



Cosmology of Non-local $f(R)$ Gravity

Ivan Dimitrijević^a, Branko Dragovich^{b,c}, Alexey S. Koshelev^d, Zoran Rakić^a, Jelena Stanković^e

^aFaculty of Mathematics, University of Belgrade, Studentski trg 16, 11000, Belgrade, Serbia

^bInstitute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

^cMathematical Institute of Serbian Academy of Sciences and Arts, Kneza Mihaila 36, Belgrade, Serbia

^dDepartamento de Física and Centro de Matemática e Aplicações, Universidade da Beira Interior, 6200 Covilhã, Portugal

^eTeacher Education Faculty, University of Belgrade, Kraljice Natalije 43, Belgrade, Serbia

Abstract. We consider a modification of GR with a special type of a non-local $f(R)$. The structure of the non-local operators is motivated by the string field theory and p -adic string theory. The spectrum is derived explicitly and the ghost-free condition for the model is formulated. We pay special attention to the classical stability of the de Sitter solution in our model and formulate the conditions on the model parameters to have a stable configuration. Relevance of unstable configurations for the description of the de Sitter phase during inflation is specifically discussed.

1. Introduction

In 2015, the gravity community celebrated the first century of the General Relativity (GR), which is viewed as one of the most beautiful and profound physical theories [1]. GR is still an acting theory of gravity usually presented by Einstein's equation of motion for the gravitational (metric) field $g_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor of matter, G is the Newtonian constant and the speed of light is taken $c = 1$. This Einstein's equation can be derived from the Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R d^4x + \int \sqrt{-g} \mathcal{L}_m d^4x, \quad (2)$$

where $g = \det(g_{\mu\nu})$ and \mathcal{L}_m is the Lagrangian of matter. In this paper we use $(- + + +)$ metric signature and stick to 4 dimensions.

2010 *Mathematics Subject Classification.* Primary 83Dxx, 83Fxx; Secondary 83C15

Keywords. nonlocal modified gravity, bounce cosmological solutions, cosmological perturbations, no-ghost conditions

Received: 30 September 2018; Accepted: 01 March 2019

Communicated by Ljubica S. Velimirović

This paper was supported in part by Ministry of Education, Science and Technological Development of the Republic of Serbia, grant No 174012. AK is supported FCT Portugal investigator project IF/01607/2015, by the grant UID/MAT/00212/2013 and COST Action CA15117 (CANTATA).

Email addresses: ivand@matf.bg.ac.rs (Ivan Dimitrijević), dragovich@ipb.ac.rs (Branko Dragovich), alexey@ubi.pt (Alexey S. Koshelev), zrakic@matf.bg.ac.rs (Zoran Rakić), jelenagg@gmail.com (Jelena Stanković)

GR has been well tested and confirmed in the Solar system, and it serves as a theoretical laboratory for gravitational investigations at other spacetime scales. It has important astrophysical implications predicting existence of black holes, gravitational lensing and gravitational waves. In cosmology, GR predicts existence of about 95% of additional matter, which makes dark side of the Universe. Namely, if GR is the gravity theory for the Universe as a whole and if the Universe has the Friedmann-Lemaître-Robertson-Walker (FLRW) metric (which is homogeneous and isotropic) at the cosmic scale, then there is about 68% of *dark energy*, 27% of *dark matter*, and only about 5% of *visible matter* in the Universe [2].

Despite remarkable phenomenological achievements and many nice theoretical properties, GR is not a complete theory of gravity. It has well known long standing problems both in UV and IR regimes. In UV or at short distances GR predicts singularities like the Big Bang or black hole ones. In particular, under rather general conditions, GR contains cosmological solutions which lead to an infinite matter density at the beginning of the Universe [3]. When physical theory contains singularity, it is an evident indication that around it such a theory has to be appropriately modified. In IR pure GR does not provide a decent explanation of the Dark Energy phenomenon since a possible cosmological term has an unnatural and unexplainable tiny value. From just a theoretical point of view GR is not a renormalizable theory even if being quantized. The ways to modify GR usually come from some more general theories like quantum gravity, string theory as well as astrophysical and cosmological observations (for a review, see [4]). Unfortunately there is no so far solid fundamental physical principle which could tell us how to find appropriate modification between infinitely many possible theoretical constructions.

In the present paper we consider an analytic non-local modification of GR. This type of non-locality has strong motivations from string field theory (SFT) [5] and p -adic string theory [6]. The term “analytic non-locality” signifies that the theory contains non-local infinite derivative operators in the form of analytic functions of covariant derivatives. In particular SFT promotes analytic functions of the d’Alembert operator \square .¹ Such a gravity modification was initially introduced in [9] and is intensively studied recently [10–17]. One of the most interesting feature of analytic non-local gravities is the presence of a non-singular ghost-free bounce. The final stage of known bouncing solutions is a de Sitter expansion. The developed models contain a non-local term of the form $R\mathcal{F}(\square)R$ and it was shown already in [11] that the late de Sitter phase after the bounce is stable. In the present paper after highlighting the general ideas regarding this modified theory we also mainly focus on the properties of a de Sitter (dS) solution. We however get as the starting point a more general non-local term of the form

$$P(R)\mathcal{F}(\square)Q(R).$$

The motivation for considering a more general non-local action is two-fold.

- First, it follows from the above cited papers that the dependence on a non-local analog of R^2 covers all possible Lagrangians with respect to the dynamics around the dS space-time as long as those starting Lagrangians depend analytically on R , not only on derivatives. In our case we allow any, including non-analytic functions P, Q .
- Second, it is important to find out whether a pure dS background can be unstable in a non-local model. This will indicate a possibility to join bounce and inflation in one setup. Without such a joint behavior just an inflationary solution, namely the Starobinsky inflation [18], was shown to successfully provide an inflationary background [14, 19] in such non-local model of gravity with $P = Q = R$. Notice, that the Starobinsky solution is not preceded by a bounce. Also it is essential that a de Sitter phase of the inflation must be unstable. This guarantees the exit from inflation without which a matter creation is impossible.

Let us note here the results of [20] in connection with the first point above. That paper is devoted to demonstrating an equivalence of quite a very general set of gravitational actions to just one action which

¹That is, there is no a direct relation to another class of non-local gravity models based on the inverse of the d’Alembert operator [7] or mixed models with positive and negative powers of the d’Alembertian like in [8].

containing exactly $P = Q = R$. We however emphasize that paper [20] considers the equivalence w.r.t. quadratic variation of the action and the corresponding spectrum. It does not touch in details the question of stability of classical perturbations and moreover those results do not account 3 and higher point correlations. Thus apart from an attempt to seek for an unstable de Sitter background the more and more active developments in measuring higher order correlations [21] will bring new constraints on models beyond the quadratic variation. An explicit computations of higher correlations is beyond the scope of the present paper but the formalism developed here is crucial in computations of those higher correlation functions. Therefore an altering of the initially proposed model may be required.

Before proceeding to the main part of our paper we would like to say that the non-local gravity models with analytic non-locality provide a modified Newtonian potential smoothing the singular limit at the origin. More specifically, the potential has a universal behavior which is a constant limit at zero distance for a very wide class of non-local functions entering the action while the standard $1/r$ falloff is naturally restored at large distances. Moreover various cosmologically interesting bounce solutions were build and thoroughly analyzed. At the perturbative level an ability of such models to accommodate inflationary scenarios and in particular the Starobinsky inflation [18, 19, 22, 23] was proven successful. In particular, an embedding of the Starobinsky model in the non-local gravity leads to in principle testable modifications of the observable parameters, such that ration of the tensor and scalar power spectra r . At the quantum level the non-locally modified gravity is shown to be renormalizable by the power-counting while the unitarity is preserved by construction. The latter means that there exist explicitly formulated conditions on non-local functions of the d'Alembert operator such that the spectrum of physical excitations is ghost-free moreover alongside with the renormalizability [24, 25] during the quantization of the model.

