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On One Problem of Connections in the Space of Non-symmetric Affine Connection and its Subspace

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Abstract. Let X_M be a submanifold of a differentiable manifold X_N ($X_M \subset X_N$). If on X_N a non-symmetric affine connection L is defined by coefficients $L^i_{jk} \neq L^i_{kj}$ and on X_M a non-symmetric basical tensor g ($g_{\alpha\beta} \neq g_{\beta\alpha}$) is given, in the present paper we investigate the problem: Find a relation between induced connection $\overline{L}$ from L_N into X_M end the connection $\overline{\Gamma}$, defined by the tensor g in X_M. The solutions is given in the Theorem 3.1., that is by the equation (3.9). Some examples are constructed.

1. Introduction

Let $L_N = (X_N, L)$ be a space of non-symmetric affine connection, where X_N is a differentiable manifold, and L^i_{jk} nonsymmetric connection. Suppose that X_M is a differentiable submanifold of X_N ($X_M \subset X_N$) and on X_M is given a non-symmetric basic tensor g ($g_{\alpha\beta} \neq g_{\beta\alpha}$). Then $GR_M = (X_M, g_{\alpha\beta})$ is so called generalized Riemannian space GR_M [1], defined on the submanifold $X_M \subset X_N$.

Let $X_M \subset X_N$ be defined in local coordinates by equations

$$x^i = x^i(u^1, \cdots, u^M) \equiv x^i(u^\alpha), \quad i = 1, \cdots, N, \quad \alpha = 1, \cdots, M. \tag{1.1}$$

The partial derivatives

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \quad (\text{rank}\,(B^i_\alpha) = M), \tag{1.2}$$

define tangent vectors on X_M.

Consider $N - M$ contravariant vectors C^i_A ($A, B, \cdots \in \{M+1, \cdots, N\}$), which are defined on X_M and are linearly independent mutually and with B^i_α. If the matrix $\begin{pmatrix} \overline{B}^\alpha_i \\ \overline{C}^A_i \end{pmatrix}$ is inverse for (B^i_α, C^i_A), the following conditions are satisfied

$$\begin{aligned} &\text{a)}\ \ B^i_\alpha \overline{B}^\beta_i = \delta^\beta_\alpha, \qquad \text{b)}\ \ B^i_\alpha \overline{C}^A_i = 0, \qquad \text{c)}\ \ \overline{B}^\alpha_i C^i_A = 0, \\ &\text{d)}\ \ C^i_A \overline{C}^B_i = \delta^B_A, \qquad \text{e)}\ \ B^i_\alpha \overline{B}^\alpha_j + C^i_A \overline{C}^A_j = \delta^i_j. \end{aligned} \tag{1.3}$$

The quantities $B^i_\alpha, \overline{B}^\alpha_i$ are *projection factors*, and $C^i_A, \overline{C}^A_i$ are *affine pseudonormals* of the submanifold X_M.

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2. Determination of GR_N on X_N

Our task is to obtain a relation between induced connection $\overline{L}$ from L_N into $X_M \subset X_N$ and connection $\overline{\Gamma}$, defined by Christoffel symbols expressed by help of non-symmetric tensor $g_{\alpha\beta}(u^1, \cdots, u^M)$, which is given on X_M, i.e. when we have $GR_M = (X_M, g_{\alpha\beta})$.

Firstly, we will show how on X_N can be defined a metric tensor G_{ij} in the manner $g_{\alpha\beta}$ to be induced one for G_{ij}. In that case we will have a generalized Riemannian space $GR_N = (X_N, G_{ij})$ and its subspace $GR_M = (X_M, g_{\alpha\beta})$. Starting from the known relation

$$\begin{aligned} &G_{ij}B^i_\alpha B^j_\beta = g_{\alpha\beta}, \quad i, j = 1, \cdots, N; \\ &\qquad \alpha, \beta = 1, \cdots, M; \quad \mathrm{rank}(B^i_\alpha) = M, \end{aligned} \tag{2.1}$$

we have (supposing a non of symmetry $g_{\alpha\beta}$ and G_{ij}) M^2 eq-s with N^2 unknowns G_{ij} (B^i_α, B^j_β are defined by (1.1) and (1.2)). Because $M < N$, in the system (2.1) $N^2 - M^2$ unknowns G_{ij} can be taken arbitrary, and the rest be ordered, under the condition $\mathrm{rank}(B^i_\alpha) = M$. In the general case we have innumerable solutions of the system (2.1) wrt G_{ij}. So, we have proved

