambda^\alpha F^h_\alpha\partial_\gamma K\xi^\gamma) + c_1KF^h_\alpha\lambda^\alpha. \end{aligned} \tag{8}$$

From the equation (7) follows that $\rho_0 = c_0 = 0$, which can be substituted to the second equation (8). Furthermore, after using the definition of Lie derivative we obtain following relation

$$L_\xi\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = -\rho_1\lambda^h + c_1KF^h_\alpha\lambda^\alpha \tag{9}$$

These equations hold true for any point and any unit vector λ^h. Analogically as described in [1, 14] from the above mentioned we obtain the equations

$$L_\xi\Gamma^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h g_{ij}, \tag{10}$$

where ψ_i is a covector and θ^h is a vector.

Similarly as in the papers [1, 14] we substitute the equations (10) in (9) and we get

$$(\delta^h_i\psi_j + \delta^h_j\psi_i + \theta^h g_{ij})\lambda^i\lambda^j = -\rho_1\lambda^h + c_1KF^h_i\lambda^i. \tag{11}$$

After contracting formula (11) with $g_{h\alpha}\lambda^\alpha$ we obtain $\theta_\alpha\lambda^\alpha = -(\rho_1 + 2\psi_\alpha\lambda^\alpha)$, where $\theta_i = g_{i\alpha}\theta^\alpha$. Therefore formula (11) has the form

$$\eta\theta^h = \theta_\alpha\lambda^\alpha\lambda^h + c_1K\cdot F^h_\alpha\lambda^\alpha, \tag{12}$$

where $\eta = g_{ij}\lambda^i\lambda^j = \pm 1$. After differentiating (12) along the curve ℓ and after detailed analysis of degrees of λ^h in such equation, we get

$$\nabla\theta^h = \theta^h(\theta_i + \nabla K/K) + \nu\delta^h_i, \tag{13}$$

where ν is a function on the space $\mathbb{V}_2$, therefore the theorem is proved. □

As it can be seen, the equations (4) have simpler form than the equations of rotary trasnformations deduced by Leiko in [4]. In Leiko's work [4] it is stated that the vector field θ which satisfies equations (13) exist only on the surfaces of revolution. This statement is not valid and we have constructed a contra example in [13].

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