The present paper is structured as follows. In Section 2 we present our model and derive equations of motion. In Section 3 we develop some general ideas of solution construction and discuss the most promising of them. In Section 4 we turn to the de Sitter solution and formulate the stability condition for linear perturbations. In Section 5 we derive explicitly the ghost-free condition for our model around a de Sitter background. In Section 6 we explore the stability condition for dS space-times in greater details and draw concluding remarks in Section 7.

2. Non-local generalization of GR and equations of motion

We consider the following nonlocal gravity action

$$S = \int d^4x \sqrt{-g} \left(\frac{M_p^2}{2} R - \Lambda + \frac{\lambda}{2} P(R) \mathcal{F}(\square) Q(R) \right), \quad (3)$$

where R is the scalar curvature, Λ is the cosmological constant, $\mathcal{F}(\square) = \sum_{n=0}^{\infty} f_n \square^n$ is an analytic function of the d'Alembert operator $\square = \nabla^\mu \nabla_\mu$ where ∇_μ is the covariant derivative. The Planck mass M_p is related to the Newtonian constant G as $M_p^2 = \frac{1}{8\pi G}$ and P, Q are scalar functions of the scalar curvature. λ is a constant and in principle can be absorbed in the rescaling of $\mathcal{F}(\square)$. However, it is a convenient tool to track the GR limit which is $\lambda \rightarrow 0$. As it is obvious we are going down the way of generalizing the results obtained earlier in the case $P = Q = R$ [?]. In the sequel we shall omit an explicit further citation of these results referring them rather as the *non-local* R^2 case.

To have physically meaningful expressions and to keep track of the non-localities one should introduce the scale of non-locality using a new mass parameter \mathcal{M} . Then the function \mathcal{F} would be expanded in Taylor series as $\mathcal{F}(\square) = \sum_{n=0}^{\infty} f_{\mathcal{M}n} \square^n / \mathcal{M}^{2n}$ with all constants $f_{\mathcal{M}n}$ dimensionless. We will return to these notations during the discussion of our results.

Varying the action (3) with respect to the metric we get the following equations of motion

$$\begin{aligned}
 -\tilde{G}_{\mu\nu} \equiv & -M_p^2 G_{\mu\nu} - g_{\mu\nu} \Lambda \\
 & + \frac{\lambda}{2} g_{\mu\nu} P \mathcal{F}(\square) Q - \lambda (R_{\mu\nu} - K_{\mu\nu}) V \\
 & + \frac{\lambda}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(P_{\mu}^{(l)} Q_{\nu}^{(n-l-1)} + P_{\nu}^{(l)} Q_{\mu}^{(n-l-1)} - g_{\mu\nu} (g^{\rho\sigma} P_{\rho}^{(l)} Q_{\sigma}^{(n-l-1)} + P^{(l)} Q^{(n-l)}) \right) = 0.
 \end{aligned}
 \tag{4}$$

Here $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $K_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \square$, $V = P_R \mathcal{F}(\square) Q + Q_R \mathcal{F}(\square) P$ where the subscript R indicates the derivative w.r.t. R (as many times as it is repeated) and

$$P^{(l)} = \square^l P, P_{\rho}^{(l)} = \partial_{\rho} \square^l P \text{ with the same for } Q, P_R, \dots$$

Provided there is a matter source as well the full equations of motion would contain $T_{\mu\nu}$ in the right hand side such that

$$\tilde{G}_{\mu\nu} = T_{\mu\nu} \tag{5}$$

Analyzing (4) we recognize that the first line is the canonical EOM for the Einstein’s GR with the cosmological constant, the second line with $\mathcal{F}(\square) = 1$ represents the extension to the local $f(R)$ type gravities while a non-constant $\mathcal{F}(\square)$ as well as the last line are unique for a higher derivative (probably non-local) modification of gravity. The trace equation is of use and we write it separately

$$M_p^2 R - 4\Lambda + 2\lambda P \mathcal{F}(\square) Q - \lambda (R + 3\square) V - \lambda \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(g^{\rho\sigma} P_{\rho}^{(l)} Q_{\sigma}^{(n-l-1)} + 2P^{(l)} Q^{(n-l)} \right) = -T_{\mu}^{\mu}. \tag{6}$$

If either P or Q is a constant in the action then effectively $\mathcal{F}(\square)$ trivializes to f_0 and we recover a local $f(R)$ gravity theory. Note that thanks to the integration by parts there is always the symmetry of an exchange $P \leftrightarrow Q$.

3. Solutions construction

3.1. Cosmological FRW solutions

In this paper the primary goal is to attack the cosmological properties of the proposed gravity model. This leads us to the cosmologically important metrics from which we focus on the Friedmann-Robertson-Walker (FRW) configurations. The latter have the following metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right), \tag{7}$$

where t is the cosmic time, $a(t)$ is the scale factor and K is the spatial curvature. Such metrics represent a homogeneous and isotropic Universe. The corresponding tensor $\tilde{G}_{\mu\nu}$ in (4) turns out to be diagonal with two distinct components: (00) and one of (ii) for some spatial index $i = 1, 2, 3$. It seems even simpler to work with the (00) and the trace equations. Hereafter we consider only the spatially-flat case $K = 0$ unless explicitly stated otherwise.

The above chosen form of the metric implies that only the matter sources of the form $T_{\nu}^{\mu} = \text{diag}(-\rho, p, p, p)$ are compatible with the equations of motion (5). Here ρ is the energy density and p is the pressure density. Such matter sources represent a wide class of physically important cases and include the most crucial set of perfect fluids which can be written as $T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu}$ where u_{μ} is the 4-velocity of the fluid.

It is common that gravity is a theory with constraints thanks to Bianchi identities. These identities manifest in the relation $\nabla^\mu G_{\mu\nu} \equiv 0$. However, one can check that given a diffeomorphism invariant action like as it is for (3) one ends up with a similar relation

$$\nabla^\mu \tilde{G}_{\mu\nu} \equiv 0.$$

This implies like in GR the canonical conservation equation for the matter

$$\nabla^\mu T_{\mu\nu} = 0.$$

In the sequel we will focus on special configurations which have the vanishing trace of Einstein equations. This can be either a vacuum such that $T_{\mu\nu} = 0$ or radiation in case of a perfect fluid. Indeed, radiation is characterized by $w = p/\rho = 1/3$ and is therefore traceless. In either situation solving the trace equation is almost enough. Namely, given that the trace is zero and there is a solution to the trace equation, and moreover accounting the FRW form of the metric we are left with just one equation, which we have chosen to be the (00). However, thanks to the Bianchi identity and the matter conservation this remaining equation can always be satisfied by adjusting the amount of radiation energy density. To be physical, the radiation energy density must be positive.

3.2. Non-local R^2 case

It is clear from the complexity of expressions (4) that constructing a general solution is a very ambitious hope. However we remind that a significant progress has been achieved in the non-local R^2 case by considering a simplifying ansatz

$$\square R = r_1 R + r_2, \tag{8}$$

with $r_{1,2}$ being constants. It is useful to repeat this procedure here in order to illuminate the steps related to our model of interest (3).