Theorem 2.1. *Let $L_N = (X_N, L)$ be a space of nonsymmetric affine connection L^i_{jk}, $GR_M = (X_M, g_{\alpha\beta})$a generalized Riemannian space and X_M a submanifold of X_N ($X_M \subset X_N$) defined by* (1.1)*. Then by means of* (2.1) *can be determined in numberless manners a tensor G_{ij} on X_N, so that $g_{\alpha\beta}$ be induced for G_{ij}.*

Example 2.1. *Find G_{ij} by virtue of* (2.1) *for $N = 3$, $M = 2$, i.e. if $X_2 \subset X_3$ is defined by eq-s*

$$x^i = x^i(u^1, u^2), \quad i = 1, 2, 3 \tag{2.2}$$

and with given $g_{\alpha\beta}$.

Solution. With respect of (2.1) we get

$$\begin{aligned} &G_{ij}B^i_1 B^j_1 = g_{11}, \quad G_{ij}B^i_1 B^j_2 = g_{12}, \\ &G_{ij}B^i_2 B^j_1 = g_{21}, \quad G_{ij}B^i_2 B^j_2 = g_{22}, \end{aligned} \tag{2.3}$$

with given $g_{\alpha\beta}$.

We have here $N^2 = 3^2 = 9$ unknowns G_{ij} and $M^2 = 2^2 = 4$ linear eq-s.

So, we can find four unknowns G_{ij} and the rest take arbitrary. For example, except G_{11}, G_{12}, G_{22}, G_{33}, take the remaining G_{ij} to be zero. Then, from (2.3) we obtain

$$\begin{aligned} &G_{11}(B^1_1)^2 + G_{12}B^1_1 B^2_1 + G_{22}(B^2_1)^2 + G_{33}(B^3_1)^2 = g_{11} \\ &G_{11}B^1_1 B^1_2 + G_{12}B^1_1 B^2_2 + G_{22}B^2_1 B^2_2 + G_{33}B^3_1 B^3_2 = g_{12} \\ &G_{11}B^1_2 B^1_1 + G_{12}B^1_1 B^2_1 + G_{22}B^2_2 B^2_1 + G_{33}B^3_2 B^3_1 = g_{21} \\ &G_{11}(B^1_2)^2 + G_{12}B^1_2 B^2_2 + G_{22}(B^2_2)^2 + G_{33}(B^3_2)^2 = g_{22}. \end{aligned}$$

From this system one obtains G_{11}, G_{12}, G_{22}, G_{33}. As a particular case of the eq-s (2.2), let us take

$$x^1 = (u^1)^2, \quad x^2 = u^1 u^2, \quad x^3 = -(u^2)^2, \tag{2.4}$$

and for $g_{\alpha\beta}$:

$$g_{11} = (u^2)^2, \quad g_{22} = -g_{21} = u^1 + u^2, \quad g_{22} = u^1 u^2. \tag{2.5}$$

Then it is

$$\begin{aligned} &B_1^1 = \partial x^1/\partial u^1 = 2u^1, \quad B_1^2 = u^2, \quad B_1^3 = 0,\\ &B_2^1 = \partial x^1/\partial u^2 = 0, \qquad B_2^2 = u^1, \quad B_2^3 = -2u^2, \end{aligned} \tag{2.6}$$

and from obtained system it follows that

$$\begin{aligned} &G_{11} = \frac{u^1 g_{11} - u^2 g_{12}}{4(u^1)^3}, \quad G_{12} = \frac{g_{12} - g_{21}}{2(u^1)^2},\\ &G_{22} = \frac{g_{21}}{u^1 u^2}, \quad G_{33} = \frac{u^2 g_{22} - u^1 g_{21}}{4(u^2)^3} \end{aligned} \tag{2.7}$$

(under condition $u^1 u^2 \neq 0$), where $g_{\alpha\beta}$ are functions of u^1, u^2, for ex. (2.5). We see that in generally is $G_{ij} \neq G_{ji}$, because of $g_{12} \neq g_{21}$. For example, $G_{21} = 0$ by supposition, and from (2.7) it is $G_{12} \neq G_{21}$ generally. Accordingly, we have obtained $GR_2 \subset GR_3$.