Consider $P = Q = R$, the case originally discussed in [9]. Then equations (5) become

$$T_{\mu\nu} = -M_p^2 G_{\mu\nu} - g_{\mu\nu} \Lambda + \frac{\lambda}{2} g_{\mu\nu} R \mathcal{F}(\square) R - 2\lambda (R_{\mu\nu} - K_{\mu\nu}) \mathcal{F}(\square) R + \frac{\lambda}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (R_\mu^{(l)} R_\nu^{(n-l-1)} + R_\nu^{(l)} R_\mu^{(n-l-1)} - g_{\mu\nu} (g^{\rho\sigma} R_\rho^{(l)} R_\sigma^{(n-l-1)} + R^{(l)} R^{(n-l)}). \tag{9}$$

Application of ansatz (8) means

$$\square^n R = r_1^n R + r_2 r_1^{n-1}, \quad n > 0 \text{ and } \mathcal{F}(\square) R = \mathcal{F}_1 R + \mathcal{F}_2, \tag{10}$$

where $\mathcal{F}_1 = \mathcal{F}(r_1)$ and $\mathcal{F}_2 = \frac{r_2}{r_1} (\mathcal{F}(r_1) - f_0)$.

As explained in the previous Subsection we will proceed with the trace equation. For the traceless matter (or for no matter at all) it becomes

$$A_1 R - \lambda \mathcal{F}^{(1)}(r_1) (2r_1 R^2 + \partial_\mu R \partial^\mu R) + A_2 = 0, \tag{11}$$

with

$$A_1 = M_p^2 - \lambda \left(4\mathcal{F}'(r_1) r_2 - 2 \frac{r_2^2}{r_1} (\mathcal{F}_1 - f_0) + 6\mathcal{F}_1 r_1 \right),$$

$$A_2 = -4\Lambda - \lambda \frac{r_2}{r_1} \left(2\mathcal{F}'(r_1) r_2 - 2 \frac{r_2^2}{r_1} (\mathcal{F}_1 - f_0) + 6\mathcal{F}_1 r_1 \right).$$

The above equation is satisfied provided $A_1 = A_2 = 0$, and $\mathcal{F}^{(1)}(r_1) = 0$. Here $\mathcal{F}^{(1)}(r_1)$ is the first derivative w.r.t. the argument evaluated at point r_1 . Simple algebra gives

$$r_2 = - \frac{r_1 [M_p^2 - 6\lambda \mathcal{F}_1 r_1]}{2\lambda [\mathcal{F}_1 - f_0]}, \quad \Lambda = - \frac{r_2 M_p^2}{4r_1}. \tag{12}$$

Upon substitution of these relations in the 00-equation one is left with the problem of positivity of a possible radiation energy density as it is explained in the previous Subsection. The sign of f_0 would control the sign of the radiation energy density.

3.3. Ansatz for equations of motion

Now we turn to our more generic model. To advance in the understanding of the structure of solutions we propose the following ansatz.

$$P = Q \text{ and } \square P = p_1 P, \tag{13}$$

where p_1 is a constant. Notice that for $P = R + \text{const}$ ansatz (8) is restored. Conditions (13) provide dramatic simplifications. Indeed, application of this ansatz yields

$$\square^n P = p_1^n P, \quad n > 0 \text{ and } \mathcal{F}(\square)P = \mathcal{F}_{p_1} P, \tag{14}$$

where $\mathcal{F}_{p_1} = \mathcal{F}(p_1)$. The further explicit substitution in the trace equation (6) yields (with $T_\mu^\mu = 0$)

$$M_p^2 R - 4\Lambda + 2\lambda \mathcal{F}_{p_1} P^2 - 2\lambda(R + 3\square)P_R \mathcal{F}_{p_1} P - \lambda \mathcal{F}^{(1)}(p_1)(g^{\rho\sigma} P_\rho P_\sigma + 2p_1 P^2) = 0. \tag{15}$$

Analogously to the non-local R^2 case a condition $\mathcal{F}^{(1)}(p_1) = 0$ simplifies the clutter a lot and we are left just with the following equation

$$M_p^2 R - 4\Lambda + 2\lambda \mathcal{F}_{p_1} (P^2 - (R + 3\square)(P_R P)) = 0. \tag{16}$$

This equation is nothing but the trace equation of a local $f(R)$ -type gravity with the following action

$$S = \int d^4x \sqrt{-g} \left(\frac{M_p^2}{2} R - \Lambda + \frac{\lambda}{2} \mathcal{F}_{p_1} P^2(R) \right). \tag{17}$$

As follows from general considerations in Subsection 3.1 the remaining (00) equation evaluated on a solution to the trace equation can be satisfied by adjusting the radiation energy density. There are still enough parameters to control its sign.

Thus, we can draw the following conclusion: any solution of a local gravity of type (17) is a solution of our non-local theory as long as an additional condition (13) is satisfied upon an appropriate adjustment of parameters in the non-local model. In particular, we must require $\mathcal{F}^{(1)}(p_1) = 0$. We note that for $P = R$ the trace equation coming from (17) and ansatz relation (13) match upon adjustment of constant parameters as was shown in [13]. In a more general case the ansatz relation (13) is indeed an extra condition and must be analyzed separately.

Then one can see that a specially simple case arises if $P = \sqrt{R + R_0}$. For such a choice of function P one gets $P_R \sim 1/P$ and as the result the latter equation can be solved by adjusting the parameters. However, there is still a non-trivial equation to be satisfied

$$\square \sqrt{R + R_0} = p_1 \sqrt{R + R_0}. \tag{18}$$

Nevertheless, presence of at least some solutions has been already preliminary shown in [26]. Interestingly, such a choice of P has no known local counterpart as it would give just a canonical Einstein-Hilbert term with a cosmological constant.

3.4. Constant curvature solutions

This is a special class of solutions which include several important cases.

Substituting $R = \text{const}$ into the trace equation (6) and using the fact that P, Q, V are now constants one gets

$$M_p^2 R - 4\Lambda + 2\lambda P f_0 Q - \lambda R V = 0. \tag{19}$$

Here V reduces to $(P_R Q + Q_R P)f_0$. Solving the latter equation is an algebraic rather than differential problem.

The very important case is the de Sitter solution as far as it plays a crucial role in the description of our Universe. In four dimensions it is characterized by

$$R_{\mu\nu} = \frac{R}{4}g_{\mu\nu} \text{ and } R = \text{const} > 0. \tag{20}$$

Notice that just a constant R does not mean the space-time is de Sitter, however. One can check that dS space-time is a vacuum solution of our model, i.e. no matter is needed to support it.

3.5. Exact analytic bounce

The following scale factor $a(t)$ in a spatially flat FLRW Universe results in an exact solution to full equations (4)

$$a = a_0 \sqrt{\cosh(\sigma t)}, \tag{21}$$

At first one can find that this solution corresponds to $R = 6\dot{H} + 12H^2 = 3\sigma^2 = \text{const}$ where as usual $H = \dot{a}/a$ and dot is the derivative w.r.t. the cosmic time t . This effectively means that all the most non-trivial third line of equation (4) vanish. As a consequence the trace of Einstein equations takes the form of (19) under an assumption that the matter is traceless. This equation in principle can be solved, at least by numeric methods. Recall that for a constant R this is an algebraic equation.

The remaining equation is the 00-component of system (4). Using that $R_{00} = -3\dot{H} - 3H^2$ and computing explicitly H this equation reads

$$-M_p^2 \frac{R}{4} (1 - 1/\cosh(\sigma t)^2) + \Lambda - \frac{\lambda}{2} P f_0 Q + \lambda V \frac{R}{4} (1 + 1/\cosh(\sigma t)^2) = T_{00}. \tag{22}$$

One immediately sees that all constant terms vanish thanks to equation (19) and one is just left with

$$\frac{R}{4} (M_p^2 + \lambda V) \frac{1}{\cosh(\sigma t)^2} = T_{00}. \tag{23}$$

As expected, this is the proper radiation energy density as it is proportional to a^{-4} . For a healthy model one must guarantee that the energy density is positive.

4. Cosmological expansion

4.1. Background

In this section we look for the cosmological spatially flat de Sitter which can be written as

$$ds^2 = -dt^2 + a_0^2 e^{2Ht} d\vec{x}^2, \tag{24}$$

with a constant H , t the cosmic time and the vector is the 3-dimensional notion. This is a particular case of a spatially flat FRW metric

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2, \tag{25}$$

with $a(t) = a_0 \exp(Ht)$. The general definition of H is $H = \dot{a}/a$ with dot being the derivative w.r.t. the cosmic time.

Some relevant background quantities (for a general a) are

$$\begin{aligned} R &= 12H^2 + 6\dot{H}, \quad \Gamma_{ij}^0 = Hg_{ij}, \quad \Gamma_{j0}^i = H\delta_j^i, \\ \square &= -\partial_t^2 - 3H\partial_t + \frac{\delta^{ij}\partial_i\partial_j}{a^2}, \end{aligned}$$

where the indexes i, j range as 1, 2, 3. On the background all quantities are space homogeneous as the metric suggests.

For perturbations in many instances we employ the conformal time τ such that

$$ad\tau = dt.$$

Then the general FRW metric (25) transforms to

$$ds^2 = a(\tau)^2(-d\tau^2 + d\vec{x}^2). \tag{26}$$

For the de Sitter background (24)

$$\tau = -\frac{1}{a_0 H} e^{-Ht} \Rightarrow a(\tau) = -\frac{1}{H\tau}.$$

So when t goes from past to future infinity, τ goes from $-\infty$ to 0_- . $t = 0$ corresponds to $\tau = -\frac{1}{a_0 H}$.

4.2. Covariant perturbations

Perturbations of equations (4) to the linear order around the dS vacuum are easy to compute since many terms drop out. The variation of the metric is as usual

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{27}$$

Hereafter bars denote the background quantities. What remains after a careful computation is

$$-m^2 \delta G_v^\mu + (\bar{R}_v^\mu - \bar{K}_v^\mu) v(\bar{\square}) \delta R = 0, \tag{28}$$

where $m^2 = M_p^2 + \lambda f_0 (\bar{P}_R \bar{Q} + \bar{Q}_{RR} \bar{P}) > 0$ and $v(\bar{\square}) = -\lambda (\bar{P}_{RR} \bar{Q} + \bar{Q}_{RR} \bar{P}) f_0 - 2\lambda \bar{P}_R \bar{Q}_R \mathcal{F}(\bar{\square})$. We have used the fact that variation of the \square acting on a scalar function is a pure differential operator and all background curvatures are constants. Indeed, varying the box we have

$$(\delta \square) f = [-h^{\mu\nu} (\partial_\mu \partial_\nu - \bar{\Gamma}_{\mu\nu}^\rho \partial_\rho) - \bar{g}^{\mu\nu} \gamma_{\mu\nu}^\rho \partial_\rho] f. \tag{29}$$

Here $\Gamma_{\mu\nu}^\rho$ is the Christoffel symbol. Also we have used that starting from (27) one gets

$$\delta g^{\mu\nu} = -h^{\mu\nu}, \quad \delta \Gamma_{\mu\nu}^\rho = \gamma_{\mu\nu}^\rho = \frac{1}{2} (\bar{\nabla}_\mu h_\nu^\rho + \bar{\nabla}_\nu h_\mu^\rho - \bar{g}^{\rho\sigma} \bar{\nabla}_\sigma h_{\mu\nu}). \tag{30}$$

So if f is a constant then $(\delta \square) f = 0$. The same is true for K_v^μ

$$(\delta K_v^\mu) f = [-h^{\mu\sigma} (\partial_\sigma \partial_\nu - \bar{\Gamma}_{\sigma\nu}^\rho \partial_\rho) - \bar{g}^{\mu\sigma} \gamma_{\sigma\nu}^\rho \partial_\rho - \delta_v^\mu \delta \square] f, \tag{31}$$

which is zero as long as f is a constant. Hence all the corresponding terms vanish.

Taking the trace of (28) we get

$$\begin{aligned} \mathcal{G}(\bar{\square}) \delta R &= [m^2 + (\bar{R} + 3\bar{\square}) v(\bar{\square})] \delta R \\ &= [m^2 - \lambda (\bar{R} + 3\bar{\square}) ((\bar{P}_{RR} \bar{Q} + \bar{Q}_{RR} \bar{P}) f_0 - 2\bar{P}_R \bar{Q}_R \mathcal{F}(\bar{\square}))] \delta R = 0. \end{aligned} \tag{32}$$

It is a homogeneous equation on δR . The general method of solving it is to use the Weierstrass factorization

$$\mathcal{G}(\bar{\square}) \delta R = - \prod_i (\bar{\square} - \omega_i^2) e^{\gamma(\bar{\square})} \delta R = 0, \tag{33}$$

where ω_i^2 are the roots of the algebraic (or perhaps transcendental) equation $\mathcal{G}(\omega^2) = 0$ and $\gamma(\bar{\square})$ is an entire function (as a consequence $e^{\gamma(\omega^2)}$ has no roots). We assume that there are no multiple roots. Such roots complicate the story slightly but still can be treated analogously [27]. The overall minus in the LHS of the

latter equation has no a particular meaning at this stage. However, it will precisely match to the minus sign upon derivation of the ghost-free condition for scalar modes below in Section 5. Then we can solve (33) for each ω_i separately

$$(\bar{\square} - \omega_i^2)\delta R = 0. \tag{34}$$

The latter is just a second order linear differential equation. It can be written explicitly as

$$\left(\partial_\tau^2 - \frac{2}{\tau} \partial_\tau + k^2 + \frac{\omega_i^2}{H^2 \tau^2} \right) \delta R = 0, \tag{35}$$

where we have taken the dS form of the background. The solution yields

$$\delta R_i = (-k\tau)^{3/2} (C_{1i} J_{\nu_i}(-k\tau) + C_{2i} Y_{\nu_i}(-k\tau)), \tag{36}$$

where J, Y are the Bessel functions of the first and second kinds, respectively, with $\nu_i = \sqrt{\frac{9}{4} - \frac{\omega_i^2}{H^2}}$ and $C_{1i,2i}$ are the integration constants.

For small values of τ which correspond to large cosmic times t the Bessel functions have the following asymptotic behavior

$$\begin{aligned} J_\nu(z) &\sim z^{\text{Re } \nu}, \\ Y_\nu(z) &\sim z^{-|\text{Re } \nu|} \text{ for } \text{Re } \nu \neq 0, \\ Y_\nu(z) &\sim \ln z \text{ for } \text{Re } \nu = 0. \end{aligned}$$

From this we conclude that δR_i are bounded irrespectively of the boundary conditions provided

$$|\text{Re } \nu_i| < \frac{3}{2}. \tag{37}$$

The full answer for δR is

$$\delta R = \sum_i \delta R_i. \tag{38}$$

where each δR_i has its arbitrary integration constants. The only thing we care about is that the total δR must be real. Note that $\delta R = 0$ is an always existing and not necessarily trivial configuration.

4.3. Scalar perturbations

We are focusing on scalar classical perturbations only since the behavior of vector and tensor classical perturbations remain the same as in GR. The corresponding equations are just scaled by a constant factor thus not affecting the dynamics at all. This is because the introduced gravity modification does not awake neither scalars nor vectors. This will become even more apparent in the next Section where the spin-2 and spin-0 modes quadratic actions around dS will be computed explicitly.