3. Relation between the connections $\overline{\mathrm{L}}$ and $\overline{\Gamma}$

We can start now to determine a relation between $\overline{L}$ and $\overline{\Gamma}$, as we have said at the beginning of the Section 2. Let $h_{\alpha\beta}$ be the symmetric part of $g_{\alpha\beta}$, i.e.

$$h_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} + g_{\beta\alpha}) \tag{3.1}$$

and $h^{\alpha\beta}$ satisfies the condition

$$h_{\alpha\beta} h^{\gamma\beta} = \delta_\beta^\gamma. \tag{3.2}$$

It is analogously

$$H_{ij} H^{kj} = \delta_i^k, \tag{3.3}$$

where H_{ij} is symmetric part of G_{ij}. We can introduce a connection Γ^i_{jk} on X_N by G_{ij} as defined above. The connection $\overline{\Gamma}^\alpha_{\beta\gamma}$ can be found starting from Christoffel symbols in GR_M:

$$\overline{\Gamma}^\alpha_{\beta\gamma} = h^{\pi\alpha}\overline{\Gamma}_{\pi.\beta\gamma} = \frac{1}{2} h^{\pi\alpha}(g_{\beta\pi,\gamma} - g_{\beta\gamma,\pi} + g_{\pi\gamma,\beta}). \tag{3.4}$$

We find corresponding derivatives in the brackets, for example

$$\begin{aligned} g_{\beta\pi,\gamma} &= \frac{\partial}{\partial u^\gamma} g_{\beta\pi} = (G_{ij} B^i_\beta B^j_\pi)_{,\gamma}\\ &= G_{ij,k} B^k_\gamma B^i_\beta B^j_\pi + G_{ij} B^i_{\beta\gamma} B^j_\pi + G_{ij} B^i_\beta B^j_{\pi\gamma}. \end{aligned}$$

In this way, by substituting into (3.4), we get

$$\overline{\Gamma}^\alpha_{\beta\gamma} = \widetilde{B}^\alpha_i (\Gamma^i_{jk} B^j_\beta B^k_\gamma + B^i_{\beta\gamma}), \tag{3.5}$$

where

$$\widetilde{B}^\alpha_i = h^{\pi\alpha} H_{pi} B^p_\pi. \tag{3.6}$$

On the other hand, the induced connection from L_N into X_M is ([2], [3]):

$$\overline{L}^{\alpha}_{\beta\gamma} = \overline{B}^{\alpha}_{i}(L^{i}_{jk}B^{j}_{\beta}B^{k}_{\gamma} + B^{i}_{\beta\gamma}). \tag{3.7}$$

We will examine a relation between $\overline{B}^{\alpha}_{i}$ and $\widetilde{B}^{\alpha}_{i}$. By substituting $\widetilde{B}^{\alpha}_{i}$ into (1.3) instead $\overline{B}^{\alpha}_{i}$ and normals N^{i}_{A} on GR_M in place of pseudonormals C^{i}_{A}, we conclude that these equations are satisfied. E.g., using (3.7,3.2), we have

$$B^{i}_{\alpha}\widetilde{B}^{\beta}_{i} \underset{(3.6)}{=} B^{i}_{\alpha}h^{\pi\beta}H_{pi}B^{p}_{\pi} = h^{\pi\beta}h_{\alpha\pi} \underset{(3.2)}{=} \delta^{\beta}_{\alpha}.$$

By the same procedure can be checked the rest eq-s from (1.3). So, the matrix $\begin{pmatrix} \widetilde{B}^{\alpha}_{i} \\ \widetilde{N}^{A}_{i} \end{pmatrix}$ is inverse for $(B^{i}_{\alpha}, N^{i}_{A})$, (in GR_N we have $\overline{C}^{A}_{i} = \overline{N}^{A}_{i} = \widetilde{N}^{A}_{i}$) and it follows that

$$\overline{B}^{\alpha}_{i} = \widetilde{B}^{\alpha}_{i}. \tag{3.8}$$

Taking in mind this equation, from (3.5), (3.7) one obtains

$$\boxed{\overline{L}^{\alpha}_{\beta\gamma} - \overline{\Gamma}^{\alpha}_{\beta\gamma} = (L^{i}_{jk} - \Gamma^{i}_{jk})\overline{B}^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma},} \tag{3.9}$$

and that is the relation we look for.