The metric for the scalar perturbations around a FRW background is defined as

$$ds^2 = a(\tau)^2 \left[-(1 + 2\phi)d\tau^2 - 2\partial_i \beta d\tau dx^i + ((1 - 2\psi)\delta_{ij} + 2\partial_i \partial_j \gamma) dx^i dx^j \right]. \tag{39}$$

Here we employ the standard ADM decomposition and the notions of spin are w.r.t. the spatial part of the metric. From 4 scalar modes only 2 are gauge invariant. The convenient gauge invariant variables (Bardeen potentials) are introduced as

$$\Phi = \phi - \frac{1}{a}(a\vartheta)' = \phi - \dot{\chi}, \quad \Psi = \psi + \mathcal{H}\vartheta = \psi + H\chi, \tag{40}$$

where $\chi = a\beta + a^2\dot{\gamma}$, $\vartheta = \beta + \gamma'$, $\mathcal{H}(\tau) = a'/a$. The prime denotes the differentiation with respect to the conformal time τ and the dot as before w.r.t. the cosmic time t .

The (1 + 3) structure suggests to represent the perturbation quantities (which can depend on all 4 coordinates) as

$$f(\tau, \vec{x}) = f(\tau, k)Y(k, \vec{x}), \tag{41}$$

where $k = |\vec{k}|$ comes from the definition of the Y -functions as spatial Fourier modes

$$\delta^{ij}\partial_i\partial_j Y = -k^2 Y. \tag{42}$$

Obviously

$$Y = Y_0 e^{\pm i\vec{k}\vec{x}}. \tag{43}$$

The relevant expressions for the background d'Alembertian is

$$\square = -\frac{1}{a^2}\partial_\tau^2 - 2\frac{a'}{a^3}\partial_\tau - \frac{k^2}{a^2} = -\partial_t^2 - 3H\partial_t - \frac{k^2}{a^2}. \tag{44}$$

Notice, that all the above expressions in this subsection are valid for a generic scale factor a .

What we want to determine are the Bardeen potentials introduced in (40). To do this we need two equations.

One equation is given by the formulation of δR in terms of Φ and Ψ accounting that the time behavior of δR itself is found above. A lengthy computation gives

$$\widetilde{\delta R} = \frac{2}{a^2} \left[k^2(\Phi - 2\Psi) - 3\frac{a'}{a}\Phi' - 6\frac{a''}{a}\Phi - 3\Psi'' - 9\frac{a'}{a}\Psi' \right], \tag{45}$$

where $\widetilde{\delta R} = \delta R - \bar{R}'(\beta + \gamma')$ is a convenient gauge invariant analog of δR . This expression is valid for a general a while its dS form is

$$\widetilde{\delta R} = -6H^2(4\Phi - \tau(\Phi' + 3\Psi') + \tau^2\Psi'') + 2\tau^2 H^2 k^2(\Phi - 2\Psi). \tag{46}$$

Another equation can be got using the following procedure. Let us exploit all the system of equation (28). First we derive the $i \neq j$ component of the system (28) which reads

$$-m^2(\Phi - \Psi) + v(\bar{\square})\widetilde{\delta R} = 0. \tag{47}$$

Second we write down the (0*i*) equation of the system (28) which is

$$2m^2(\Psi' + \mathcal{H}\Phi) + (v(\bar{\square})\widetilde{\delta R})' - \mathcal{H}v(\bar{\square})\widetilde{\delta R} = 0. \tag{48}$$

Third, we deduce the (00) equation of system (28) which yields

$$-2m^2(k^2\Psi + 3\mathcal{H}\Psi' + 3\mathcal{H}^2\Phi) - 3\mathcal{H}(v(\bar{\square})\widetilde{\delta R})' - \left(k^2 - \frac{3}{\tau^2}\right)v(\bar{\square})\widetilde{\delta R} = 0, \tag{49}$$

where the last term proportional to $1/\tau^2$ accounts the fact that the background is a dS space. Finally, multiplying (47) by k^2 , (48) by $3\mathcal{H}$ and summing these results all together with (49) as well as accounting that for the dS space-time $\mathcal{H}^2 = 1/\tau^2$ we get

$$-m^2 k^2(\Phi + \Psi) = 0, \tag{50}$$

which is our required another equation. Obviously the latter constraint simplifies the succeeding computations considerably. Since $\widetilde{\delta R}$ is provided independently by (38) one readily expresses from (47)

$$2\Phi = \frac{1}{m^2} \sum_i v(\omega_i^2) \delta R_i. \tag{51}$$

Notice that $\widetilde{\delta R}$ coincides with δR if \bar{R} is a constant or on a more general basis in the longitudinal (Newtonian) gauge $\beta = \gamma = 0$. Taking into account (37) we see that Bardeen potentials are vanishing as long as $|\text{Re } v_i| < 3/2$ for each i . If $\text{Re } v_i = 0$ for some i then the perturbations become frozen. At last, if for at least one i we have $|\text{Re } v_i| > 3/2$ then perturbations grow.

This is in perfect agreement with [11] and the comparison should be done as follows. In that reference a more general class of solutions was studied which asymptote to the de Sitter background at late times while the non-local part of the Lagrangian is given by $R\mathcal{F}(\square)R$. In the present paper we stick to the de Sitter solutions from the very beginning but the non-local part of the Lagrangian is more general. Assuming in our case $P(R) = Q(R) = R$ we retrieve the results exactly as in Section 4 in [11] where the late time regime is studied.

There is a special case $m^2 = 0$ as we loose the possibility to find out the Bardeen potentials separately. The compatibility of system (32,47) requires that either $\delta R = 0$ or there is a root of $v(\square)$. However, neither of equations coming from (28) would help in this case as all of them lack information about individual Bardeen potentials if $m^2 = 0$. Physically this reflects the fact that effectively the Einstein-Hilbert term vanishes and one reduces the number of propagating degrees of freedom.

5. No-ghost conditions

5.1. Physical excitations

In this Subsection we find out the physical excitations and their respective quadratic Lagrangians around dS background for the model (3). dS is a maximally symmetric space and this helps a lot in technical calculations. This can be accomplished in a fully covariant way using (27). To work out the task we use the covariant mode decomposition introduced as [28]

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + (\bar{\nabla}_\mu A_\nu + \bar{\nabla}_\nu A_\mu) + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{\bar{g}_{\mu\nu}}{4} \bar{\square} \right) B + \bar{g}_{\mu\nu} \frac{h}{4}. \tag{52}$$

Here $\tilde{h}_{\mu\nu}$ is a transverse and traceless tensor (spin-2), A_μ is a transverse vector (spin-1), B and h are scalars (spin-0) and the operator acting on B is traceless. Notice that here we use the fully covariant 4-dimensional decomposition and notions of spin are w.r.t. the full 4-dimensional Poincaré group. From a pure group representation arguments modes of different spins do not mix at the linearized level and therefore we can analyze them separately.

Let us first introduce few intermediate notations:

$$\sqrt{-\bar{g}}\delta_{EH} = \delta^2(\sqrt{-\bar{g}}R), \quad \delta_g = \frac{h^2}{8} - \frac{h_{\mu\nu}h^{\mu\nu}}{4},$$

where specifically around the dS background we have

$$\delta_{EH} = \left(\frac{1}{4}h_{\mu\nu}\bar{\square}h^{\mu\nu} - \frac{1}{4}h\bar{\square}h + \frac{1}{2}h\bar{\nabla}_\mu\bar{\nabla}_\rho h^{\mu\rho} + \frac{1}{2}\bar{\nabla}_\mu h^{\mu\rho}\bar{\nabla}_\nu h^{\nu\rho} \right) - \frac{\bar{R}}{24}(4h_{\mu\nu}^2 - h^2). \tag{53}$$

This can be checked using the results of [29]. We also need a variation of the scalar curvature which is

$$\delta R = (-\bar{R}_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu - \bar{g}_{\mu\nu}\bar{\square})h^{\mu\nu}. \tag{54}$$

This expression above is valid for any background.

We are now ready to compute the second variation of the full action (3) around dS background. To start with we note that this task for $P = Q = R$ was accomplished already in [30, 31]. Generically this is a very tough task but the constancy of the background scalar curvature helps a lot. It is performed by considering $\mathcal{F}(\square)$ as a Taylor series and performing a number of summations, integrations by parts and algebraic simplifications as explained in the just mentioned references. Essentially here we just need to generalize to arbitrary P and Q . To simplify the things we are using the background equation (19) wherever possible. The result of these considerations is

$$\delta^2 S = \int d^4x \sqrt{-\bar{g}} \left[\left(\frac{M_P^2}{2} + \frac{\lambda}{2} (P_R Q + Q_R P) f_0 \right) \left(\delta_{EH} - \frac{\bar{R}}{2} \delta_g \right) + \frac{\lambda}{2} \frac{1}{2} (P_{RR} Q + P Q_{RR}) f_0 (\delta R)^2 + \frac{\lambda}{2} P_R Q_R \delta R \mathcal{F}(\bar{\square}) \delta R \right], \tag{55}$$

where all P, Q quantities are evaluated on the solution to equation (19). Note also that Λ is not explicitly here due to the use of relations (12).

It is very important for the subsequent computation that around the dS background the terms in $h_{\mu\nu}$ which contain A_μ and $\bar{\nabla}_\mu \bar{\nabla}_\nu B$ do not contribute neither to $\delta_{EH} - \frac{\bar{R}}{2} \delta_g$ nor to δR . This implies that formula (52) reduces around the dS space to

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{4} g_{\mu\nu} \phi, \quad \phi = -\square B + h. \tag{56}$$

This looks like a tremendous simplification.

The spin-2 excitation $\tilde{h}_{\mu\nu}$ does not appear in the linearization of the non-local term $P\mathcal{F}Q$ apart from the constant term in the Taylor series expansion of \mathcal{F} . Simply δR evaluated on $h_{\mu\nu} = \tilde{h}_{\mu\nu}$ vanish. Therefore the spin-2 propagator remains like in a local R^2 gravity. Its careful evaluation reveals the following result

$$\delta^2 S(\hat{h}_{\mu\nu}) = \int d^4x \sqrt{-\bar{g}} \frac{1}{2} \hat{h}_{\mu\nu} \left(\bar{\square} - \frac{\bar{R}}{6} \right) \left[1 + \frac{2}{M_P^2} \frac{\lambda}{2} f_0 (PQ)_R \right] \hat{h}^{\mu\nu}, \tag{57}$$

where f_0 is the zero Taylor coefficient the $\mathcal{F}(\square)$ decomposition and the hat signifies that we have canonically normalized the field (i.e. we would have $\frac{1}{2} \hat{h}_{\mu\nu} \square \hat{h}^{\mu\nu}$ as the dynamical term for the field in GR). We notice that the expression in brackets is a purely background quantity and is therefore a constant. We just need to guarantee it is positive to preserve the no-ghost spectrum.

The spin-0 mode ϕ is a bit less trivial as it does contribute to the variations of the non-local term, which is a natural expectation. δR evaluated on $h_{\mu\nu} = \frac{1}{4} \bar{g}_{\mu\nu} \phi$ is just $-\frac{1}{4} (3\bar{\square} + \bar{R}) \phi$. One readily arrives at

$$\delta^2 S(\phi) = - \int d^4x \sqrt{-\bar{g}} \frac{\hat{\phi}}{2} \left(\bar{\square} + \frac{\bar{R}}{3} \right) \left[1 + \frac{\lambda}{M_P^2} f_0 (PQ)_R - \frac{\lambda}{M_P^2} (f_0 (P_{RR} Q + P Q_{RR}) + 2 P_R Q_R \mathcal{F}(\bar{\square})) (3\bar{\square} + \bar{R}) \right] \hat{\phi}. \tag{58}$$

where as before the hats denote a canonically normalized field.

5.2. Notes on spin-0, 2 modes dynamics

We see that the derived above result for the Lagrangians of scalar and tensor modes have a very neat structure. Namely, they clearly resemble the canonical factors known from GR (or from local R^2 modification of gravity) but on top of this the quadratic form of the scalar mode has a new non-local components.

First we stress that in order to preserve the nature of the tensorial mode we must arrange that

$$\mathcal{T} = 1 + \frac{\lambda}{M_P^2} f_0 (PQ)_R > 0. \tag{59}$$

This impose certain restrictions on functions P, Q as well as signs of parameters λ and f_0 . For example we notice that for $P = 1/Q$ the above combination is always 1, i.e. greater than zero. This can effortlessly be generalized to $PQ = \text{const}$ configuration. In order to connect this result with the previous Section we note that the 3-dimensional spin-2 excitations form a subset of the 4-dimensional spin-2 modes. This is why the dynamics of tensor perturbations in the ADM formalism does not change in our model.

The situation is more involved for the scalar mode. The first factor $\bar{\square} + \bar{R}/3$ corresponds to the standard GR scalar degree of freedom. The second factor can generate new degrees of freedom. As many as roots it has w.r.t. the d'Alembert operator. Given $\mathcal{F}(\square) = f_0$ we observe the standard Brans-Dicke scalar mode of the $f(R)$ gravity. In a general situation with an arbitrary (analytic) function $\mathcal{F}(\square)$ we must require that no more than one root exist.

Having said this we formalize the statement as follows. We demand that the following representation takes place:

$$1 + \frac{2}{M_p^2} \frac{\lambda}{2} f_0(PQ)_R - \frac{\lambda}{M_p^2} f_0(P_{RR}Q + PQ_{RR})(3\bar{\square} + \bar{R}) - \frac{\lambda}{M_p^2} 2P_R Q_R(3\bar{\square} + \bar{R})\mathcal{F}(\bar{\square}) = -(\sigma\bar{\square} - \omega^2)e^{\gamma(\bar{\square})}, \quad (60)$$

with $\gamma(\square)$ being an entire function resulting in no roots from the exponential factor. We observe at this stage that the LHS of the latter equation is exactly $\mathcal{G}(\bar{\square})/M_p^2$ where the operator $\mathcal{G}(\square)$ is defined in (32). The minus sign on the right hand side accounts for the overall minus sign in (58). This means that for $\sigma = 1$ and $\omega^2 > 0$ we have a good residue sign +1 in this new pole and keep the wrong residue sign -1 for the standard gravity scalar (the latter is required to arrange the proper IR behavior of the full graviton propagator). Hence, we demand that $\sigma = 0$ (no roots) or $\sigma = 1$ when the Brans-Dicke scalar appears. ω^2 is a real positive mass squared of the scalar mode so that it is not a tachyon.

In order to satisfy the above relation one must adjust correspondingly function $\mathcal{F}(\square)$. The initial assumption that $\mathcal{F}(\square)$ is analytic is obviously fulfilled. To gain more constraints we can check for example the consistency of the above expression at $\bar{\square} = 0$. Indeed, evaluating at zero one gets

$$1 + \frac{\lambda}{M_p^2} f_0(PQ)_R - \frac{\lambda}{M_p^2} f_0 \bar{R}(PQ)_{RR} = \omega^2 e^{\gamma(0)} > 0.$$

First we notice that in analogy with the spin-2 excitation given $PQ = \text{const}$ the latter inequality is satisfied.

However, a bit more general consideration comes from considering the latter inequality as a differential equation w.r.t. PQ as the function of \bar{R} . Generically we must analyze all possible classes of functions which may satisfy the inequality. This will be hopefully accomplished in a separate paper in conjunction with many other open questions related to the advertised here model. Nevertheless, we want to point out the following two interesting configurations.