From exposed it follows the next theorem

Theorem 3.1. *Let $L_N = (X_N, L)$ be a space of nonsymmetric affine connection, defined by coefficients L^{i}_{jk} on a differentiable manifold X_N and $GR_M = (X_M, g_{\alpha\beta})$ a generalized Riemannian space defined by means of nonsymmetric basic tensor $g_{\alpha\beta}$ on the submanifold $X_M \subset X_N$, which is defined by* (1.1). *Then the equation* (3.9) *gives the relation between induced connection $\overline{L}^{\alpha}_{\beta\gamma}$ from L_N into X_M and the connection defined in X_M on the base of Christoffel symbols $\overline{\Gamma}^{\alpha}_{\beta\gamma}$ obtained wrt $g_{\alpha\beta}$, where $B^{j}_{\beta} = \partial x^{j}/\partial u^{\beta}$, and $\overline{B}^{\alpha}_{i}$ is defined by eq-s* (1.3), (3.6) *and* (3.8).

Example 3.1. *Suppose that, as in the Example 2.1., $X_2 \subset X_3$ be defined by eq-s* (2.4), *$g_{\alpha\beta}$ by* (2.5), *L^{i}_{jk} have values*

$$L^{1}_{11} = x^{1}, \quad L^{1}_{12} = x^{1}x^{2}, \quad L^{1}_{21} = x^{1} + x^{2}, \quad \textit{the rest} \quad L^{i}_{jk} = 0, \tag{3.10}$$

and the values of C^{i}_{A} $(A = 3)$ are given as follows

$$C^{1}_{3} \equiv C^{1} = u^{1}, \quad C^{2}_{3} \equiv C^{2} = 0, \quad C^{3}_{3} \equiv C^{3} = 1. \tag{3.11}$$

Find components of induced connection $\overline{L}^{\alpha}_{\beta\gamma}$ from L_3 into X_2 using (3.7).

Solution. In order to apply (3.7), we firstly find $\overline{B}^{\alpha}_{i}, \overline{C}^{A}_{i}$. In the present case is

$$\begin{aligned} \mathcal{M} = (B^{i}_{\alpha}, C^{i}_{A}) &= \begin{pmatrix} B^{1}_{\alpha} & C^{1}_{A} \\ B^{2}_{\alpha} & C^{2}_{A} \\ B^{3}_{\alpha} & C^{3}_{A} \end{pmatrix} = \begin{pmatrix} B^{1}_{1} & B^{1}_{2} & C^{1} \\ B^{2}_{1} & B^{2}_{1} & C^{2} \\ B^{3}_{1} & B^{3}_{1} & C^{3} \end{pmatrix} \\ &= \begin{pmatrix} 2u^{1} & 0 & u^{1} \\ u^{2} & u^{1} & 0 \\ 0 & -2u^{2} & 1 \end{pmatrix}, \end{aligned} \tag{3.12}$$

$$|\mathcal{M}| = \det \mathcal{M} = 2(u^1)^2 - 2u_1(u^2)^2, \tag{3.13}$$

$$\mathcal{M}^{-1} = \begin{pmatrix} \overline{B}_i^\alpha \\ \overline{C}_i^A \end{pmatrix} = \begin{pmatrix} \overline{B}_1^\alpha & \overline{B}_2^\alpha & \overline{B}_3^\alpha \\ \overline{C}_1 & \overline{C}_2 & \overline{C}_3 \end{pmatrix} = \begin{pmatrix} \overline{B}_1^1 & \overline{B}_2^1 & \overline{B}_3^1 \\ \overline{B}_1^2 & \overline{B}_2^2 & \overline{B}_3^2 \\ \overline{C}_1 & \overline{C}_2 & \overline{C}_3 \end{pmatrix}. \tag{3.14}$$

On the other hand wrt (3.12) is

$$\mathcal{M}^{-1} = \frac{1}{|\mathcal{M}|} \begin{pmatrix} u^1 & -2u^1u^2 & -(u^1)^2 \\ u^2 & -2u^1 & -u^1u^2 \\ -2(u^2)^2 & 4u^1u^2 & 2(u^1)^2 \end{pmatrix} \tag{3.15}$$

By comparing of (3.14) and (3.15), we conclude:

$$\overline{B}_1^1 = \frac{u^1}{|\mathcal{M}|}, \quad \overline{B}_2^1 = -\frac{2u^1u^2}{|\mathcal{M}|}, \quad \cdots \quad \overline{C}_3 = \frac{2(u^1)^2}{|\mathcal{M}|}. \tag{3.16}$$

To find $\overline{L}_{\beta\gamma}^\alpha$ by virtue of (3.7), remark that B_α^i are given in (2.6), $\overline{B}_i^\alpha$ in (3.16), L_{jk}^i in (3.10), where x^i have the values (2.4).

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