- Requiring that $(PQ)_R - \bar{R}(PQ)_{RR}$ is a constant we immediately see that $(PQ)_R = 2\alpha\bar{R} + \beta$. Consequently, $PQ = \alpha\bar{R}^2 + \beta\bar{R} + \gamma$ favoring generalized R^2 models of the form $R\mathcal{F}R$ provided $\alpha \neq 0$.
- Second and even more intriguing case is when $(PQ)_R = \beta$. This is in a sense a sub-case of the above more general formula but it has a very interesting interplay with the whole model. First, this results in $(PQ)_{RR} = 0$ and a very simple constraint on parameters. However, on top of this special constraint results in $PQ = \beta\bar{R} + \gamma$. This in turn accommodates explicitly a possibility $P = Q = P_0 \sqrt{\bar{R} + \bar{R}_0}$. Overall this looks like a de-localized EH gravity. The main point to make here though is that as we have seen above in Section 3.3 such a peculiar form of P, Q functions leads to a possibility to solve the full Einstein equations in this model. An extended analysis of this special configuration will be undertaken in a future publication [32].

6. Stability of the constant curvature backgrounds

As the main result of Section 4 the question of stability of the de Sitter vacuum is narrowed to the satisfactory solution to eq. (37). ν in turn depends on the structure of the non-local operator $\mathcal{G}(\square)$ such that

$$\nu = \sqrt{\frac{9}{4} - \frac{\omega^2}{H^2}}, \quad (61)$$

and $\mathcal{G}(\omega^2) = 0$.

Following equation (60) and the comment thereafter we have an explicit form for the operator \mathcal{G} . Moreover, as we have already understood in the previous Section the requirement that the system does not lead to ghosts demands that there is no more than one such a root ω^2 for the operator \mathcal{G} .

To make the analysis of roots tractable we specialize to monomials [33]

$$P(R) = R^p, \quad Q(R) = R^q, \tag{62}$$

for some nonzero p and q . At the background level this results in the following modification of equation (19)

$$M_p^2 R - 4\Lambda + \lambda f_0 R^{p+q}(2 - p - q) = 0. \tag{63}$$

This equation can be solved in general w.r.t. R as long as p, q are integers and $-3 \leq p + q \leq 4$. As in the previous Section it is useful to analyze $\mathcal{G}(0)$ to see whether the stability condition can be reached. Indeed, \mathcal{G} is analytic by construction but a compatibility condition must be fulfilled

$$M_p^2 + \lambda R^{p+q-1}(p + q)(2 - p - q)f_0 = \omega^2 e^{\nu(0)}. \tag{64}$$

It is obvious from (61) that as long as ω^2 is real it should be at least positive in order to satisfy (37). The latter condition (64) clearly shows that ω^2 is indeed real and therefore reduces to the following necessary inequality

$$M_p^2 + \lambda R^{p+q-1}(p + q)(2 - p - q)f_0 > 0. \tag{65}$$

Satisfactory solution to this relation is a necessary stability condition. From here it is obvious that two special cases, namely $p + q = 0$ and $p + q = 2$ always have a stable de Sitter phase.

In a general situation we have to understand equation (63) together with the latter inequality (65). As a side turn one can simplify (65) using the background equation (63) to

$$M_p^2 R(p + q - 1) < 4\Lambda(p + q). \tag{66}$$

One can see from this representation that for $p + q = 1$ (directly related to (18)) one can have a stable de Sitter phase given a positive Λ . The latter condition is not improbable as long as $\lambda f_0 > 0$.

In an attempt to solve the system (63) and (65) one can rewrite it as

$$1 - s + u = 0, \quad 1 + uz > 0, \tag{67}$$

where $s = \frac{4\Lambda}{M_p^2 R}$, $z = p + q$, $u = \frac{\lambda f_0}{M_p^2} R^{z-1}(2 - z)$. This latter system looks rather simple but unfortunately does not provide immediate new interesting solutions from the physical point of view.

7. Discussion and conclusion

The main subject of our investigation was the generalized gravity theory represented by (3) with the emphasis on the de Sitter solution. This theory contains higher derivatives and as such has a potentially dangerous property to generate ghosts. However, the ghost-free condition can be clearly formulated and is translated to the following statements: (i) the operator $\mathcal{G}(\square)$ (eq. (32) or explicitly the LHS of equation (60) multiplied by M_p^2) has no more than one root; (ii) one should arrange $\sigma = 0$ or 1 and ω^2 is positive in (60); (iii) \mathcal{T} in (59) must be positive. Going further we find that simple algebraic conditions (63) and (65) determine a possibility to have stable dS configurations.

Namely, satisfying the inequality (65) one guarantees that the de Sitter phase is stable. Interestingly, for $P = R^p, Q = R^q$ two special cases $p + q = 0$ and $p + q = 2$ always lead to stable de Sitter solutions. The first case is a generalized “cosmological constant” (since $PQ = 1$). The second case is the generalized R^2 theory and for now is the mainly developed candidate for a renormalizable theory of gravity [25].

Generically one would expect such a non-local model to be a candidate for the UV gravity completion. In this case a possible de Sitter phase would be attributed to the inflationary stage of the Universe evolution. This way the de Sitter phase should not be eternally stable as the inflation should stop. This implies for example that the presently found bounce solutions in non-local gravity with $P = Q = R$ [11, 34] cannot be straightforwardly continued to a viable cosmological inflation. This is because those solutions end up with a dS expansion which is eternally stable as it corresponds to $p + q = 2$ in (65). Even though the Starobinsky inflation is here, in the model with $P = Q = R$, this solution has no a bounce phase while an idea to construct a joint bounce and inflation is an intriguing open question on the table.

On contrary one can try to use non-local models in an attempt to challenge the present day slowly accelerated expansion of the Universe. In this case one has to arrange a stable evolution for a long period of time to be compatible with the presently observed Universe. In this regime models with $P = Q = 1$ seem to be absolutely suitable.

One more observed special case is a generalized Einstein-Hilbert term with $p = q = 1/2$. This leads to a possibility to solve completely the equations of motion provided we can find a solution to equation (18). This solution will automatically be a solution of a local gravity action (17). However, finding a solution to (18) turns out to be a complicated problem and we keep it in a separate study [32].

Generically presence or absence of stable de Sitter configurations in model (3) with monomial P and Q has been reduced to a really simple algebraic system (67). In a general situation both stable and unstable regimes can be organized based on the parameters of the model. In all regimes we can control the order of the non-locality scale \mathcal{M} using other parameters in the theory.

Some other cosmological solutions of modified gravity with analytic nonlocality can be found in [35].

References

- [1] R. Wald, *General Relativity* (University of Chicago Press, 1984).
- [2] P. A. R. Ade et al. [Planck Collaboration], *Planck 2013 results. XVI. Cosmological parameters*, *Astron. Astrophys.* (2014) [1303.5076 [astro-ph.CO]].
- [3] A. Borde, A. H. Guth and A. Vilenkin, *Inflationary space-times are incomplete in past directions*, *Phys. Rev. Lett.* **90** (2003) 151301 [gr-qc/0110012].
- [4] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, *Modified gravity and cosmology*, *Phys. Rep.* **513**, 1–189 (2012) [1106.2476v2 [astro-ph.CO]]; S. Nojiri and S. D. Odintsov, *Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models*, *Phys. Rep.* **505**, 59–144 (2011) [1011.0544v4 [gr-qc]]; T. P. Sotirou and V. Faraoni, *$f(R)$ theories of gravity*, *Rev. Mod. Phys.* **82** (2010) 451–497 [0805.1726v4 [gr-qc]].
- [5] I. Ya. Aref'eva, D.M. Belov, A.A. Giryavets, A.S. Koshelev, P.B. Medvedev, *Noncommutative Field Theories and (Super)String Field Theories*, hep-th/0111208; W. Taylor and B. Zwiebach, *D-branes, Tachyons, and String Field Theory*, [hep-th/0311017].
- [6] V.S. Vladimirov, I.V. Volovich, I.V. Zelenov, *p -Adic Analysis and Mathematical Physics* (Singapore: World Scientific (1994)); B. Dragovich, A. Yu. Khrennikov, S. V. Kozyrev and I. V. Volovich, *On p -Adic Mathematical Physics*, *p -Adic Numbers Ultrametric Anal. Appl.* **1**, 1–19 (2009) [0904.4205 [math-ph]].
- [7] S. Deser, R.P. Woodard, *Nonlocal Cosmology*, *Phys. Rev. Lett.* **99**, 111301 (2007) [0706.2151 [astro-ph]].
- [8] S. Jhingan, S. Nojiri, S. D. Odintsov, M. Sami, I. Thongkool, S. Zerbini, *Phantom and non-phantom dark energy: The Cosmological relevance of non-locally corrected gravity*, *Phys. Lett.* **B663** (2008) 424–428, [0803.2613, [hep-th]]; S. Nojiri, S. D. Odintsov, *Modified non-local- $F(R)$ gravity as the key for the inflation and dark energy*, *Phys. Lett.* **B659** (2008) 821–826, [0708.0924 [hep-th]].
- [9] T. Biswas, A. Mazumdar and W. Siegel, *Bouncing universes in string-inspired gravity*, *JCAP* **0603** (2006) 009 [hep-th/0508194].
- [10] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, *Towards singularity and ghost free theories of gravity*, *Phys. Rev. Lett.* **108** (2012) 031101 [1110.5249 [gr-qc]].
- [11] T. Biswas, A. S. Koshelev, A. Mazumdar and S. Y. Vernov, *Stable bounce and inflation in non-local higher derivative cosmology*, *JCAP* **1208** (2012) 024 [1206.6374 [astro-ph.CO]].
- [12] A. S. Koshelev, *Stable analytic bounce in non-local Einstein-Gauss-Bonnet cosmology*, *Class. Quant. Grav.* **30** (2013) 155001 [1302.2140 [astro-ph.CO]].
- [13] A. S. Koshelev, S. Y. Vernov, *Cosmological Solutions in Nonlocal Models*, 1406.5887 [gr-qc];
- [14] B. Craps, T. De Jonckheere and A. S. Koshelev, *Cosmological perturbations in non-local higher-derivative gravity*, *JCAP* **1411**, no. 11, 022 (2014) [1407.4982 [hep-th]].
- [15] G. Calcagni, L. Modesto and P. Nicolini, *Super-accelerating bouncing cosmology in asymptotically-free non-local gravity*, [1306.5332 [gr-qc]]; L. Modesto, *Phys.Rev.* **D86**, 044005 (2012), 1107.2403; L. Modesto, J. W. Moffat and P. Nicolini, *Black holes in an ultraviolet complete quantum gravity*, *Phys. Lett. B* **695**, 397 (2011).

- [16] B. Dragovich, *Towards p -adic matter in the universe*, Springer Proc. Math. Stat. **36**, 13–24 (2013) [[1205.4409](#) [hep-th]].
- [17] A. Conroy, A. Mazumdar, S. Talaganis and A. Teimouri, *Non-local gravity in D -dimensions: Propagator, entropy and bouncing Cosmology*, [1509.01247](#) [hep-th].
- [18] A. A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, Phys. Lett. B **91**, 99 (1980) [Phys. Lett. **91B**, 99 (1980)] [Adv. Ser. Astrophys. Cosmol. **3**, 130 (1987)].
- [19] A. S. Koshelev, L. Modesto, L. Rachwal and A. A. Starobinsky, *Occurrence of exact R^2 inflation in non-local UV-complete gravity*, JHEP **1611**, 067 (2016) [arXiv:1604.03127 [hep-th]].
- [20] T. Biswas, T. Koivisto and A. Mazumdar, *Nonlocal theories of gravity: the flat space propagator*, arXiv:1302.0532 [gr-qc].
- [21] R. Gualtieri et al. [SPIDER Collaboration], *SPIDER: CMB Polarimetry from the Edge of Space*, arXiv:1711.10596 [astro-ph.CO].
- [22] A. A. Starobinsky, *Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations*, Phys. Lett. **117B**, 175 (1982).
- [23] A. S. Koshelev, K. Sravan Kumar and A. A. Starobinsky, *R^2 inflation to probe non-perturbative quantum gravity*, JHEP **1803**, 071 (2018) [arXiv:1711.08864 [hep-th]].
- [24] K. Stelle, Phys.Rev. D**16**, 953 (1977).
- [25] S. Talaganis, T. Biswas and A. Mazumdar, *Towards understanding the ultraviolet behavior of quantum loops in infinite-derivative theories of gravity*, [1412.3467](#) [hep-th];
P. Don, S. Giaccari, L. Modesto, L. Rachwal and Y. Zhu, *Scattering amplitudes in super-renormalizable gravity*, JHEP **1508**, 038 (2015), [[1506.04589](#) [hep-th]].
- [26] I. Dimitrijevic, B. Dragovich, J. Stankovic, A. S. Koshelev and Z. Rakic, *On nonlocal modified gravity and its cosmological solutions*, Springer Proc. Math. Stat. **191**, 35–51 (2014), [arXiv:1701.02090 [hep-th]].
- [27] S. Y. Vernov, *Localization of Non-local Cosmological Models with Quadratic Potentials in the case of Double Roots*, Class. Quant. Grav. **27**, 035006 (2010) [arXiv:0907.0468 [hep-th]].
- [28] E. D’Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, *Graviton and gauge boson propagators in $AdS(d+1)$* , Nucl. Phys. B **562**, 330 (1999) [hep-th/9902042].
- [29] S. M. Christensen and M. J. Duff, *Quantizing Gravity with a Cosmological Constant*, Nucl. Phys. B **170**, 480 (1980).
- [30] T. Biswas, A. S. Koshelev and A. Mazumdar, *Gravitational theories with stable (anti-)de Sitter backgrounds*, Fundam. Theor. Phys. **183**, 97 (2016) [arXiv:1602.08475 [hep-th]].
- [31] T. Biswas, A. S. Koshelev and A. Mazumdar, *Consistent higher derivative gravitational theories with stable de Sitter and anti de Sitter backgrounds*, Phys. Rev. D **95**, no. 4, 043533 (2017) [arXiv:1606.01250 [gr-qc]].
- [32] I. Dimitrijevic, B. Dragovich, A. S. Koshelev, Z. Rakic and J. Stankovic, work in progress.
- [33] I. Dimitrijevic, *Cosmological solutions in modified gravity with monomial nonlocality*, Applied Mathematics and Computation **285**, 195–203 (2016).
- [34] A. S. Koshelev and S. Y. Vernov, *On bouncing solutions in non-local gravity*, Phys. Part. Nucl. **43**, 666 (2012) [arXiv:1202.1289 [hep-th]].
- [35] I. Dimitrijevic, B. Dragovich, J. Grujic and Z. Rakic, *Some Cosmological Solutions of a Nonlocal Modified Gravity*, Filomat **29** (3), (2015), 619–628 [[1508.05583](#) [hep-th]];
I. Dimitrijevic, B. Dragovich, J. Grujic and Z. Rakic, *Some Power-Law Cosmological Solutions in Nonlocal Modified Gravity*, Springer Proc. Math. Stat. **111**, 241 (2014);
I. Dimitrijevic, B. Dragovich, J. Stankovic, A. S. Koshelev and Z. Rakic, *On Nonlocal Modified Gravity and Its Cosmological Solutions*, Springer Proc. Math. Stat. **191**, 35–51 (2016);
I. Dimitrijevic, B. Dragovich, Z. Rakic and J. Stankovic, *On Nonlocal Gravity with Constant Scalar Curvature*, Publications de l’Institut Mathematique, **103(117)**, 53–59 (2018).